Decoherence of the field state in the single-mode laser

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The rate at which an initially pure field state goes into mixture is calculated for the single-mode laser under the detailed balance condition. It is shown that the field state similar to the coherent state originally suggested by M. O. Scully and W. E. Lamb, Jr. is selected as the most stable state among squeezed states, and that the speed of whole decoherence process is strongly dependent on the quality factor Q of a laser cavity. In a laser cavity with extremely high quality factor, squeezed states other than the Scully-Lamb ansatz remain stable on the observation time scale.

Introduction

While there is no doubt about preciseness of the quantum theory especially in the atomic scale and below, the relation between the quantum theory and the classical reality we usually experience in a macroscopic scale is still an issue of controversy. The most counter-intuitive part in the formalism of the quantum theory may be the superposition principle, which allows a quantum system to be in two or more states simultaneously. This paradoxical aspect of the quantum theory is often illustrated with the infamous "Schrödinger's cat".

In the last two decades, a theory of decoherence [1,2] has been developed as a solution to explain the obvious discontinuity between the quantum world and the classical reality. It is plausible to state that the system of interest in the macroscopic scale should be treated not as an isolated system but as an open system continuously interacting with surrounded systems with many quantum degrees of freedom (the environment). According to the decoherence theory, it is exactly this interaction which destroys the annoying superposition states, and as a result of the negative selection process the classical reality spontaneously emerges out of the quantum states expressed in the Hilbert space.

W. H. Zurek and his collaborators identified classicality of states with predictability and characterized the effectiveness of decoherence process in terms of the *predictability sieve*. They defined the "predictability sieve" as a method to search for classical *preferred states* (the observable states stable against the environment on the observation time scale) by measuring the decoherence rate of pure initial states to mixtures. The initial states which are least affected by the environment are singled out as the best candidates for the "preferred states", while the states unstable compared to the observation time are excluded as non-observable ones. They studied the decoherence mechanism in a specific environment model, where a quantum harmonic oscillator undergoes quantum Brownian motion in thermal equilibrium.[3] They demonstrated by means of linear entropy that the coherent states are singled out as the maximally predictive states.

In this presentation, the single-mode laser is investigated as an extended general example of the environment model studied by Zurek and collaborators.[3] The laser is known to be a macroscopic open system in non-thermal equilibrium. In a laser cavity, the electromagnetic field interacts with a laser medium in steady energy flux, which is poured in by pumping and poured out as transmission loss through a half mirror. In this hot open system, the electromagnetic field is a system of interest and other systems interacting with the field correspond to an environment. The laser field is often assumed to be a coherent state, but it is rather a coherent state with phase diffusion in a mixed state than the Glauber coherent state in a pure state,[4] since the field is continuously subject to decoherence from its environment. In the context of the decoherence theory, it is easily inferred that the state similar to the coherent state with phase diffusion should have the longest life time and would be selected as the most stable states in the decoherence process, whereas other states are excluded as transient non-observable states.

The decoherence theory will be applied to the Scully-Lamb single-mode laser model above threshold.[5] Time evolution of the field is numerically calculated under the detailed balance condition on the photon number, where the photon statistics of the field is in steady-state. Pure states with squeezing parameter are initially prepared and each decoherence rate is computed by measuring increase of the uncertainty area of the field quadrature-components.

Comparison to the previous studies

The Scully-Lamb single-mode laser master equation is

$$\begin{aligned}\frac{d\rho_{n,m}}{dt} = &-\left(\frac{N'_{n,m}A}{1+N_{n,m}\frac{B}{A}}\right)\rho_{n,m}\\ &+\left(\frac{\sqrt{nm}A}{1+N_{n-1,m-1}\frac{B}{A}}\right)\rho_{n-1,m-1}\\ &-\frac{1}{2}\frac{\omega}{Q}(n+m)\rho_{n,m}\\ &+\frac{\omega}{Q}\sqrt{(n+1)(m+1)}\rho_{n+1,m+1},\end{aligned} \tag{1}$$

where ω is the mode frequency, Q is the Q factor of the cavity, A is the linear gain coefficient, B is the self-saturation coefficient and $N'_{n,m}$ and $N_{n,m}$ are

$$N'_{n,m} = \frac{1}{2}(n+1+m+1)+\frac{1}{8}\frac{B}{A}(n-m)^2 \tag{2}$$

$$N_{n,m} = \frac{1}{2}(n+1+m+1)+\frac{1}{16}\frac{B}{A}(n-m)^2. \tag{3}$$

[6,7]

$\omega \sim 2\pi\times 10^{15} rad\cdot t^{-1}$. The laser master equation (1) well above threshold may be written in the general framework of the Lindblad form.[8] The Lindblad form of evolution is written as

$$L[\rho] = \frac{1}{\mathrm{i}\hbar}[H,\rho] + \frac{1}{2\hbar}\sum_j([V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger]). \tag{4}$$

The environment effects in the Lindblad form master equation has recently been studied in terms of the predictability.[9,10] But the previous studies treated only the case of $\{V_j\}$ linear in position $\hat{x}$ and momentum $\hat{p}$,

$$\sum_j([V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger])$$
$$V_j = a_j\hat{p} + b_j\hat{x} \qquad (a_j, b_j : \text{complex numbers}),$$

which is known as an analytically solvable model,[11] whereas in the case of the laser model well above threshold, the gain terms (the first and the second terms) of Eq. (1) include $\{V_j\}$ non-linear in position and momentum, or $\{V_j\}$ non-linear in the quadrature-components of the field amplitude $\hat{X} \equiv (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{Y} \equiv (\hat{a} - \hat{a}^\dagger)/(2\mathrm{i})$[7] in the terminology of quantum optics, as

$$\int_0^\infty ds([V(s)\rho, V(s)^\dagger] + [V(s), \rho V(s)^\dagger])$$
$$V(s) = \hat{a}^\dagger \exp\left(-\frac{s}{2}\hat{a}\hat{a}^\dagger\right),$$

where $\hat{a}$ is the annihilation operator for photons defined by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \mathrm{i}\frac{1}{m\omega}\hat{p}\right). \tag{5}$$

The non-linearity of $\{V_j\}$ in the laser model comes from dependence of the denominator on the photon number n and m in the gain terms. This photon number dependence in the gain terms is essential for the laser model, because it gives rise to non-thermal equilibrium of the laser field, *i.e.*, the Poissonian photon statistics, in steady-state, which is the notable feature of lasers. Hence the laser model may be regarded as an extended case of the previously studied model [9,10] to a highly relevant non-linear system of the Lindblad form.

Decoherence rate under the detailed balance condition

In this presentation the laser system is supposed to be in energy steady-state operation, where the detailed balance condition on the photon number states,

$$\frac{An}{1+\frac{B}{A}n}\rho_{n-1,n-1} - Cn\rho_{n,n} = 0, \tag{6}$$

is satisfied at any time t. In order to simplify calculations we shall define parameters α, β and C such as $\alpha = A$, $\beta = A/B$ and $C = \omega/Q$, where α is dependent on the pumping rate and C represents the photon loss velocity of a laser cavity. The standard value of β is given as $\beta = 3\times 10^7$.[12] From Eq. (6) the steady-state solution of diagonal elements above threshold ($\alpha/C > 1$) is obtained as

$$\rho_{n,n} = \frac{\exp(-\beta-\bar{n})(\beta+\bar{n})^{\beta+n}}{(\beta+n)!}, \tag{7}$$

where the average photon number is $\bar{n} = \beta[(\alpha/C) - 1]$.

In the master equation (1), diagonal elements have non-zero steady-state solution Eq. (7), while off-diagonal elements do not have non-zero steady-state solution.[5] It means in steady-state $\langle\hat{E}\rangle = 0$ and the field state has no phase coherency, which is apparently against high coherency observed in the laser field. This contradictory aspect of the laser field is now understood by well-known analogy of ferromagnetism.[5,6] In ferromagnetism, the expectation value of magnetization is zero in steady-state due to the nature of its openness. But the magnetization deteriorates into steady-state mixture on a global time scale, which makes it look constant on the observation time scale. The same argument is applied to the laser model. Given that the field state is initially a pure state being quite similar to the Glauber coherent state, the expectation value of the electric field decays very slowly compared to the observation time scale. Due to the stability of the initial field state on the observation time scale, the laser field is considered to have remarkably high phase coherency. This means that the ansatz at the initial time suggested by M. O. Scully and W. E. Lamb, Jr. in their original paper [5] is crucial to explain phase coherency of the laser field.

The field state is supposed to be a pure state at $t = 0$ and be under the detailed balance condition at all time $t \geq 0$, just as the Scully-Lamb ansatz is. In this case, the solution for Eq. (1) is written in the form

$$\rho_{n,n-k}(t) = \sqrt{\rho_{n,n}\rho_{n-k,n-k}} \times \exp[\mathrm{i}(\theta_n - \theta_{n-k})]T_n^{(k)}(t) \tag{8}$$

$$T_{n-k}^{(-k)}(t) = T_n^{(k)}(t)^* \qquad (t \geq 0) \tag{9}$$

$$T_n^{(k)}(0) = 1, \tag{10}$$

where $\rho_{n,n}$ is Eq. (7), $\alpha/C > 1$ and θ_n determines an initial field state.

The *phase order parameter* Δ,

$$\Delta \equiv (\theta_{n+2} - \theta_{n+1}) - (\theta_{n+1} - \theta_n) \quad (n = 0, 1, 2\ldots), \tag{11}$$

is introduced for intial pure states with phase dependence on n^2. When phase is linear on n, *i.e.*, $\Delta = 0$, Eq. (8) corresponds to the Scully-Lamb ansatz whose density matrix elements are all real, except that Eq. (8) has an arbitrary phase constant. θ_n is

$$\theta_n = \theta_0 + \Theta n + \Delta\frac{(n-1)n}{2} \qquad (n = 0, 1, 2\ldots), \tag{12}$$

where $\Theta \equiv \theta_1 - \theta_0$ is an arbitrary phase constant.

Q function [13] of the field states expressed by Eqs. (8) and (11) at $t = 0$ is

$$\begin{aligned} &Q(|\alpha|\cos\theta_\alpha, |\alpha|\sin\theta_\alpha) \\ &\quad\equiv \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle \\ &\quad= \frac{1}{\pi}\left[\left\{\sum_n \sqrt{f_n(\alpha)g_n} \times \cos\left[n(\Theta - \theta_\alpha) + \frac{\Delta}{2}(n-1)n\right]\right\}^2\right. \\ &\qquad\left. + \left\{\sum_n \sqrt{f_n(\alpha)g_n} \times \sin\left[n(\Theta - \theta_\alpha) + \frac{\Delta}{2}(n-1)n\right]\right\}^2\right], \end{aligned} \tag{13}$$

where

$$f_n(\alpha) \equiv \exp(-|\alpha|^2)\frac{(|\alpha|^2)^n}{n!} \tag{14}$$

$$g_n \equiv \exp(-\beta - \bar{n})\frac{(\beta+\bar{n})^{\beta+n}}{(\beta+n)!}, \tag{15}$$

$|\alpha\rangle$ is a coherent state and $\alpha = |\alpha|\exp(i\theta_\alpha)$.

As is shown in Fig. 1, the field states with Δ are squeezed states when $|\Delta|$ is of the order $\bar{n}^{-1}$. Δ determines both the squeezing degree and the direction of squeezing at once, because the detailed balance condition restricts the quantum fluctuation in the direction of coherent excitation as constant, where the photon number variance is $\langle(\Delta n)^2\rangle = \bar{n}+\beta$ at any Δ.

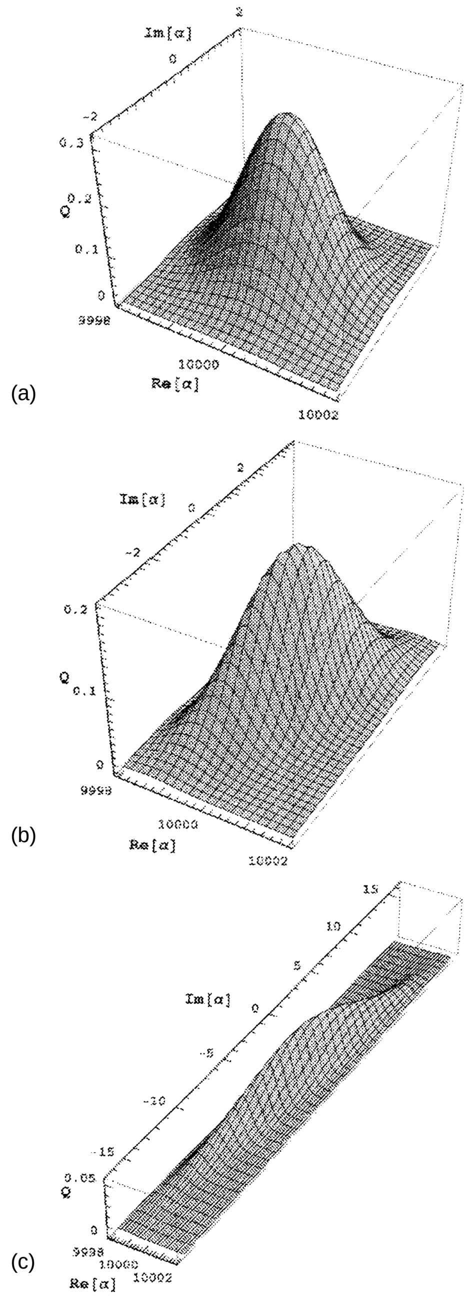

Fig. 1. Q function of the squeezed states expressed with $\mathbf{\Delta}$. The photon number average, $\bar{n} = 1\times 10^8$. (a) The Scully-Lamb ansatz, $\mathbf{\Delta} = 0$. (b) The squeezed state at $\mathbf{\Delta} = 1\times\bar{n}^{-1}$. (c) The squeezed state at $\mathbf{\Delta} = 5\times\bar{n}^{-1}$.

By inserting Eqs. (8) and (12) into Eq. (1), the solution for Eq. (1) is found in the form

$$T_n^{(k)}(\Delta,t) = \exp\left([-p_n^{(k)}(\Delta) + \mathrm{i}q_n^{(k)}(\Delta)]t\right), \tag{16}$$

where

$$\begin{aligned} p_n^{(k)}(\Delta) = C\Bigg\{ & \frac{\left(n+1-\frac{k}{2}+\frac{k^2}{8\beta}\right)(\beta+\bar{n})}{\beta+n+1-\frac{k}{2}+\frac{k^2}{16\beta}} + \left(n-\frac{k}{2}\right) \\ & - \left[\frac{\sqrt{n(n-k)(\beta+n)(\beta+n-k)}}{\beta+n-\frac{k}{2}+\frac{k^2}{16\beta}}\right. \\ & \left. + \sqrt{\frac{(n+1)(n+1-k)}{(\beta+n+1)(\beta+n+1-k)}}(\beta+\bar{n})\right] \\ & \times \cos(k\Delta)\Bigg\} \end{aligned} \tag{17}$$

$$\begin{aligned} q_n^{(k)}(\Delta) = C\Bigg[& \sqrt{\frac{(n+1)(n+1-k)}{(\beta+n+1)(\beta+n+1-k)}}(\beta+\bar{n}) \\ & - \frac{\sqrt{n(n-k)(\beta+n)(\beta+n-k)}}{\beta+n-\frac{k}{2}+\frac{k^2}{16\beta}}\Bigg]\sin(k\Delta), \end{aligned} \tag{18}$$

when $T_n^{(k)}$ may be approximated as

$$T_{n-1}^{(k)}(\Delta,t) \simeq T_n^{(k)}(\Delta,t) \simeq T_{n+1}^{(k)}(\Delta,t). \tag{19}$$

Equation (19) is valid when $|\Delta| \ll 1$ and $Ct \ll 1$. Note that $T_n^{(0)}(\Delta,t) = 1$ regardless of any Δ or t, *i.e.*, the detailed balance condition Eq. (6) is rigidly satisfied at any Δ.

The linear entropy is calculated as

$$\begin{aligned} s(\rho) &\equiv Tr(\rho-\rho^2) \\ &= 1 - \sum_{n,m}\rho_{n,n}\rho_{m,m}\exp[-2p_n^{(n-m)}(\Delta)t]. \end{aligned} \tag{20}$$[2,3]

Since Eq. (17) has a relation $p_n^{(k)}(\Delta) \geq p_n^{(k)}(0) > 0$, the Scully-Lamb ansatz produces the linear entropy at the minimum rate, thus becoming the most stable field state in energy steady-state laser operation.

For the Hermitian amplitude operators of the two quadrature phases $\hat{X} \equiv (\hat{a}+\hat{a}^\dagger)/2$ and $\hat{Y} \equiv (\hat{a}-\hat{a}^\dagger)/(2\mathrm{i})$, we find

$$\begin{aligned} \langle\hat{X}\rangle(\Theta,t) = & \sum_{n=0}^{\infty} g_n\sqrt{\frac{(\beta+\bar{n})(n+1)}{(\beta+n+1)}}\exp(-p_{n+1}^{(1)}t) \\ & \times \cos(\Theta+\Delta n+q_{n+1}^{(1)}t) \end{aligned} \tag{21}$$

$$\langle\hat{Y}\rangle(\Theta,t) = \langle\hat{X}\rangle\left(\Theta-\frac{\pi}{2},t\right) \tag{22}$$

$$\begin{aligned} & \langle(\Delta X)^2\rangle(\Theta,t) \\ & = \frac{1}{4}\Bigg\{1+2\sum_{n=0}^{\infty}\Bigg[g_n(\beta+\bar{n})\sqrt{\frac{(n+2)(n+1)}{(\beta+n+2)(\beta+n+1)}} \\ & \quad \times\exp(-p_{n+2}^{(2)}t)\cos[2\Theta+\Delta(2n+1)+q_{n+2}^{(2)}t]\Bigg] \end{aligned}$$

$$+2\bar{n} - 4\left[\sum_{n=0}^{\infty}\left[g_n\sqrt{\frac{(\beta+\bar{n})(n+1)}{\beta+n+1}}\right.\right.$$
$$\left.\left.\times \exp(-p_{n+1}^{(1)}t)\cos(\Theta+\Delta n+q_{n+1}^{(1)}t)\right]\right]^2\Bigg\} \tag{23}$$

$$\langle(\Delta Y)^2\rangle(\Theta,t) = \langle(\Delta X)^2\rangle\left(\Theta-\frac{\pi}{2},t\right). \tag{24}$$

$\Theta_0(t)$ is defined to satisfy

$$\langle(\Delta X)^2\rangle(\Theta_0(t),t) \le \langle(\Delta X)^2\rangle(\Theta,t) \tag{25}$$

for all Θ. If $\sqrt{\langle(\Delta X)^2\rangle(\Theta_0(t),t)} < 1/2$, the field is a squeezed state at time t. The angle $\chi(t)$ between the direction of squeezing and the direction of coherent excitation is written as

$$\chi(t) = \tan^{-1}\left(\frac{\langle\hat{Y}\rangle}{\langle\hat{X}\rangle}\right)(\Theta_0(t),t).^{13)} \tag{26}$$

χ satisfies $-\pi/2 \le \chi < \pi/2$.

Conclusion

As a result, it is shown that the Scully-Lamb ansatz is singled out as the most stable state among squeezed states under the detailed balance condition. Figure 2 is a result of numerical calculation for the uncertainty area of the field quadrature components, where the Scully-Lamb ansatz is clearly picked out as the most stable state in the decoherence process. Figure 3 shows that the squeezed states remain squeezed for times $t \ll C^{-1}$. It is shown in Figs. 2 and 3 that the speed of the whole decoherence process strongly depends on the photon loss parameter C, which is inversely proportional to the quality factor Q of a laser cavity.

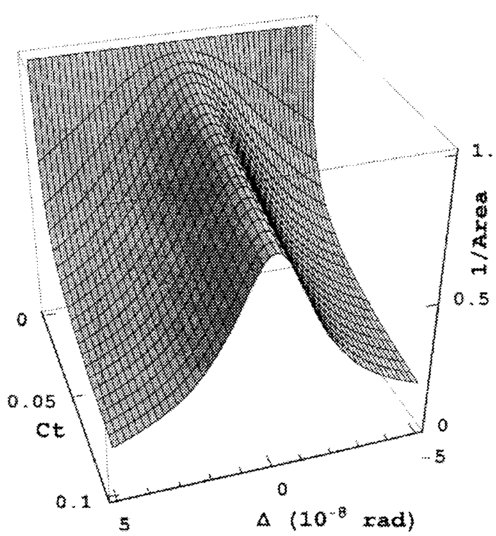

Fig. 2. The inverse of the uncertainty error area as a function of both $\mathbf{\Delta}$ and Ct. "Area" is $\sqrt{\langle(\Delta X)^2\rangle\langle(\Delta Y)^2\rangle(\Theta=\Theta_0)}$. $\bar{n} = 1\times10^8$. The Scully-Lamb ansatz, $\mathbf{\Delta}=0$, is clearly picked out as time goes on.

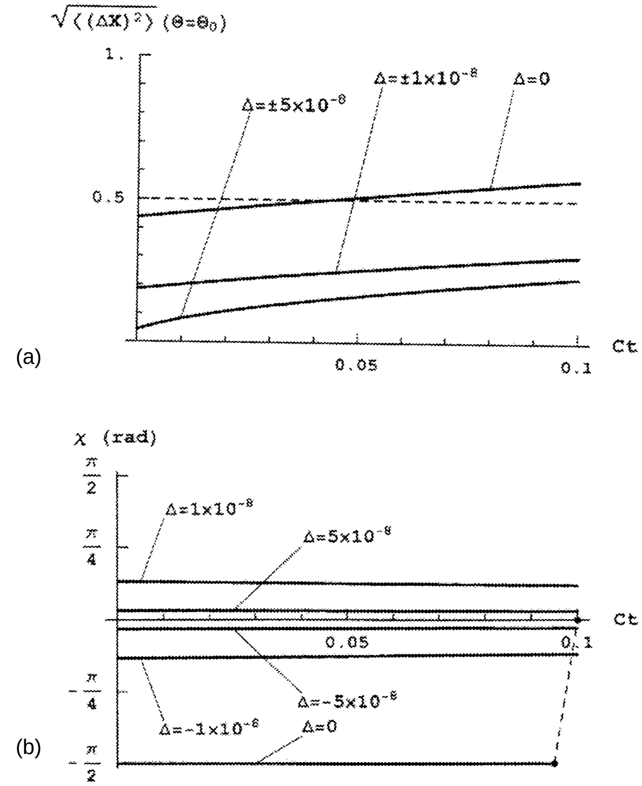

Fig. 3. Time evolution of squeezed states. $\bar{n} = 1\times10^8$. $\mathbf{\Delta} = 0, \pm1\times\bar{n}^{-1}, \pm5\times\bar{n}^{-1}$. (a) The minimum variance of $\hat{X}$ on rotation around the origin in phase space. A dashed line is the vacuum fluctuation level. (b) The angle χ between the direction of squeezing and the direction of coherent excitation. $\mathbf{\Delta}=0$ has a discontinuous point from $\chi=-\pi/2$ to $\chi=0$ between $Ct=0.095$ and $Ct=0.1$, because during the time the quantum fluctuation in the direction orthogonal to the direction of coherent excitation exceeds the fluctuation in the direction of coherent excitation.

The photon loss velocity C is approximately evaluated as

$$C = \frac{c(1-R)}{L}, \tag{27}$$

where c is the light velocity, R is the reflectance of a cavity mirror and L is the length of a cavity.[12)] The conventional standard value of C was given as more than of the order $10^5 s^{-1}$ (assumed as $R=99.95\%$, $L=1\,\mathrm{m}$). In this case, squeezed states are all washed away in the decoherence process for much shorter times than the observation time scale (assumed of the order $10^{-3}s$), and the Scully-Lamb ansatz is the only stable state we can observe. Nonetheless, a present laser cavity for a gravitational wave detector, *e.g.*, a laser interferometer of VIRGO at Cascina in Italy, is about to reach C of the order $10^0 s^{-1}$ (assumed as $R=99.999\%$, $L=3000\,\mathrm{m}$). In such a cavity with extremely high quality factor Q, squeezed states can exist for times long compared to the observation time scale, thus becoming the second best candidates for the "preferred states" in the "predictability sieve" of the single-mode laser.

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