16. The Singularity Method in Boundary Value Problems of the Theory of Elasticity

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Summary

Using the Maxwell-Betti's reciprocal theorem, the general formulation transforming the fundamental differential equation to the integral equation in the theory of elasticity is deduced. In similar ways as in hydrodynamics, boundary integral equations which are represented by either source and sink or doublet distribution are derived. The formulations of the boundary integral equation for various problems, i.e. the two dimensional elastostatics, the two dimensional elastodynamics, the three dimensional elastostatics and the plate bending problems are shown and studied especially regarding the property of their kernel function in concern with their numerical integration. To verify its usefullness and accuracy, some numerical examples are shown. The proposed equations are especially usefull for stress concentration problems and diffraction problems during passage of elastic waves in the infinietly extended elastic medium.

1. Introduction

In the recent years, the singularity method, meaning a method of solving boundary value problems by integral equations in regard to the boundary singularity (so-called the Boundary Element Method; BEM), has been widely applied to the boundary value problems in the theory of elasticity, and a number of related reports and text books have been published.[1]

The idea to solve the boundary value problem in the theory of elasticity by transformation the fundamental defferential equation into the integral equation has already appeared in the early 1900's. Love[2] sketches that the integral approach has been exploited by Betti and Fredholm.

The outstanding feature of the BEM is that it reduces of a dimension of problem. The system of equations resulted from discretization of the boundary integral equation becomes smaller than the system in the finite element method (FEM), in which the domain of the problem under consideration is in discretized into a number of elements. The FEM may be not appropriate to treat the problems of an infinite domain because its dimensions becomes very large and also it is difficult to defines boundary condition for infinity. The BEM has no such difficulties in solving these problems since the discretization is needed only along the boundary of the body not as like in its whole domain as the FEM.

However this method has not been tried positively until very close days. The reason may lie upon the facts that the kernel functions of the boundary integral equation are complex and have strong singularities.

In this report, we develope the BEM for various problems in the theory of elasticity and show its usefullness. It is most important in applying the BEM that the boundary should be discretized in sufficient number of segments and the singularities of kernel functions shold be properly integrated.

In chapter 2 and 3, we show the general

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formulation by transforming the fundamental differential equation in the theory of elasticity into the integral equation and show the related important theorems. We propose the simplified boundary integral equation of displacement field making use of simple singularities similar to source and sink or doublet in hydrodynamics. In the following, we show concrete formulations of the BEM for various problems, i.e. the two dimensional elastostatics, the two dimensional elastodynamics, the three dimensional elastostatics and the plate bending problems and investigate the numerical property of the kernel functions. Also we make some remarks on the numerical computation.

2. Fundamental Equations in the Theory of Elasticity

Before formulating the boundary equations, we summarise the field equations.[3)]

Let D be a domain which may be interior or exterior on the boundary C, as shown in Fig. 1. Under the basic assumptions of small displacement, homogeneous and isotropic linear elastic material, the equation of motion, Hook's law and the strain-displacement relation are expressed as follows;

$$\rho\frac{\partial^2\bar{u}_i}{\partial t^2}=\sum_{j=1}^{3}\frac{\partial\bar{\sigma}_{ij}}{\partial x_j}+\bar{X}_i \tag{2.1}$$

$$\bar{\sigma}_{ij}=\lambda\delta_{ij}\sum_{k=1}^{3}\bar{e}_{kk}+2G\bar{e}_{ij} \tag{2.2}$$

$$\bar{e}_{ij}=\frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right) \tag{2.3}$$

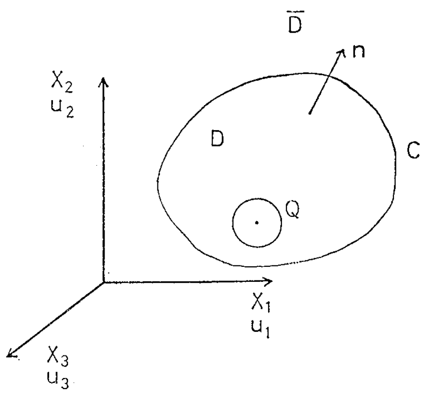


Fig. 1 Coordinate systems

$i=1,2,3$ and $j=1,2,3$

where $\bar{u}_j$, $\bar{\sigma}_{ij}$, $\bar{e}_{ij}$ and $\bar{x}_j$ are the displacements, stresses, strains and body forces which are function of position in space and time, ρ, λ and G is density, Lame's constant and shear modulus and δ_{ij} is Kronecker's delta.

We use the expression either $x_1, x_2, x_3, \cdots$ or $x, y, z, \cdots$, according to the circumstances, but there will be no confusion.

Substituting Eq. (2.2) and Eq. (2.3) into Eq. (2.1), we obtain the Navier-Cauchy equation,

$$\rho\frac{\partial^2\bar{u}_i}{\partial t^2}=G\Delta\bar{u}_i+(\lambda+G)\frac{\partial\bar{\gamma}}{\partial x_i} \tag{2.4}$$

where

$$\bar{\gamma}=\sum_{j=1}^{3}\frac{\partial\bar{u}_j}{\partial x_j}$$

$$\Delta=\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial x_3{}^2}$$

Δ is the Laplacian, assuming that the body forces dose not exist.

If the displacement field is in time harmonics with circular frequency, we usually use abbreviation such as;

$$\bar{u}_i(x_1,x_2,x_3,t)=R_e[u_i(x_1,x_2,x_3)e^{i\omega t}] \tag{2.5}$$

where $u_i(x_1, x_2, x_3)$ are complex vaiables which is independent of time. $R_e[\ ]$ denotes the real part of the complex expression, and $i=\sqrt{-1}$.

Substituting Eq. (2.5) into Eq. (2.4), we will have the governing equation for complex displacements,

$$(\Delta+k^2)u_i+\alpha\frac{\partial\gamma}{\partial x_i}=0 \tag{2.6}$$

where

$$\left.\begin{aligned} k^2&=\frac{\rho\omega^2}{G}\\ \alpha&=\frac{\lambda+G}{G}=\frac{k^2}{K^2}-1\\ K^2&=\frac{\rho\omega^2}{\lambda+2G}\end{aligned}\right\} \tag{2.7}$$

k and K are the wave numbers of transverse (SV wave) and longitudinal (P wave) waves respectively.

In the special case of $\omega=0$ in Eq. (2.6) represents the basic equilibrium equation in terms of displacements in the elastostatics. We note that $u_i(x_1, x_2, x_3)$ can be considered as a real function in the elastostatics.

Now, the problem is to solve the fundamental differential equation (2.6) or Eq. (2.8) under the given boundary conditions.

When the displacements are known, the stersses can be obtained by following differentiation.

$$\sigma_{ij}=\lambda\delta_{ij}\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_k}+G\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{2.9}$$

Surface tractions are expressed as

$$\tau_i=\sum_{j=1}^{3}\sigma_{ij}\frac{\partial n}{\partial x_j} \tag{2.10}$$

where n is the outward normal vector on the boundary, as shown in Fig. 1.

3. Reciprocal Theorem and Boundary Integral Equation

3.1 *Reciprocal theorem*

Let us consider the two elastodynamic states. They have and the same medium and vary sinsiodally in time with the same circular frequency. These two states are distinguished by the superscripts (1) and (2) respectively. Then following functional is introduced,

$$\tilde{L}(u^{(1)}, u^{(2)})=\int_D[U_K-U_D]\,dv \tag{3.1}$$

where

$$\left.\begin{aligned}U_K&=\sum_{j=1}^{3}\frac{Gk^2}{2}u_j^{(1)}u_j^{(2)}\\U_D&=\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}e_{ij}^{(1)}\sigma_{ij}^{(2)}\end{aligned}\right\} \tag{3.2}$$

If $u^{(1)}$ are equal to $u^{(2)}$, then U_K stands for the kinematic energy stored in D and U_D stands for the complex elastic deformation energy and their difference is to be Lagrangean but they are not the time averaged value. So that $\tilde{L}$ is not accurate Lagrangean. We call it the modefied Lagrangian function. It is clear that there exists the reciprocal relation in Eq. (3.1), that is,

$$\tilde{L}(u^{(1)}, u^{(2)})=\tilde{L}(u^{(2)}, u^{(1)}) \tag{3.3}$$

Integrating Eq. (3.1) by parts, we have

$$\tilde{L}(u^{(1)}, u^{(2)})=\frac{1}{2}\int_C\sum_{j=1}^{3}u_j^{(1)}\tau_j^{(2)}\,ds+\frac{Gk^2}{2}\int_D\sum_{j=1}^{3}\left[u_j^{(1)}u_j^{(2)}+\frac{1}{k^2}u_j^{(1)}\cdot\left\{\Delta u_j^{(2)}+\alpha\frac{\partial\gamma^{(2)}}{\partial x_j}\right\}\right]dv$$

If $u^{(1)}$ and $u^{(2)}$ are regular in D and satisfies the fundamental differential equation (2.1), then we have

$$\tilde{L}(u^{(1)}, u^{(2)})=\frac{1}{2}\int_C\sum_{j=1}^{3}u_j^{(1)}\tau_j^{(2)}\,ds$$

and Eq. (3.3) can be written in the following form as the reciprocity between the boundary displacements and tractions.

$$\int_C\sum_{j=1}^{3}u_j^{(1)}\tau_j^{(2)}\,ds=\int_C\sum_{j=1}^{3}u_j^{(2)}\tau_j^{(1)}\,ds \tag{3.4}$$

This formula is well known as Maxwell-Betti's reciprocal theorem.[14),15)]

3.2 *Boundary integral equations and boundary conditions*

Let us introduce the fundamental solutions which satisfies the Navier-Cauchy equation (2.6) and has an appropriate singularity;

$$U_j^m(P, Q)$$

where

$$P\equiv(x_{P1}, x_{P2}, x_{P3})$$
$$Q\equiv(x_{Q1}, x_{Q2}, x_{Q3})$$

$U_j^m(P, Q)$ means the influence function which represents the displacement at the point Q in the x_m direction when the concentrated force of unit magnitude acts at the point P in the x_m direction when the concentrated force of unit magnitude acts at the point P in the x_j direction. Then we can obtain the function $T_j^m(P, Q)$ substituting $U_j^m(P, Q)$ into Eq. (2.8) and Eq. (2.9). $T_j^m(P, Q)$ means the influence function which represents the displacement at the point Q in the x_m direction when the displacement of unit magnitude acts at the point P in the x_j direction.

Let us now apply the reciprocal theorem, Eq. (3.4), in a domain D, bounded by the surface C and a small circle around the point Q, taking two states as follows:

$$\{u_j^{(1)}, \tau_j^{(1)}\}=\{u_j, \tau_j\}$$
$$\{u_j^{(2)}, \tau_j^{(2)}\}=\{U_j^m(P,Q),\ T_j^m(P,Q)\}\ .$$

Then we obtain the following representation of displacements;

$$u_m(Q)=\int_C \sum_{j=1}^{3}[u_j(P)T_j^m(P,Q) -\tau_j(P)U_j^m(P,Q)]\,ds_P \qquad (3.5)$$

$m=1, 2, 3$

This equation permits us to solve the boundary value problem in the theory of elasticity by integral equation.

Eq. (3.5) is the basic equation of BEM. The following boundary value problems are classified in the first place: the interior problems and the exterior problems where the boundary C consist of several boundaries.

In the interior problems, we may apply Eq. (3.5). For the steady-state elastodynamics, Niwa et. al.[16),17)] applied the boundary equation (3.5) to eigenfrequency analysis. Eigenvalues can be obtained as a parameter for which a non-trivual solution of homogeneous boundary integral equation exists.

In the exterior problems, there is some difference in the solution due to the property of material between domain D and $\bar{D}$, which means complementary domain of D. In this case, the displacements must be continuous and the tractions must be in equilibrium on the boundary C.

i) Assuming the rigid boundaries, we obtain the tractions $\tau_j^{(d)}(P)$ when the external forces act on the boundary.

ii) Forcing the boundary C to deform itself by the displacements $u_j(P)$, we define the traction $\tau_j^{(in)}(P)$ on the interior domain D and $\tau_j^{(out)}(P)$ on the exterior domain $\bar{D}$, respectively

Then the equiriburium condition on C is as follow.

$$\tau_j^{(in)}(P)+\tau_j^{(out)}(P)+\tau_j^{(d)}(P)=0 \quad \text{on } C \qquad (3.6)$$

Differentiating and transforming the Eq. (3.5) to the representation of boundary tractions, we can determine the boundary displacements.

3.3 *Representation by a singularity similar to source and sink or doublet in hydrodynamics*

When we investigate the exterior problem using previous boundary integral equation, it will be complex and inconvenient because two kernel functions $U_j^m(P,Q)$ and $T_j^m(P,Q)$ are nessesary to treat it.

Let us try to simplify these equations.

Let us consider the configuration shown in Fig. 2. Let D be a exterior domain and let $\bar{D}$ be the complementary domain of D. Normal vector n has the same direction as defined in Fig. 1.

Changing the sign of Eq. (3.5), the displacements at point Q are represented respectively in D and $\bar{D}$ as follows;

$$-\int_C \sum_{j=1}^{3}[u_j(P)T_j^m(P,Q)-\tau_j(P)U_j^m(P,Q)]\,ds_P \left.\begin{array}{ll} =u_m(Q) & \text{in } D \\ =0 & \text{in } \bar{D}\end{array}\right\} \qquad (3.7)$$

Assuming that the material of domain $\bar{D}$ is same as of domain D and function $u_j^{(0)}(P)$ is regular in $\bar{D}$, the integration around the point

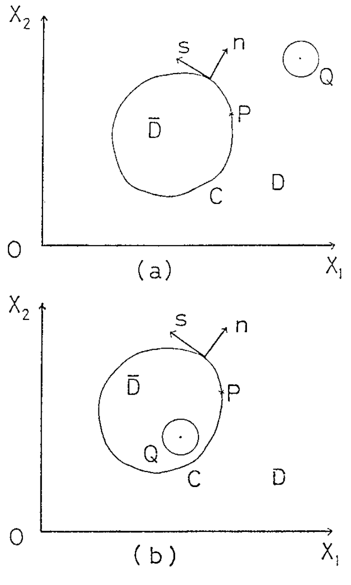


Fig. 2 Exterior domain and Interior domain

Q vanishes by the Green's theorem. If the point Q exists in the domain D, we have the following formula,

$$-\int_C \sum_{j=1}^{3}[u_j^{(0)}(P)T_j^m(P,Q)-\tau_j^{(0)}(P)U_j^m(P,Q)]\,ds_P \begin{cases} =0 & \text{in } D \\ =-u_m^{(0)}(Q) & \text{in } \bar{D} \end{cases} \tag{3.8}$$

Adding Eq. (3.7) to Eq. (3.8) and assuming the total tractions vanishes on the boundary C, then

$$\tau_j(P)+\tau_j^{(0)}(P)=0 \qquad \text{on } C \tag{3.9}$$

We have

$$u_m(Q)=-\int_C \sum_{j=1}^{3}\{u_j(P)+u_j^{(0)}(P)\}\cdot T_j^m(P,Q)\,ds_P \tag{3.10}$$

On the other hand, assuming the total displacement, instead of tractions, vanishes on the boundary C,

$$u_j(P)+u_j^{(0)}(P)=0 \qquad \text{on } C \tag{3.11}$$

We have

$$u_m(Q)=\int_C \sum_{j=1}^{3}\{\tau_j(P)+\tau_j^{(0)}(P)\}U_j^m(P,Q)\,ds_P \tag{3.12}$$

On the analogy of the hydrodynamics, we may call Eq. (3.10) and Eq. (3.12) the representation of source and sink or doublet distribution.[4),5),9)]

In the elastodynamics, we may concider the plane harmonic waves as the regular function $u_j^{(0)}(P)$ and $\tau_j^{(0)}(P)$ in D. Then, the boundary condition shown by Eq. (3.9) is equal to the one of a cavity hole. Furthermore, if we add $u_j^{(0)}(P)$ to both sides of Eq. (3.10), we obtain the Fredholm's equation of the second kind whose unknouwns are the total displacements $u_j(P)+u_j^{(0)}(P)$ on C.

On the contrary, the boundary condition shown by Eq. (3.10) is equal to the one of a rigid inclusions and if we add $u_j^{(0)}(P)$ to both sides of Eq. (3.12), we obtain the Fredholm's equation of the first kind whose unknowns are the total tractions $\tau_j(P)+\tau_j^{(0)}(P)$ on C.

In the elastostatics, we consider the uniform stress fields as the regular functions $u_j^{(0)}(P)$ and $\tau_j^{(0)}(P)$ in D. Then we can investigate the stress concentration problem around a cavity or around a rigid inclusion using Eq. (3.10) or Eq. (3.12).

3.4 *Numerical procedure*

Numerical procedure to solve boundary value problems by making use of, for example, Eq. (3.5) is as follows. At first the boundary may be divided into N elements. (If the problem is three dimentional, the boundary elements are parts of the external surface of the body. If the problem is two dimensions, they are line elements.) Assuming the unknown quantities are constant for each element and giving the boundary condition at the middle point of each element, we can solve them as simultaneous equations.

Recently, a trend toward the higher order element which means that the unknown quantities are not constant for each element has appeared. Howevere, if the integration of the kernel functions over each element are done exactly, we might obtain accurate enough solotion by using constant elements.

Generally speaking, the accuracy of the BEM analysis depend on the number of elements and on the accuracy of numerical integration, especially, of singular kernel. We can not obtain the exact value of calculations near the boundary if higher singular integral are not carried out carefully.

In this report, we also study the numerical property of kernel functions for various problems and improve the accuracy of its integration.

4. Formulation for Various Problems and Numerical Examples

4.1 *Two dimensional elastostatics*[10)]

4.1.1 Representation of Airy's stress function

In the two dimensional elastostatics, Airy's stress function can be applied. Let f be Airy's stress function. The fundamental differential equation to be satisfied is as follows,

$$\Delta^2 f=0 \tag{4.1}$$

The fundamental solution of Eq. (4.1) may be taken as follows;

$$f=S(P,Q)=\frac{E}{8\pi}R^2\log R \tag{4.2}$$

where

$R=\overline{PQ}=[(x_Q-x_P)^2+(y_Q-y_P)^2]^{1/2}$

E: Young's modulus

Relations between the stress function and the displacements and the tractions are derived as follows,

$$\left.\begin{aligned}
\tau_1&=T_1(P,Q)=\frac{\partial^2}{\partial s_P\partial y_P}S(P,Q)\\
\tau_2&=T_2(P,Q)=-\frac{\partial^2}{\partial s_P\partial x_P}S(P,Q)\\
u_1&=U_1(P,Q)=\Xi(P,Q)-\frac{1}{2G}\frac{\partial}{\partial x_P}S(P,Q)\\
u_2&=U_2(P,Q)=H(P,Q)-\frac{1}{2G}\frac{\partial}{\partial y_P}S(P,Q)\\
\frac{\partial}{\partial x_P}&\Xi(P,Q)=\frac{\partial}{\partial y_P}H(P,Q)=\frac{1}{E}\Delta_P S(P,Q)
\end{aligned}\right\} \tag{4.3}$$

where $\partial/\partial s_P$ denotes the differentiation tangential to boundary at the point P.

We take the two states in Eq. (3.4) as:

$\{U_j(P,Q),\ T_j(P,Q)\}=\{u_j^{(2)},\tau_j^{(2)}\}$ and $f=\{u_j^{(1)},\tau_j^{(1)}\}$,

we obtain as follows

$$f(Q)=\int_C\sum_{j=1}^{2}[u_j(P)T_j(P,Q)-\tau_j(P)U_j(P,Q)]\,ds_P \tag{4.4}$$

Although $U_j(P,Q)$ and $T_j(P,Q)$ are the displacements and the tractions by the singularity $S(P,Q)$ respectively from above equation. They are also stress functions at the point Q when the unit magnitude of tractions and displacements act on the point P respectively.

We can easilly introduce the representation of displaerments from Eq. (4.4) as same as Eq. (3.5) (See appendix A).

The representation like as source and sink or doublet distribution are as follows,

$$f(Q)=\int_C\sum_{j=1}^{2}\{\tau_j(P)+\tau_j^{(0)}(P)\}U_j(P,Q)\,ds_P \tag{4.5}$$

where

$u_j(P)+u_j^{(0)}(P)=0$ on C

and

$$f(Q)=-\int_C\sum_{j=1}^{2}\{u_j(P)+u_j^{(0)}(P)\}T_j(P,Q)\,ds_P \tag{4.6}$$

where

$\tau_j(P)+\tau_j^{(0)}(P)=0$ on C

Differentiating Eq. (4.5) which is used in obtaining stresses, Nishitani's equation[18] can be derived.

Lastly, integrating Eq. (4.6) by parts, we have

$$f(Q)=\int_C\left[\varepsilon_s(P)\frac{\partial}{\partial n_P}S(P,Q)-\delta(P)\frac{\partial}{\partial s_P}S(P,Q)\right]ds_P \tag{4.7}$$

where

$$\varepsilon_s=\frac{\partial}{\partial s}(u_s+u_s^{(0)})$$

$$\delta=\frac{\partial}{\partial s}(u_n+u_n^{(0)})$$

u_n and u_s indicate the normal and tangential components of displacements on the boundary respectively. It allows us to directly calculate the stress concentration around the cavity hole by solving the Eq. (4.7).

4.1.2 Property of the singularity of the kernel function

a) Kernel functions of displacements

As shown in appendix A, the kernel functions are very complex and the integration must be done carefully. To investigate the property of singularity of the kernel functions, we will study the integration over the arbitrary element $\overline{P_nP_{n+1}}$ on the boundary as shown in Fig. 3.

At first, all kernel functions, $T_j^m(P,Q)$, are tangential derivative of some functions so that the integration can be done easily. For instance,

$$\int_{P_n}^{P_{n+1}}T_1^1(P,Q)\,ds_P=\int_{P_n}^{P_{n+1}}\frac{\partial}{\partial s_P}\left[\frac{1}{2\pi}\theta+\frac{1+\nu}{8\pi}\sin 2\theta\right]ds_P$$

$$=\frac{1}{2\pi}(\theta_{n+1}-\theta_n)$$
$$+\frac{1+\nu}{8\pi}(\sin 2\theta_{n+1}-\sin 2\theta_n)$$

The integration of $T_1^1(P, Q)$ is shown in Fig. 4 for the example. The range of integration is taken between the point P_n $(-1.0,0.0)$ and the point P_{n+1} $(1.0,0.0)$ on the x-axis. The observation point $Q(O, y)$ is moved from the origin to a arbitrary point of y-axis. As shown in the figure, the horizontal axis denotes the distance of the observation point Q from the origin. The vertical axis denotes the integrated values. Taking the limit from the interior domain to the boundary, the integration of the first term of $T_1^1(P, Q)$, $\theta_{n+1}-\theta_n$, reaches π. From this result, it is obvious that the behaviour of is the mist dominant factor in the kernel functions $T_1^1(P, Q)$. For the another kernel function $U_j^m(P, Q)$, by transforming the variable suitablly, the integration can be done analytically. For example, the integration of log R will be as follows:

$$\int_{P_n}^{P_{n+1}}\log R\, ds_P$$
$$=\int_{\theta_n}^{\theta_{n+1}}\log\left\{\frac{-h}{\cos(\theta-\alpha_n)}\right\}\frac{h\,d\theta}{\cos^2(\theta-\alpha_n)}$$

where

$$h=-R\cos(\theta-\alpha_n)$$
$$s_P=h\tan(\theta-\alpha_n)$$

therefore

$$\int_{P_n}^{P_{n+1}}\log R\, ds_P$$
$$=h\left[\tan t\left\{\log\left(\frac{-h}{\cos t}\right)-1\right\}+t\right]_{\theta_n-\alpha_n}^{\theta_{n+1}-\alpha_n}$$
$$=-R_{n+1}\sin(\theta_{n+1}-\alpha_n)(\log R_{n+1}-1)$$
$$+R_n\sin(\theta_n-\alpha_n)(\log R_n-1)$$
$$+h(\theta_{n+1}-\theta_n)$$

Taking the range of integration same as in the case of $T_1^1(P, Q)$, we show the integration of $U_1^1(P, Q)$ in Fig. 5. In this case, the most dominant singurality of kernel function U_1^1 (P, Q) is the term of log R.

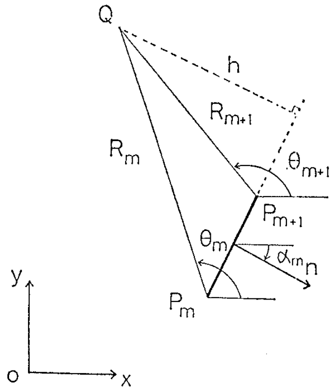


Fig. 3 Local coordinate systems for calculation of kernel functions

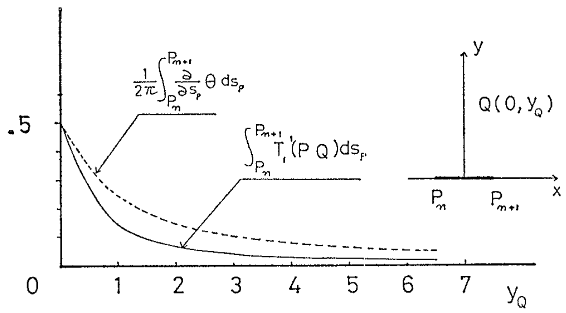


Fig. 4 Integration of kernel function $T_1^1(R, Q)$

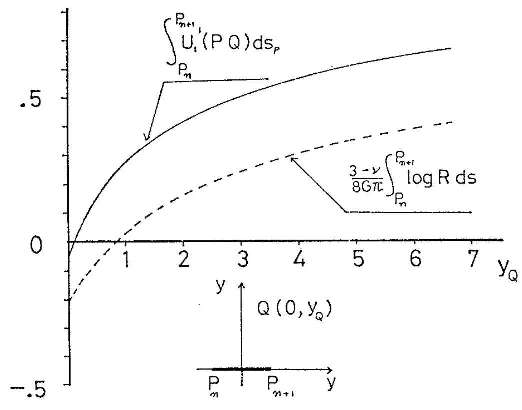


Fig. 5 Integration of kernel function $U_1^1(R, Q)$

b) Kernel functions of stresses

After displacements and tractions are obtained, stresses can be calculated by the following formulas.

$$\left.\begin{aligned}\sigma_x(Q)&=\frac{\partial^2}{\partial y_Q^2}f(Q)\\&=\int_C\sum_{j=1}^{2}\left[u_j(P)\frac{\partial^2}{\partial y_Q^2}T_j(P,Q)\right.\\&\qquad\left.-\tau_j(P)\frac{\partial^2}{\partial y_Q^2}U_j(P,Q)\,ds_P\right.\\\sigma_y(Q)&=\frac{\partial^2}{\partial x_Q^2}f(Q)\\\tau_{xy}(Q)&=-\frac{\partial^2}{\partial x_Q\partial y_Q}f(Q)\end{aligned}\right\}\qquad(4.8)$$

In this case, assuming the displacements and the tractions are constant on the boundary elements, it will be difficult to obtain the exact values for the stresses on the boubdary. One of the reasons is that the dominant part of the singularity of the kernel function $\partial^2/\partial y_Q^2\cdot T_1^1(P,Q)$ etc. is the order of $1/R^2$ and it is difficult to evaluate it numerically. Therefore, to obtain the exact solution, we have to assume that the displacements are varied as linear or higher order over each elements or to take any other similar means.

Here, we propose another convenient method which permits us to obtain accurate stresses on the boundary and in the interior domain by constant elements. It is a method to integrate Eq. (4.8) by parts, namely

$$\begin{aligned}\sigma_x(Q)=-\int_C\Big[&\frac{\partial}{\partial s_P}u(P)\frac{\partial^3}{\partial y_P\partial y_Q^2}T_1(P,Q)\\&-\frac{\partial}{\partial s_P}v(P)\frac{\partial^3}{\partial x_P\partial y_Q^2}T_2(P,Q)\\&+\tau_1(P)\frac{\partial^2}{\partial y_Q^2}U_1(P,Q)\\&+\tau_2(P)\frac{\partial^2}{\partial y_Q^2}U_2(P,Q)\Big]\,ds_P\end{aligned}\qquad(4.9)$$

Then, the order of the singularity of the krrnel functions is reduced to $1/R$. Integration of this type can be done analytically in the same manner as in the previous section. We can obtain exact stresses on the boundary by Eq. (4.9) and $\partial u_j/\partial s_P\cdot(P)$ in Eq. (4.9) by numerical differentiation.

4.1.3 Two dimensional semi-infinite elastic space problems[3)]

The one of the merits of the BEM analysis is that the problems can be investigated by distributing the singularities on the finite boundary only. Hence, when the boundary has an infinite part, this merit may be lost. In the case of infinite the straight boundary which may occure frequently in practice, if we choose the fundamental solution to satisfy the boundary condition on the interface identically, we may not need to put elements on the surface.

For such fundamental solution of the stress function to satisfy the boundary condition on the interface we have the followings:

$$S(P,Q)=\frac{E}{8\pi}R^2\log\left(\frac{R}{R'}\right)\qquad(4.10)$$

$$R=\overline{PQ}=[(x_Q-x_P)^2+(y_Q-y_P)^2]^{1/2}$$
$$R'\quad=[(x_Q-x_P)^2+(y_Q+y_P)^2]^{1/2}$$

where the interface is placed along the x axis. The singularity of this kernel is the same as the one of Eq. (4.2) and the various representation is the same as in that case. (Kernel functions are in appendix B.) Although the solution of the semi-finite elastic space problem has been obtained by Nishitani[18)], the kernel functions are very complicated.

4.1.4 Numerical examples

Fig. 6 shows the stresses around a circular hole in an infinite elastic plate under the uniform loading. The accuracy of the computation of the stress concentration factor of of an elliptic hole for various aspect ratio with the number of division N (the part of a quadrant) is plotted in Fig. 7. The relation between the accuracy of computation and the number of division is linear on a log scale. Therefore, the error of computation decreases reciprocally as the number of division increases.

Fig. 8 shows the stress concentration factors of a semi-elliptic notch in a semi-infinite elastic plate under the uniform loading. The marks in the figure are results of Nishitani's caliculation[20)]. Both results are in good

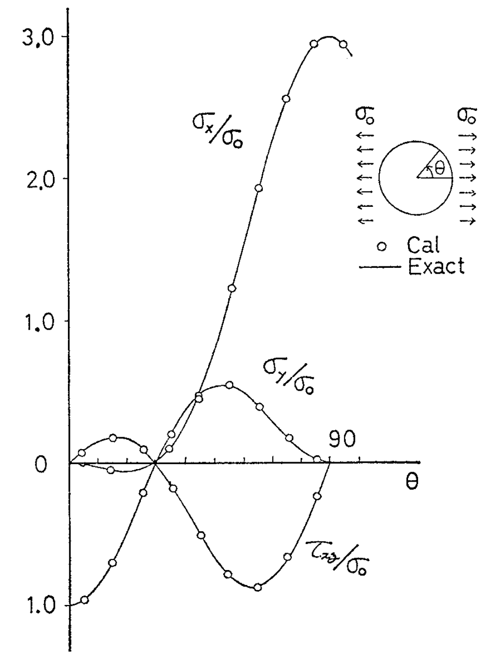


Fig. 6 Stresses around a circular hole in an infinite elastic plate under uniform loading

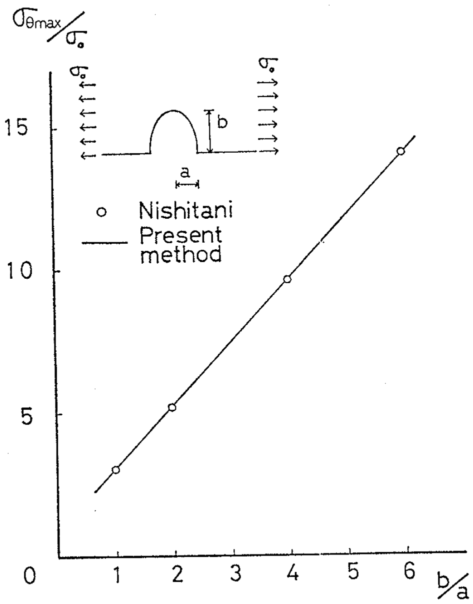


Fig. 8 Stress concentrations of a semi-elliptic notch in a semi-infinite elastic plate under uniform loading

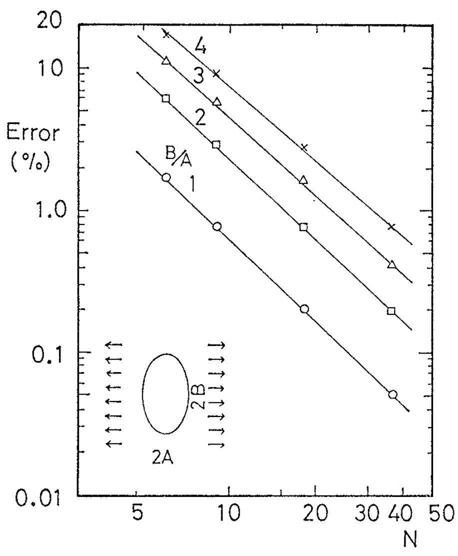


Fig. 7 Accuracy of the computation of the stress concentration factor of an elliptic hole for various aspect ratio with the number of division N (the part of a quadrant)

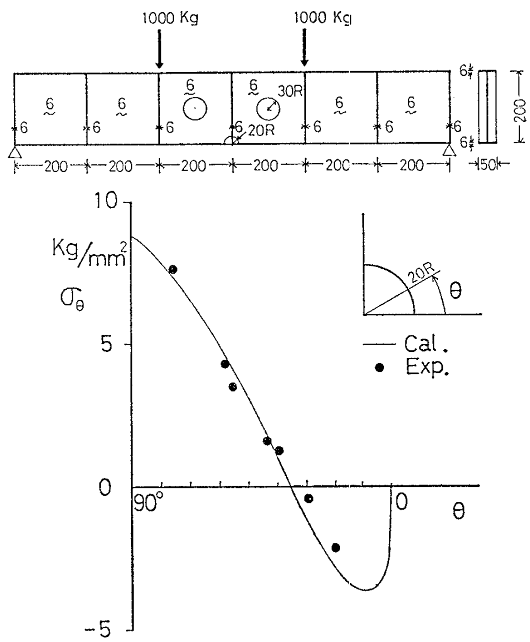


Fig. 9 Numerical and experimental results obtained form I-shape beam under uniform bending

agreement.

Finally, Fig. 9 shows the numerical and the experimental results obtained from I-shape beam under the uniform bending. The assumption of calculation is that the beam is considered as the solid structure constructed by the plate members and the displacemens of the contact lines between the web and flanges are dependent on the components of the displacements along the contact line and independent of the components of the one vertical to that line. Calculations are in good agreement with experimental results.

4.2 *Two dimensional elastodynamic problems*[13)]

4.2.1 Fundamental solutions

Let us introduce the fundamental solutions satisfying the Navier-Cauchy's equation of two dimensional elastodynamics and the radiation condition as follows,

$$U_j^m(P,Q)=\frac{i}{4G}\left[H_0^{(2)}(kR)\delta_{jm}+\frac{1}{k^2}\frac{\partial^2}{\partial x_{P_j}\partial x_{P_m}}\cdot\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}\right] \quad (4.11)$$

where $H_0^{(2)}$ is the zero-th order Hankel function of the second kind.

Substituting Eq. (4.11) into Eq. (2.9) and Eq. (2.10), we may obtain the kernel functions $T_j^m(P, Q)$ (See appendix C).

To ivestigate the property of the kernel functions $U_j^m(P, Q)$ and $T_j^m(P, Q)$, we begin with the property of the Hankel function. In the vicinity of the origin, it behave like as shown in reference 15),

$$H_0^{(2)}(kR)\xrightarrow[kR\ll 1]{}\frac{2}{\pi i}\log R$$

$$H_0^{(2)}(kR)-H_0^{(2)}(KR)\xrightarrow[\substack{kR\ll 1\\ KR\ll 1}]{} i\frac{1+\nu}{4\pi}k^2R^2\log R$$

Substituting these approximations into Eq. (4.11), we obtain,

$$U_j^m(P,Q)\xrightarrow[\substack{kR\ll 1\\ KR\ll 1}]{}\frac{1}{2\pi G}\log R\,\delta_{jm}-\frac{1+\nu}{16\pi G}\frac{\partial^2}{\partial x_{P_j}\partial x_{P_m}}(R^2\log R)$$

This becomes the same kernel function with elastostatics. In the same way as mentioned above, it is obvious that the property of kernel functions $T_j^m(P, Q)$ for small arguments are the same with that of elastostatics.

Applying these properties to the integration of the kernel functions, we propose an accurate integration method as follows. Namely, let us separate the singularity term from the kernel function of the elastodynamics as follows.

$$U_j^m(P,Q)=[U_j^m(P,Q)-U_j^m(P,Q)_{(\text{static})}]+U_j^m(P,Q)_{(\text{static})} \quad (4.12)$$

where $U_j^m(P, Q)_{(\text{static})}$ denotes the kernel functions of the elastostatics.

Then, the integration of the first term of the right hand side of Eq. (4.12) can be done easily by the simple numerical integration such as trapezoidal rule, because those of singularities are eliminated and then the integration of the last term of the right side of Eq. (4.12) can be done analytically by the same way as in the elastostatic case. For a simple example, we show the integration of the imaginary part of $H_0^{(2)}(KR)$. In the same mammer as in the representation of Eq. (4.12),

$$Y_0(KR)=\left[Y_0(KR)-\frac{2}{\pi}\log R\right]+\frac{2}{\pi}\log R$$

where $Y_0(\ )$ denotes the zero-th order Bessel function of the second kind.

Fig. 10 shows the accuracy of integration of this method.

4.2.2 Numerical examples

a) Rigid circular inclusion

We try to calculate the stresses around a

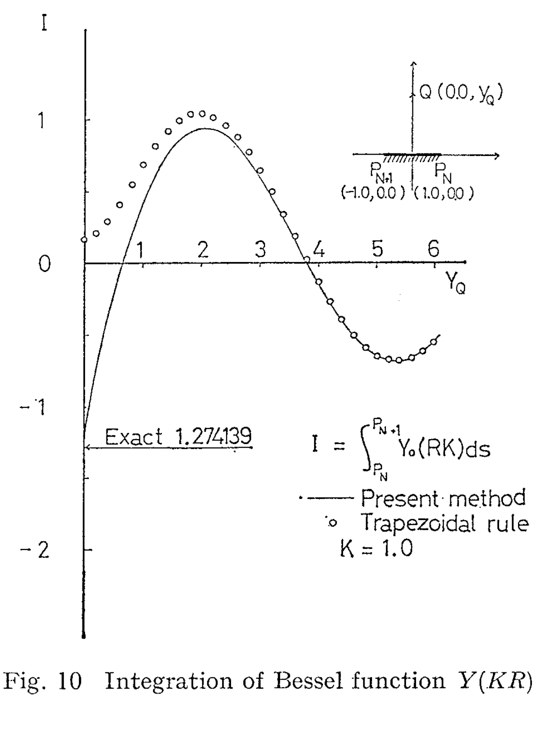
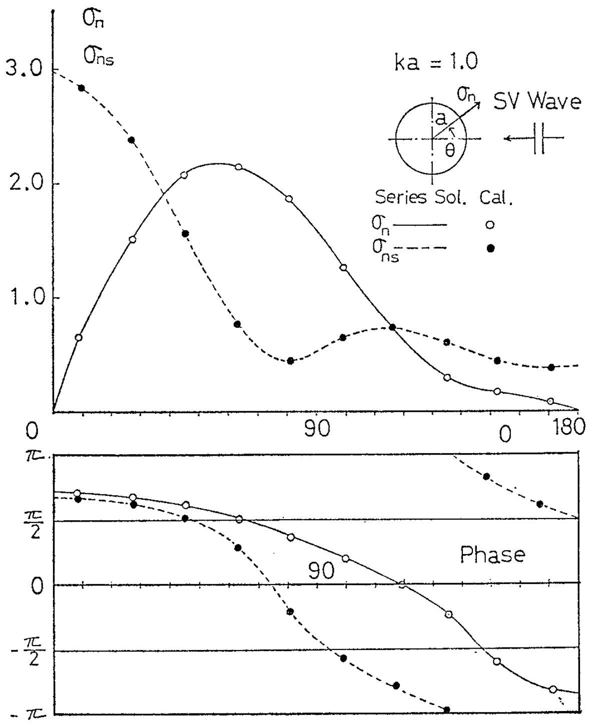


Fig. 10 Integration of Bessel function $Y(KR)$



(a)

(b)

Fig. 11 Stresses around a rigid circular inclusion of radius a subject to the incident plane harmonic P and SV wave

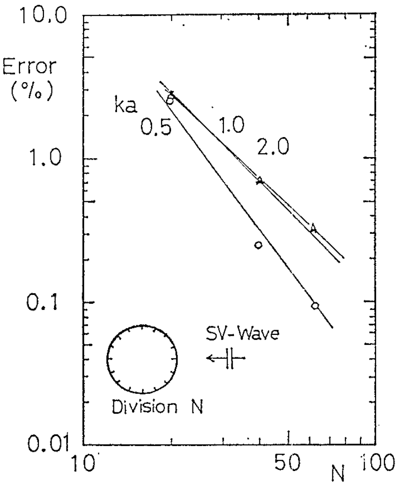


Fig. 12 Accuracy of the computation of the stress concentration factor of a rigid circular inclusion with the number of division N (divided allover length) for various wave numbers (at $\theta=0°$, SV wave)

rigid circular inclusion of radius a subject to the incident plane harmonic P and SV wave. Fig. 11 shows the principal stresses due to the incident P and SV wave. Curved lines are analytical solutions[22]. Fig. 12 shows the relation between the accuracy of the computation of the stress concentration factors (at $\theta=0°$) and the number of division N (divide overall length) for various wave numbers. Although there are some deviations in the accuracy under certain circumstances. The calculation errors are under 1 per cent when the number of division is more than 40 in the case of the short wave length of $Ka=1.0$.

b) Circular cavity

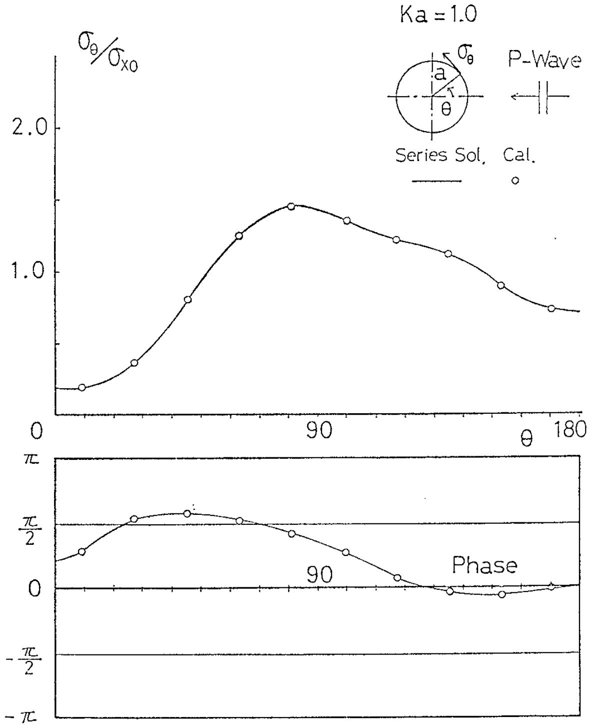
Fig. 13 shows the principal stresses around a circular cavity of radius a due to the incident P and SV wave.

Fig. 14 shows the principal stresses versus the wave number of the incident P and SV waves. The results are also in good agreement with the analytical solutions[21),22)].

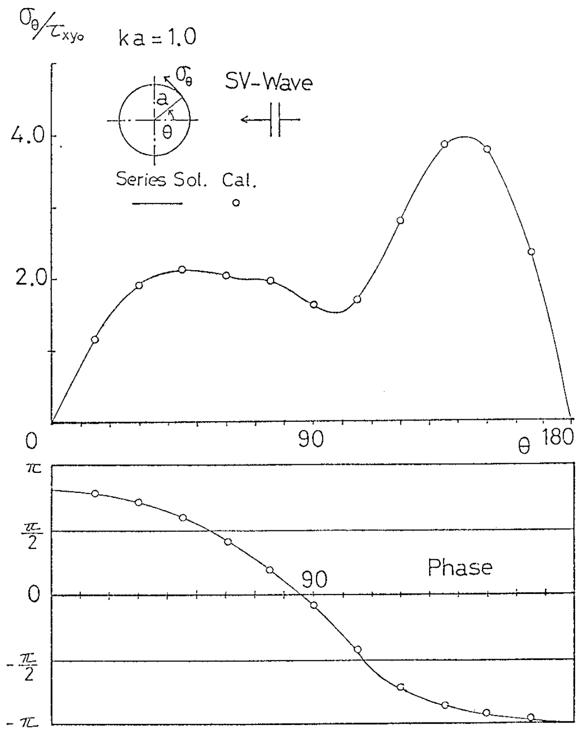
4.3 *Three dimensional elastostatic ploblems*[4)]

4.3.1 Fundamental solutions

The fundamental solutions of Navier-Cauchy's equation are as follows,

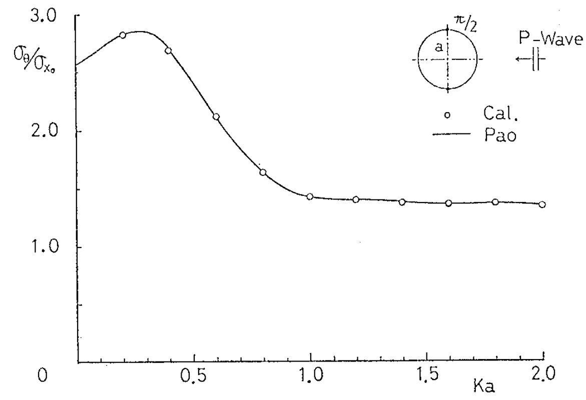


(a)

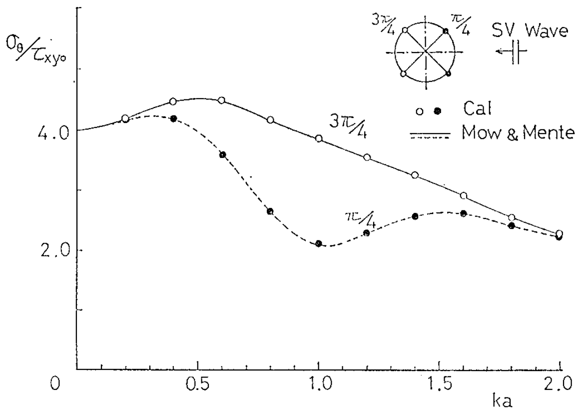


(b)

Fig. 13 Stresses around a circular cavity hole of radius a subject to the incident plane harmonic P and SV wave



(a)



(b)

Fig. 14 Principal Stresses versus wave number of the incident P and SV waves

$$U_j^m(P,Q)=\frac{1}{4\pi G}\left[\frac{1}{R}\delta_{jm}-\frac{1}{4(1-\nu)}\frac{\partial^2 R}{\partial x_{P_j}\partial x_{P_m}}\right] \tag{4.13}$$

Substituting Eq. (4.13) into Eq. (2.9) and Eq. (2.10), the kernel functions $T_j^m(P,Q)$ becomes,

$$T_j^m(P,Q)=\frac{1}{4\pi}\left[\delta_{jm}\frac{\partial}{\partial n_P}\left(\frac{1}{R}\right)+\frac{\partial x_{P_m}}{\partial n_P}\frac{\partial}{\partial x_{P_j}}\left(\frac{1}{R}\right)+\frac{\nu}{1-\nu}\frac{\partial x_{P_j}}{\partial n_P}\frac{\partial}{\partial x_{P_m}}\left(\frac{1}{R}\right)-\frac{1}{2(1-\nu)}\frac{\partial}{\partial n_P}\left(\frac{\partial^2 R}{\partial x_{P_j}\partial x_{P_m}}\right)\right] \tag{4.14}$$

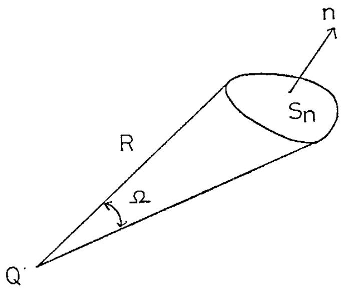


Fig. 15 Solid angle

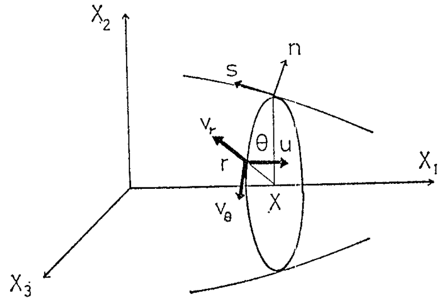


Fig. 16 Cylindrical coordinate systems

The singular properties of the kernel functions are similar as in the two dimensional elastostatics and $1/R$ and $\partial(1/R)/\partial n_P$ is the main parts of the singularity of the kernel function $U_j^m(P, Q)$ and $T_j^m(P, Q)$ respectively.

Niwa et al.[23], Manori[24], Webster[25] and Bai et al.[26] investigated the integration of the kernel function analytically. For example, we may show the integration of $\partial(1/R)/\partial n_P$ over the arbitrary element.

$$\int_{S_n}\frac{\partial}{\partial n_P}\left(\frac{1}{R}\right)ds_P$$
$$=\int_{S_n}\sum_{j=1}^{3}\frac{\partial x_{P_j}}{\partial n_P}\frac{\partial}{\partial x_{P_j}}\left(\frac{1}{R}\right)ds_P$$
$$=\int_{S_n}\frac{\cos(\vec{n},\vec{R})}{R}ds_P=\int_{S_n}d\Omega$$

where $d\Omega$ denotes the solid angle which the area ds is seen from the point Q, as shown in Fig. 15.

If Q lies on the boundary Sn considered, the solid angle becomes to 2π. In similar way, integration of $1/R$ and its derivatives can be done analytically.

4.3.2 Axi-symmetric elastic problems

Let us consider the stress analysis problems for an axi-symmetric body around the x-axis subject to axi-symmetric loads or displacements.

The axi-symmetric elasticity problems can be divided into two parts of problems. The one is the expansion and contraction problem and the other is the torsion problem.

We introduce the cylindrical coodinate systems, as shown in Fig. 16.

$$\left.\begin{aligned}&x_1=x\\&x_2=r\cos\theta\\&x_3=r\sin\theta\\&P=(x_P,r_P,\theta_P),\quad Q=(x_Q,r_Q,\theta_Q)\end{aligned}\right\}\tag{4.14}$$

Because of the symmetry of the problems, the displacements and tractions are constant quantities on a circumference of radius r. And they become the function of x and r only.

a) Expansion and contraction problems

Let the displacements and tractions under consideration be as shown in Fig. 16.

$$\left.\begin{aligned}&u_1=u(x,r)\quad,\quad \tau_1=\tau_x(x,r)\\&u_2=v_r(x,r)\cos\theta\ ,\quad \tau_2=\tau_r(x,r)\cos\theta\\&u_3=v_r(x,r)\sin\theta\ ,\quad \tau_3=\tau_r(x,r)\sin\theta\end{aligned}\right\}\tag{4.15}$$

Substituting Eq. (4.15) into Eq. (3.5), we obtain the boundary integral equation of unknown u, v_r, τ_x, τ_r,

$$\left.\begin{aligned}u(Q)=\int_C&[u(P)T_x^1(P,Q)+v_r(P)T_r^1(P,Q)\\&-\tau_x(P)U_x^1(P,Q)\\&-\tau_r(P)U_r^1(P.Q)]r_P\,ds_P\\v_r(Q)=\int_C&[u(P)T_x^2(P,Q)+v_r(P)T_r^2(P,Q)\\&-\tau_x(P)U_x^2(P,Q)\\&-\tau_r(P)U_r^2(P,Q)]r_P\,ds_P\end{aligned}\right\}\tag{4.16}$$

where

$$U_x^1(P,Q)=\int_0^{2\pi}U_1^1(P,Q)\,d\theta_P$$

$$U_r({}^1P,Q)=\int_0^{2\pi}[U_2^1(P,Q)\cos\theta_P+U_3^1(P,Q)\sin\theta_P]\,ds_P \qquad (4.17)$$

etc.

b) Torsion problems

The relations between $\{u_j, \tau_j\}$ and $\{v_\theta, \tau_\theta\}$ are as follows,

$$\left.\begin{aligned} u_1&=0 , & \tau_1&=0\\ u_2&=-v_\theta(x,r)\sin\theta, & \tau_2&=-\tau_\theta(x,r)\sin\theta\\ u_3&=v_\theta(x,r)\cos\theta , & \tau_3&=\tau_\theta(x,r)\cos\theta\end{aligned}\right\} \qquad (4.18)$$

Substituting Eq. (4.18) into Eq. (3.5), we obtain

$$v_\theta(Q)=\int_C[v_\theta(P)T_\theta(P,Q)-\tau_\theta U_\theta(P,Q)]\,r_P\,ds_P \qquad (4.19)$$

where

$$\left.\begin{aligned} U_\theta(P,Q)&=\int_0^{2\pi}[-U_2^3(P,Q)\sin\theta_P+U_3^3(P,Q)\cos\theta_P]\,d\theta_P\\ T_\theta(P,Q)&=\int_0^{2\pi}[-T_2^3(P,Q)\sin\theta_P+T_3^3(P,Q)\cos\theta_P]\,d\theta_P\end{aligned}\right\} \qquad (4.20)$$

The integration of the kernel functions are composed of following integrals;

$$S=\frac{1}{4\pi}\int_0^{2\pi}\frac{d\theta_P}{R}=\frac{1}{2}\int_0^\infty e^{-|x_Q-x_P|t}J_0(tr_P)J_0(tr_Q)\,dt$$

$$S^*=\frac{1}{4\pi}\int_0^{2\pi}R\,d\theta_P=\frac{r_P}{2}\int_0^\infty e^{-|x_Q-x_P|t}J_1(tr_P)J_0(tr_Q)\frac{dt}{t}+\frac{r_Q}{2}\int_0^\infty e^{-|x_Q-x_P|t}J_0(tr_P)J_1(tr_Q)\frac{dt}{t}+\frac{x_Q-x_P}{2}$$

$$P=\frac{1}{4\pi}\int_0^{2\pi}\frac{\cos\theta_P}{R}\,d\theta_P=\frac{r_P}{r_Q}S-\frac{1}{r_Q}\frac{\partial}{\partial s_P}S^*$$

$$P^*=\frac{1}{4\pi}\int_0^{2\pi}R\cos\theta_P\,d\theta_P=\frac{1}{2}\int_0^\infty e^{-|x_Q-x_P|t}J_1(tr_P)J_1(tr_Q)\,dt$$

These integrals S, S^*, P, P^* can be expressed by the complete elliptic integrals[7] as shown in appendix D and the kernel functions shown in appendix E.

The properties of the kernel functions are in the same manner as the two dimensional elastostatics. The main part of the singularity of the kernel functions $U_j^m(P,Q)$ is logarithmic, that of $T_j^m(P,Q)$ is the normal derivative of $\log R$.

On the integration of the term $\partial S/\partial n_P$, the new function T may be introduced,

$$T=\frac{r_P}{2}\int_0^\infty e^{-|x_Q-x_P|t}J_1(tr_P)J_0(tr_Q)\,dt$$

Then S and T have a relation as follows;

$$r_P\frac{\partial S}{\partial n_P}=-\frac{\partial T}{\partial s_P}$$

Thus, the integration of $\partial S/\partial n_P$ over the element is carried out directly as;

$$\int_{P_n}^{P_{n+1}}r_P\frac{\partial S}{\partial n_P}ds_P=-\int_{P_n}^{P_{n+1}}\frac{\partial T}{\partial s_P}ds_P=-\Big[T\Big]_{P_n}^{P_{n+1}}$$

For example, taking the range of integration between the point P_n (1.0, 5.0) and the point P_{n+1} (-1.0, 5.0) and moving the observation point Q $(0, r_Q)$ from the origin to a arbitrary

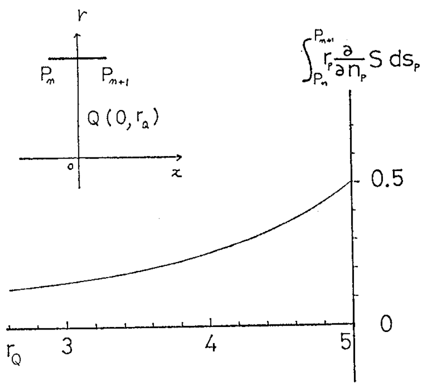


Fig. 17 Integration of $r\,\partial S/\partial n$

point of r-axis, we may calculate the above integral as shown in Fig. 17.

4.3.3 Numerical examples

Fig. 18 shows the stress around a spherical cavity in an infinite elastic medium under the uniform loading. The accuracy of computation of stress concentration factor of elliptic cavity for various aspect ratio is plotted with the number of division N of a quadrant in Fig. 19. Fig. 20 shows displacements of a solid cylinder under the uniform tension along the x-axis. Fig. 21 shows the stress around a spherical cavity in an infinite elastic medium under the twisting load. Results are in good agreement with the exact solutions.

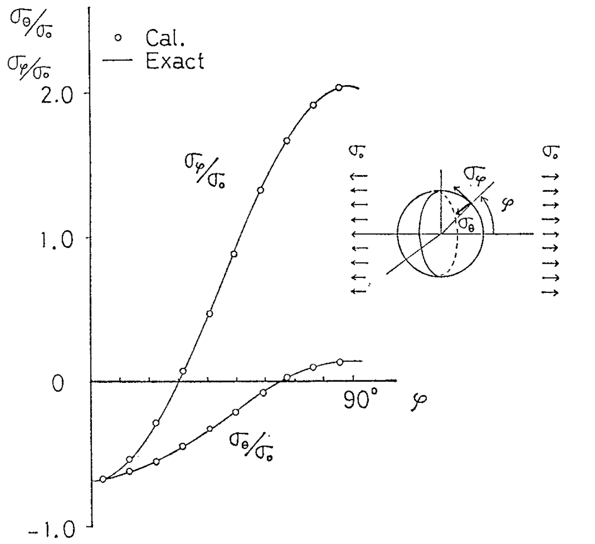


Fig. 18 Stresses around a spherical cavity hole in an infinite elastic medium under uniform loading

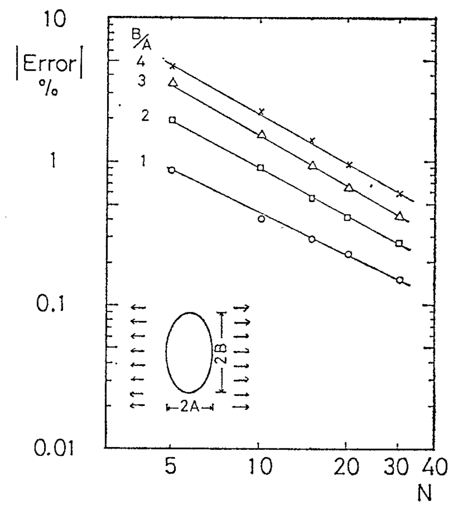


Fig. 19 Accuracy of computation of stress concentration factor of elliptic cavity hole for various aspect ratio with the number of division N

4.4 *Plate bending problems*[11)]

4.4.1 Boundary integral equations

Let D_0 denotes the domain on which the lateral load $q(x_P, y_P)$ acts. The boundary integral equations for deflection w are given as follows[9)]

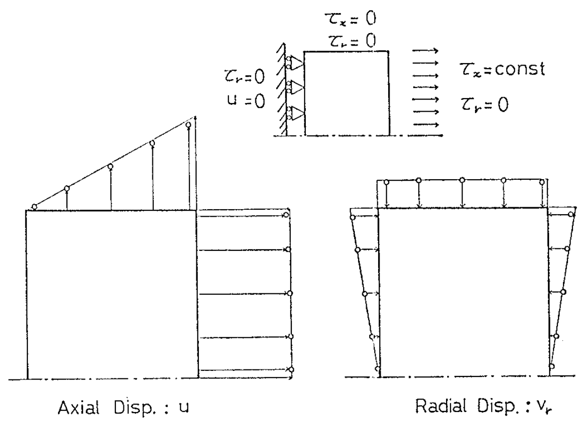


Fig. 20 Displacements of a solid cylinder under uniform tension along x-axis

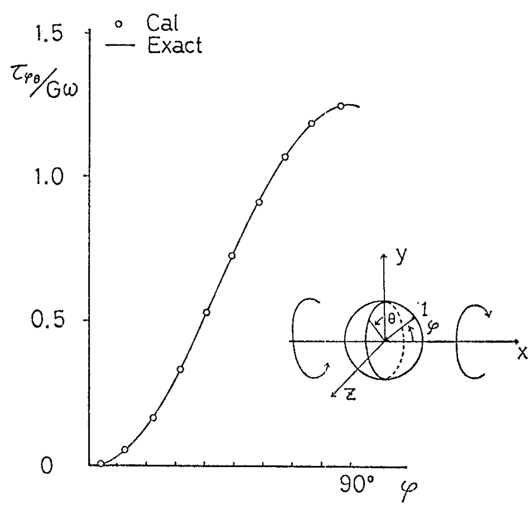


Fig. 21 Stresses around a spherical cavity hole in an infinite elastic medium under twisting load

$$w(Q)=\iint_{D_q} q(x_P,y_P)S(x_P,y_P,Q)\,dx_P\,dy_P + \int_C \Big[m_n(P)\frac{\partial}{\partial n_P}S(P,Q) - v_n(P)S(P,Q) - \frac{\partial}{\partial n_P}w(P)M_n(P,Q) + w(P)V_n(P,Q) \Big] ds_P \quad (4.21)$$

where

$$S(x_P,y_P,Q)=S(P,Q)=\frac{1}{8\pi D}R^2\log R$$

D denotes the flexural rigidity. m_n and v_n is the bending moment and the equivalent transverse shear force respectively. The kernel function $M_n\cdots$etc. are shown in appendix F.

The fundamental solution $S(P,Q)$ is the same as that of the two dimensional elastostatics. Therefore, the properties of singularity of the kernel functions are also the same.

4.4.2 Representation like as source and sink or doublet distribution in hydrodynamics

Investigate the exterior ploblems, Eq. (4.21) is complicated and inconvenient because they are composed of two sets of kernel functions. Let us try to simplify the equation to be represented by one set of kernel function.

Now, the deflection in D and $\bar{D}$ becomes as follows, respectively

$$-\int_C \Big[m_n(P)\frac{\partial}{\partial n_P}S(P,Q) - v_n(P)S(P,Q) - \frac{\partial}{\partial n_P}w(P)M_n(P,Q) + w(P)V_n(P,Q) \Big] ds_P \begin{cases} =-w(Q) & \text{in } D \\ =0 & \text{in } \bar{D} \end{cases} \quad (4.22)$$

Notation is the same as Fig. 2 and it is assumed that the lateral load is absent. Taking the regular function $w^{(0)}(P)$ in the domain $\bar{D}$, the following integral vanishes in D,

$$-\int_C \Big[m_n^{(0)}(P)\frac{\partial}{\partial n_P}S(P,Q) - v_n^{(0)}S(P,Q) - \frac{\partial}{\partial n_P}w^{(0)}(P)M(P,Q) + w^{(0)}(P)V_n(P,Q) \Big] ds_P = 0 \quad \text{in } D \quad (4.23)$$

Adding Eq. (4.22) to Eq. (4.23) and assuming the following boundary conditions, we derive

$$\left. \begin{array}{l} w(P)+w^{(0)}(P)=0 \\ \dfrac{\partial}{\partial n_P}w(P)+\dfrac{\partial}{\partial n_P}w^{(0)}(P)=0 \end{array} \right\} \text{ on } C \quad (4.24)$$

We obtain the representation of deflection as follows.

$$w(Q)=-\int_C \Big[\{m_n(P)+m_n^{(0)}(P)\}\frac{\partial}{\partial n_P}S(P,Q) - \{v_n(P)+v_n^{(0)}(P)\}S(P,Q) \Big] ds_P \quad (4.25)$$

On the other hand, when the boundary conditions are putted on tractions as follows;

$$\left. \begin{array}{l} m_n(P)+m_n^{(0)}(P)=0 \\ v_n(P)+v_n^{(0)}(P)=0 \end{array} \right\} \text{ on } C \quad (4.26)$$

We obtain

$$w(Q)=\int_C \Big[\Big\{ \frac{\partial}{\partial n_P}w(P)+\frac{\partial}{\partial n_P}w^{(0)}(P) \Big\} M(P,Q) - \{w(P)+w^{(0)}(P)\}V_n(P,Q) \Big] ds_P \quad (4.27)$$

If we take the regular function $w^{(0)}$ as the uniform bending field, Eq. (4.25) and Eq. (4.27) are appropriate to use for stress analysis around a rigid inclusion and a cavity hole respectively in an infinite plate under uniform bending.

4.4.3 Numerical computation for the uniform latatal loading

The numerical computation of Eq. (4.21) is the same as in the two dimensional elastostatics. When the lateral load acts the plate, the integration over the domain D_0 is needed. Then we have to do the numerical integration over the domain D_0, such as in the FEM. No matter what unknowns are on the boundary, the calculations must be done over the whole interior domain D_0 and the advantage of the BEM is lost.

However, the lateral load q is usually uniform and then the following method is applicable.

Now, the deflection w is governed by the differential equation as follows,

$$\Delta^2 w = \frac{q}{D} \tag{4.28}$$

where q is constant over the plate.

This inhomogeneous equation has a general solution in the following form,[8]

$$w = w_0 + w_1$$

where

$$\Delta^2 w_0 = \frac{q}{D} \tag{4.29}$$

$$\Delta^2 w_1 = 0 \tag{4.30}$$

where w_0 is a particular solution for Eq. (4.28) and w_1 is the homogeneous solution.

We may find a particular solution w_0 for a uniform loading for Eq. (4.29), for example, as follows,

$$w_0 = \frac{q}{64D}(x^2+y^2)^2 \tag{4.31}$$

Then, the problem is to obtain the solution w_1 which satisfy the boundary condition as a whole.

For example, in the case of the clamped plate, the boundary conditions are given as follows,

$$\left.\begin{array}{l} w=0 \\ \partial w/\partial n = 0 \end{array}\right\} \quad \text{on } C \tag{4.32}$$

So, the boundary condition of w becomes as follows,

$$\left.\begin{array}{l} w_1 = -w_0 = -\dfrac{q}{64D}(x^2+y^2)^2 \\ \dfrac{\partial w_1}{\partial n} = -\dfrac{\partial w_0}{\partial n} = -\dfrac{q}{64D}\left\{\cos\alpha(x^3+xy^2) \right. \\ \qquad\qquad \left. +\sin\alpha(x^2y+y^3)\right\} \end{array}\right\} \tag{4.33}$$

where

$$\cos\alpha = -\frac{\partial x}{\partial n}, \quad \sin\alpha = \frac{\partial y}{\partial n}$$

Accordingly, the solution w_1 can be obtained from the differential equation (4.30) under the boundary condition (4.33). The boundary integral equations are as follows,

$$\int_C \left[m_{n1}(P)\frac{\partial}{\partial n_P}S(P,Q) - v_{n1}(P)S(P,Q)\right] ds_P$$
$$= -w_0(Q) + \int_C \left[\frac{\partial}{\partial n_P} w_0(P) M_n(P,Q) - w_0(P)V_n(P,Q)\right] ds_P$$

where unknowns are m_{n1} and v_{n1} on the boundary.

4.4.4 Numerical examples

Fig. 22 shows the stresses around a circular hole in an infinity extended plate under the uniform bending. Fig. 23 shows the deflection and the bending moment of a circular plate under the uniform lateral load.

Both results in good agreement with the exact solutions.

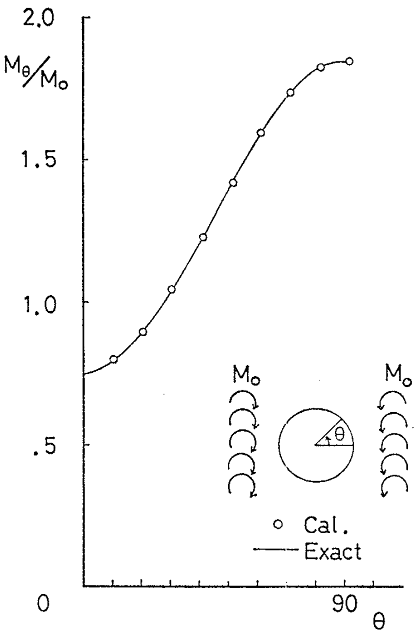
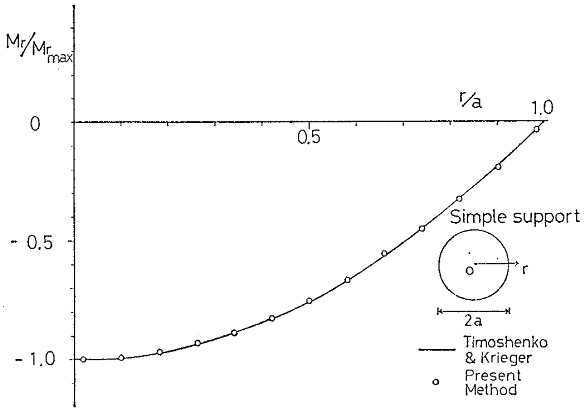
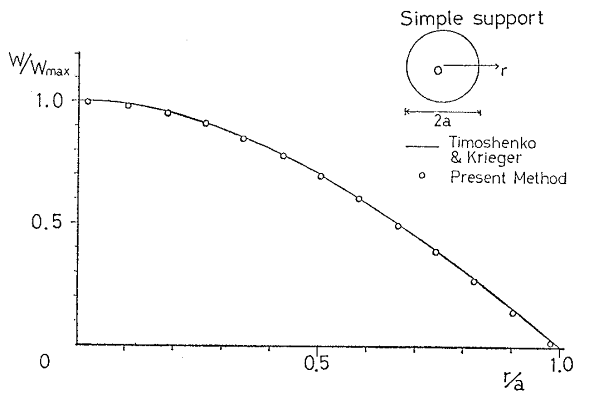


Fig. 22 Stresses around a circuler hole in an infinite plate under uniform bending



(a)



(b)

Fig. 23 Deflections and bending moments of a circular plate under uniform lateral load

5. Conclusions

In this report, the general formulation of the singularity method (so-called the boundary element method) which transform the fundamental differential equation in the theory of elasticity into the integral equation is studied and summarised as follows:

(1) There exists many alternative boundary integral equations to be solved, and we may select the favorable one to solve the problem considered.

(2) The usefull one of such integral equations are derived and represented by one species of singularity as the unknown like as source and sinks or doublet in hydrodynamics.

(3) The formulation of the boundary integral equations for various problems, i.e. the two dimensional elastostatics, the two dimensional elastodynamics, the three dimensional elastostatics and the plate bending problems, are shown,

(4) The singular properties of the kernel function for each problem are studied especially with regard to its analytical and numerical integration.

(5) To verify its usefullness and accuracy, some numerical examples are shown in each problem.

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Appendix A

Kernel function of two dimensional elastostatics

$$T_1{}^1(P,Q)=\frac{\partial}{\partial s_P}\left[\frac{1}{2\pi}\theta+\frac{1+\nu}{8\pi}\sin 2\theta\right]$$

$$T_1{}^2(P,Q)=\frac{\partial}{\partial s_P}\left[\frac{1-\nu}{4\pi}\log R-\frac{1+\nu}{8\pi}\cos 2\theta\right]$$

$$T_2{}^1(P,Q)=\frac{\partial}{\partial s_P}\left[\frac{-1+\nu}{4\pi}\log R-\frac{1+\nu}{8\pi}\cos 2\theta\right]$$

$$T_2{}^2(P,Q)=\frac{\partial}{\partial s_P}\left[\frac{1}{2\pi}\theta-\frac{1+\nu}{8\pi}\sin 2\theta\right]$$

$$U_1{}^1(P,Q)=\frac{1}{G}\left[\frac{3-\nu}{8\pi}(\log R+1)-\frac{1+\nu}{16\pi}\cos 2\theta\right]$$

$$U_1{}^2(P,Q)=-\frac{1+\nu}{16\pi G}\sin 2\theta$$

$$U_2{}^1(P,Q)=U_1{}^2(P,Q)$$

$$U_2{}^2(P,Q)=\frac{1}{G}\left[\frac{3-\nu}{8\pi}(\log R+1)+\frac{1+\nu}{16\pi}\cos 2\theta\right]$$

Appendix B

Kernel function of two dimensional semi-infinit elastic space problems

$$f(Q)=\int_C\sum_{j=1}^{2}[u_j(P)T_j(P,Q)-\tau_j(P)U_j(P,Q)]\,ds_P$$

$$T_1(P,Q)=\frac{E}{4\pi}\frac{-\partial}{\partial s_P}\left[(y_Q-y_P)\log\frac{R}{R'}+\frac{y_Q-y_P}{2}-\frac{(y_Q-y_P)R^2}{2R'^2}\right]$$

$$T_2(P,Q)=\frac{E}{4\pi}\frac{\partial}{\partial s_P}\left[(x_Q-x_P)\log\frac{R}{R'}+\frac{x_Q-x_P}{2}-\frac{(x_Q-x_P)R^2}{2R'^2}\right]$$

$$U_1(P,Q)=\frac{-1}{2\pi}\left[\frac{1-\nu}{2}(x_Q-x_P)\log\frac{R}{R'}-(y_Q-y_P)(\theta+\theta')-(1-\nu)(x_Q-x_P)\frac{y_Q y_P}{R'}\right]$$

$$U_2(P,Q)=\frac{-1}{2\pi}\left[\frac{1-\nu}{2}(y_Q-y_P)\log\frac{R}{R'}+(x_Q-x_P)(\theta+\theta')-\frac{5+\nu}{2}y_Q+(1+\nu)(y_Q+y_P)\frac{y_Q y_P}{R'^2}\right]$$

$$\theta'=\tan^{-1}\frac{y_Q+y_P}{x_Q-x_P}$$

Appendix C

Kernel function of two dimensional elasto-dynamics

$$T_1{}^1(P,Q)=\frac{i}{4}\left[\left(\frac{\partial y_P}{\partial n_P}\right)^2\frac{\partial}{\partial n_P}H_0^{(2)}(kR)+\left(\frac{\partial x_P}{\partial n_P}\right)^2\frac{\partial}{\partial n_P}H_0^{(2)}(KR)+\frac{\partial x_P}{\partial n_P}\frac{\partial y_P}{\partial n_P}\frac{\partial}{\partial s_P}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}\right]$$
$$-\frac{i}{2k^2}\frac{\partial^3}{\partial s_P\partial x_P\partial y_P}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}$$

$$T_2{}^1(P,Q)=-\frac{i}{4}\left[\left(\frac{\partial y_P}{\partial s_P}\right)^2\frac{\partial}{\partial s_P}H_0^{(2)}(kR)+\left(\frac{\partial x_P}{\partial s_P}\right)^2\frac{\partial}{\partial s_P}H_0^{(2)}(KR)+\frac{\partial x_P}{\partial s_P}\frac{\partial y_P}{\partial s_P}\frac{\partial}{\partial n_P}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}\right]$$
$$+\frac{i}{2}\frac{\partial}{\partial s_P}\left[H_0^{(2)}(kR)+\frac{1}{k^2}\frac{\partial^2}{\partial x_P^2}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}\right]$$

$$T_1{}^2(P,Q)=\frac{i}{4}\left[\left(\frac{\partial x_P}{\partial s_P}\right)^2\frac{\partial}{\partial s_P}H_0^{(2)}(kR)+\left(\frac{\partial y_P}{\partial s_P}\right)^2\frac{\partial}{\partial s_P}H_0^{(2)}(KR)+\frac{\partial x_P}{\partial s_P}\frac{\partial y_P}{\partial s_P}\frac{\partial}{\partial n_P}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}\right]$$
$$-\frac{i}{2}\frac{\partial}{\partial s_P}\left[H_0^{(2)}(kR)+\frac{1}{k^2}\frac{\partial^2}{\partial y_P^2}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}\right]$$

$$T_2{}^2(P,Q)=\frac{i}{4}\left[\left(\frac{\partial x_P}{\partial n_P}\right)^2\frac{\partial}{\partial n_P}H_0^{(2)}(kR)+\left(\frac{\partial y_P}{\partial n_P}\right)^2\frac{\partial}{\partial n_P}H_0^{(2)}(KR)-\frac{\partial x_P}{\partial n_P}\frac{\partial y_P}{\partial n_P}\frac{\partial}{\partial s_P}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}\right]$$
$$+\frac{i}{2k^2}\frac{\partial^3}{\partial s_P\partial x_P\partial y_P}\{H_0^{(2)}(kR)-H_0^{(2)}(KR)\}$$

Appendix D

Integrals S,S^*, $P'P^*$ expressed by the complete elliptic integrals

$$S=\frac{kK(k)}{2\pi(r_Pr_Q)^{1/2}}$$

$$S^*=\frac{2(r_Pr_Q)^{1/2}E(k)}{\pi k}$$

$$P=\frac{1}{\pi k(r_Pr_Q)^{1/2}}\left[\left(1-\frac{1}{2}k^2\right)K(k)-E(k)\right]$$

$$P^*=-\frac{(x_Q-x_P)k}{4\pi(r_Pr_Q)^{1/2}}K(k)-\frac{\Lambda(\alpha,\beta)}{4}+\frac{1}{2}\qquad r_P>r_Q$$
$$=-\frac{(x_Q-x_P)k}{4\pi r_P}K(k)+\frac{1}{4}\qquad r_P=r_Q$$
$$=-\frac{(x_Q-x_P)k}{4\pi(r_Qr_P)^{1/2}}K(k)+\frac{\Lambda(\alpha,\beta)}{4}\qquad r_P<r_Q$$

$$k^2=\frac{4r_Pr_Q}{(x_Q-x_P)^2+(r_Q-r_P)^2}$$

$$\alpha=\sin^{-1}k$$

$$\beta=\sin^{-1}[(x_Q-x_P)/\{(x_Q-x_P)^2+(r_Q-r_P)^2\}^{1/2}]$$

$K(\ )$ and $E(\ )$ denotes the complete elliptic integral of the first and second kind respectively. $\Lambda(\ )$ is the Hewman's ramda function

Appendix E

Kernel function of axi-symmetric elastic porblem

$$U_x{}^1(P,Q)=S-\frac{1}{4(1-\nu)}\frac{\partial^2S^*}{\partial x_P^2}$$

$$U_r{}^1(P,Q)=-\frac{1}{4(1-\nu)}\frac{\partial^2S^*}{\partial x_P\partial r_P}$$

$$U_x{}^2(P,Q)=-\frac{1}{4(1-\nu)}\left\{(x_Q-x_P)\frac{\partial P}{\partial r_P}+\frac{1}{r_P}(x_Q-x_P)P\right\}$$

$$U_r{}^2(P,Q)=\frac{1}{4(1-\nu)}\left\{(3-4\nu)P+(x_Q-x_P)\frac{\partial P}{\partial x_P}\right\}$$

$$T_x{}^1(P,Q)=\frac{\partial S}{\partial n_P}+\frac{1}{1-\nu}\frac{\partial x_P}{\partial n_P}\frac{\partial S}{\partial x_Q}-\frac{1}{2(1-\nu)}\frac{\partial}{\partial n_P}\left(\frac{\partial^2S^*}{\partial x_P^2}\right)$$

$$T_r{}^1(P,Q)=\frac{\partial x_P}{\partial n_P}\frac{\partial S}{\partial r_P}+\frac{\nu}{1-\nu}\frac{\partial r_P}{\partial n_P}\frac{\partial S}{\partial x_P}$$

$$-\frac{1}{2(1-\nu)}\frac{\partial}{\partial n_P}\left(\frac{\partial^2 S^*}{\partial x_P \partial r_P}\right)$$

$$T_x{}^2(P,Q)=\frac{\partial r_P}{\partial n_P}\frac{\partial P}{\partial x_P}+\frac{\nu}{1-\nu}\frac{\partial x_P}{\partial r_P}\left(\frac{\partial P}{\partial r_P}+\frac{1}{r_P}P\right)$$

$$-\frac{1}{2(1-\nu)}\frac{\partial}{\partial n_P}\left(\frac{\partial^2 P^*}{\partial x_P \partial r_P}+\frac{1}{r_P}\frac{\partial P^*}{\partial x_P}\right)$$

$$T_r{}^2(P,Q)=\frac{\partial P}{\partial n_P}-\frac{1}{1-\nu}\frac{\partial x_P}{\partial n_P}\frac{\partial P}{\partial x_P}$$

$$+\frac{\nu}{1-\nu}\frac{\partial r_P}{\partial n_P}\frac{1}{r_P}P$$

$$+\frac{1}{2(1-\nu)}\frac{\partial}{\partial n_P}\left(\frac{\partial^2 P^*}{\partial x_P^2}\right)$$

$$U_Q(P,Q)=P$$

$$T_Q(P,Q)=\frac{\partial P}{\partial n_P}-\frac{\partial r_P}{\partial n_P}\frac{P}{r_P}$$

Appendix F

Kernel function of plate bending problem

$$S(P,Q)=\frac{1}{8\pi D}R^2\log R$$

$$\frac{\partial}{\partial n_P}S(P,Q)=-\frac{1}{8\pi D}(2R\log R+R)\cos(\theta-\alpha_P)$$

$$\frac{\partial}{\partial n_Q}S(P,Q)=\frac{1}{8\pi D}(2R\log R+R)\cos(\theta-\alpha_Q)$$

$$\frac{\partial^2}{\partial n_P \partial n_Q}S(P,Q)=-\frac{1}{8\pi D}[(2\log R+1)$$

$$\cdot\cos(\alpha_P-\alpha_Q)$$

$$+2\cos(\theta-\alpha_P)\cos(\theta-\alpha_Q)]$$

$$M_n(P,Q)=\frac{1+\nu}{4\pi}\log R+\frac{1-\nu}{8\pi}\cos 2(\theta-\alpha_P)$$

$$+\frac{1+3\nu}{8\pi}$$

$$V_n(P,Q)=\frac{\partial}{\partial s_P}\left[\frac{1}{2\pi}\theta+\frac{1-\nu}{8\pi}\sin 2(\theta-\alpha_P)\right]$$

$$\frac{\partial}{\partial n_Q}M_n(P,Q)=\frac{1+\nu}{4\pi}\frac{\cos(\theta-\alpha_Q)}{R}$$

$$+\frac{1-\nu}{4\pi}\frac{\sin 2(\theta-\alpha_P)\sin(\theta-\alpha_Q)}{R}$$

$$\frac{\partial}{\partial n_Q}V_n(P,Q)=\frac{\partial}{\partial s_P}\left[\frac{-1}{2\pi}\frac{\sin(\theta-\alpha_Q)}{R}\right.$$

$$\left.-\frac{1-\nu}{4\pi}\frac{\cos 2(\theta-\alpha_P)\sin(\theta-\alpha_Q)}{R}\right]$$