The Risk Parity Portfolio and the Low-Risk Asset Anomaly*

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abstract

Recently a portfolio management method called risk parity has been attracting attention for its high performance. This method levels out portfolios' risk allocations, but very few explanations have been provided for the high performance.

One of the exceptions is Asness et al. [1]. Asness et al. [1] links the discounting of low-risk assets by leverage aversion to risk parity portfolios. But risks and risk allocations should not be regarded as the same. On the other hand, it is also possible to show that investor overconfidence is a discounting factor for low-risk assets. In this case, as the size of risk allocations in the market is directly linked to either overvaluation or undervaluation, we could get a result supporting risk parity portfolios, which suggests that it is desirable to level out risk allocations by constantly comparing them with the market portfolio.

Here we seek the more appropriate explanation for risk parity portfolios between leverage aversion and overconfidence. As a result, we choose the latter as the more plausible explanation. First, we summarize the relationship between the respective implications of the two theories and risk parity portfolios. Next, we conduct research on bond markets where we could detect some difference between leverage aversion and overconfidence. Our empirical study shows that risk parity portfolios demand explanations other than leverage aversion.

Key words: risk allocation, low-risk asset anomaly, CAPM, market portfolio
JEL classification: G11, G12

1. Introduction

The market portfolio is important not only in academics but also in practice. For example, considering a category of securities as a single market, such as domestic equity or government bond, investors often manage their portfolio based on the market-value weighted portfolio. The reason why they stick to the market portfolio is that they expect it has a preferred risk and return property. However, in recent years, apart from anomalies which have been pointed out previously, there have been some reports insisting on strategies to obtain high efficiency

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rather than the market portfolio without special skills or information.

One of such strategies is risk parity portfolio (which is described as RP portfolio hereafter) which attempts to equalize risk contributions or risk allocations across all assets included in a portfolio. Now, when you choose a portfolio in this way, you should keep in mind that it depends on the form of securities issued. For example, the investment ratio in a company may increase when it splits into two companies, and the ratio decreases when two companies merge. Any portfolio can be formed through the integration and division of companies or securities, then it can be argued that we should consider the division of the securities when comparing the portfolio selection methods[1]. However, the issues about the division of the securities is not discussed in this paper.

One of the reasons why RP portfolio is attracting attention in practice would be its intuitive clarity. In the operation of institutional investors such as pension funds, they are always conscious of diversification. In order to diversify their portfolios, they determine a diversified asset allocation as a benchmark and they maintain their portfolio to keep the deviations from the benchmark so as not to be large. Although a typical equity ratio of these allocations is around 50%, performance of this portfolio will depend mostly on the results of the equity market because of the significantly high volatility of equity. It is pointed out that the portfolio cannot be regarded as a well-diversified portfolio because its performance depends on the results of a single asset. Risk parity is appealed as a solution to this situation or as a method of being easy to understand diversified portfolios.

Another reason why RP portfolio is worth noting is its performance. In a variety of portfolio selection problems, from the selection of individual securities, sectors or asset classes, PR portfolios have shown better performances than market-value weighted indices considered as the market portfolio. Therefore, risk parity products have appeared one after the other[2]. In the *Journal of Investing,* where practical operational ideas are mainly discussed, issues in the autumn of 2012 and the spring of 2011 covered risk parity intensively.

Although RP portfolio has attracted attention in this way, the reasons of its performance have not seemed to be described well. As one of the few, Asness et al. [1] has been recognized as the consistent description of RP portfolio.

They explain the superior performance which RP portfolio achieves in various investments with an investor's leverage aversion. The asset pricing model that takes into account the leverage constraints they cited as a basis is from Frazzini and Pedersen [12], where there is a low-risk asset anomaly that the lower the risk the asset has, the bigger alpha which is not explained by market risk exists. Asness et al. [1] claims that RP portfolio contains a larger amount of low-risk assets and that cause of superiority is the low-risk asset anomaly.

[1] Because the market portfolio is not changed by the division of the securities, as an extreme example, it can be devised how to split the securities so market portfolio has the property of risk parity.

[2] In recent years, asset management companies such as the following have offered risk parity funds. Bridgewater All Weather Fund, which is regarded as a pioneer, has grown up to $46bil. (2011): AQR Capital Management, Aquila Capital, BlackRock, Bridgewater, First Quadrant, Invesco, Lyxor Asset Management, PanAgora Asset Management, Putnam Investments, State Street Global Advisors, Wellington, etc.

But we can easily make a counter-example to this claim. RP portfolio is intended to smooth out risk allocations. It does not necessarily have a lot of the low-risk assets. If there is much supply of low-risk assets in the market, then the market portfolio holds more such assets than RP portfolio. In this case, RP portfolio would lose to the market if Frazzini and Pedersen's [12] low-risk asset anomaly worked. Conversely, if RP portfolio was excellent, the low-risk assets anomaly caused by leverage constraints might not exist.

Given this, the performance of RP portfolio could not be explained by low risk effect. Frazzini and Pedersen [12] make a lot of empirical tests in a broad range of assets, and find the low risk effect. However, it might be the only true reason for the superior performance in such a market that RP portfolio held a lot of low-risk assets by chance. Other portfolio selection methods to bias in low-risk assets would also outperform well. To explain RP portfolio, it is required to justify smoothing risk allocation.

By the way, the explanation of the low-risk assets anomaly is not limited to the leverage constraint. For example, similar results can be derived from an investor's tendency to be overconfident about the information they obtained[3]. Empirical results about an investor's overconfidence in psychology are so robust that it has been applied to a variety of asset pricing models. An investor's overconfidence leads to different results about risk allocation from those that leverage constraint leads.

With respect to these two factors of low-risk assets anomalies, in this paper, we arrange the implications related to the RP portfolio and show that the explanation by overconfidence is more plausible. In equilibrium under the constraint of leverage, leveling the risk allocation more than the market portfolio may be undesirable. On the other hand, under the overconfidence of investors, it is always desirable that a portfolio is modified from market portfolio in the direction to smooth out the risk allocation. Then, we look at the bond markets where such differences appear. From the empirical results, the explanation of RP portfolio by leverage constraint is denied.

In the next section, we review the previous research and describe the characteristics of RP portfolio. We create a model that assumes the overconfidence of investors and borrowing constraints in Section 3. In Section 4, we examine the nature of the equilibrium and clarify the difference between implications to the RP portfolio. The market for the abundance of low-risk assets is interesting. Then, focusing on the bond market such as the interesting market, we make a simple empirical analysis in Section 5. Section 6 is the conclusion. Proof of the proposition necessary to prove is shown in the Appendix.

2. Risk parity portfolio and previous research

Ray Dalio from Bridgewater Associates, who is considered as a pioneer of RP portfolio, writes in Dalio [8]:

3 There are some hypotheses to explain the superior performance of low-risk assets, such as investment management commissioned based on a benchmark, the equity's payoff similar to an option, etc.

In the drive to solve their two biggest problems --- inadequate expected returns and over concentration in equities ···

As a countermeasure, he proposes the RP portfolio, where the risk contributions of each asset to the entire risk of a portfolio are equalized.

It will be formulated as follows. Suppose there are S risky assets ($s = 1, \cdots, S$). Let portfolio vector be $x = (x_1, \cdots, x_S)'$, and vector where s-th element is x_s and the others are 0 be $\hat{x}_s = (0, \cdots, 0, x_s, 0, \cdots, 0)'$, and covariance matrix be Ω. Then the risk σ_p expressed in standard deviation of the portfolio is,

$$\sigma_p = \sqrt{x' \ \Omega x} = \sum_{s=1}^{S} \frac{\hat{x}_s' \ \Omega x}{\sigma_p}. \tag{1}$$

The risk can be broken down into *s* asset's risk contribution $RC_s \equiv \hat{x}_s' \ \Omega x / \sigma_p$ as the last term shows. In this paper, we focus on the risk allocation RC_s / σ_p which is ratio of the risk contriburion. Because it can also be regarded as the budget allocated to each asset from total risk of the portfolio, these are also referred to as risk budget[4].

In the RP portfolio, the risk allocations are even:

$$\frac{RC_s}{\sigma_p} = \frac{1}{S}. \tag{2}$$

To calculate the RP portfolio, ingenuity is required because the problem is not the standard mean-variance optimization. But, the existence of a solution was already confirmed (Maillard et al. [17]) and efficient algorithms have been proposed (Chaves et al. [6]).

There have been many reports which say that RP portfolio achieves good performance although it is a simple technique that does not use the expected return. Chaves et al. [5] compares some portfolio selection methods as well as mean-variance optimization in asset classes's allocation problems, and finds the RP portfolio was good in the Sharpe ratio. Asness et al. [1] also analyzes allocation problems of asset classes and reports that the RP portfolio was good, even if the scope of the underlying assets changes by data availability.

Applied to the problem of the U.S. equity sector allocation, Lee [15] has confirmed RP portfolio showed good performance. Chow et al. [7] compared the various portfolios' selection criteria for individual stock selection problems. Although the RP portfolio is not included in their analysis, apparently because of a large number of underlying assets, mean-variance optimal portfolio, under the assumption that Sharpe ratios of each asset are equal, has been investigated. As described later, this assumption is one of the conditions for RP portfolio may be found on the frontier. In the empirical analysis, this portfolio was superior

4 The risk contribution relates marginal change of portfolio's risk, from the fact that $RC_s = x_s \frac{\partial \sigma_p}{\partial x_s}$ is satisfied. Therefore, the optimal portfolio can be characterized with RC_s.

to the market value-weighted portfolio. Maillard et al. [17] invests in the commodity markets. They report that although RP portfolio may be inferior to the global minimum variance method depending on the period, the degree of concentration of investment in RP portfolio is smaller and this characteristic is favorable.

There has been some criticism against the RP portfolio for the practical difficulty of leverage (for example, Inker [13]). Because the risk of RP portfolio is determined dependently by the assets incorporated, the adjustment should be made by the rent or loan of safe assets. To let the level of risk of the portfolio be as same as the traditional portfolios, you will need to leverage with borrowing. Because of the various constraints of the borrowing reality, RP portfolio is criticized. But it might be a matter of investor risk appetite and investment constraints, this does not seem to be a criticism of the portfolio selection method.

On the other hand, RP portfolio has been reported to have favorable properties when considering the actual operation. Kaya and Lee [14] write that changes of inputs such as covariance make little change to the RP portfolio. Even when there is fat tail distribution or an estimated error exits, it has been reported that RP portfolio is relatively effective in many cases than the mean-variance optimal portfolio. There are many reports on the efforts of operations also. One approach is equalizing the risk contribution of risk factor, not assets (for example, Lohre et al. [16], Roncalli and Weisang [19]). This might resolve the problem that RP portfolio is dependent heavily on assets included.

RP portfolio has attracted attention by intuitive risk diversification and the results beyond the traditional approach. However, it is difficult to explain why the performance is good. In general, the results provided by the portfolio are those obtained from the input parameters assumed at the time of selection such as expected returns or covariance. It should be impossible to discuss relative merits only with the results. With awareness of these issues, Lee [15] has emphasized the importance to recheck the inputs and to try to evaluate portfolio selection methods that are based on risks including RP portfolio.

So, we will discuss the optimality of RP portfolio in mean-variance analysis. When the correlation coefficients of all asset pairs are the same and the Sharpe ratios of all assets are the same, RP portfolio is the tangency portfolio. This confirmation is easy, as shown in the Appendix. Kaya and Lee [14] report, about major asset classes, although there are large disparities among Sharpe ratios in 10 years, the median of 10 year Sharpe ratios for 75 years are approximately equal.

However, if such conditions are right, it will be strange. If all investors trade in this way, the price determined as a result is not going to meet the expectations. That is, when the assets are priced so that the correlation coefficients are equal and the Sharpe ratios are equal, as shown in the Appendix, investor demand for securities are inverse of the standard deviation. Since the sum of all investor demand for securities does not generally commensurate with the supply, the price assumed as before does not hold.

To describe the risk parity, it is required that the price is consistent with investor behavior. Previous studies that were introduced here seem to underscore the performance and there are a few discussions described by factors that are consistent in equilibrium. One of such discussions is Asness et al. [1]. In this paper, their description will be investigated.

3. Pricing Model

Frazzini and Pedersen's [12] model which is used to explain the performance of RP portfolio in Asness et al. [1] is described in this section. In this paper, we simplify the setting of the model[5], but we introduce overconfident investors to analyze the effect in the same setting. Overconfident investors are modeled after Daniel et al. [10] because their setting is similar to Frazzini and Pedersen [12] and it is easy to introduce them.

3.1. Setting

There are investors $(i = 1, \dots, I)$ who have wealth W^i at time 0, living until time 1. Investors trade securities $(s = 1, \dots, S)$. Payoffs of securities can't be replicated by each other. Investors can borrow and lend at the risk free rate of interest.

Security s pays the dividend $\tilde{\delta}_s$ at time 1 which is drawn from normal distribution, then the price goes to zero. Investors have common prior beliefs about the dividend $\tilde{\delta}_s \sim N(\delta_s, V_s)$. Issuance volume can be set to arbitrary because there is a degree of freedom to the size of the dividend, all securities are issued I, that is, one per investor. Securities are divisible into arbitrary small packets. P_s is the relative price of the security s to the risk-free asset.

Having homogeneous preference, all investors choose their portfolio $x^i = (x^i_1, \cdots, x^i_S)'$ to maximize $E^i[-\exp(-A\tilde{c}^i)]$. $\tilde{c}^i$ is the consumption at time 1, which is equal to the payoff of the portfolio and $E^i[\cdot]$ is the expectation based on i's beliefs.

Investor i is constrained:

$$\gamma^i \sum_s x^i_s P_s \le W^i \tag{3}$$

If leverage is not allowed, $\gamma^i = 1$, if it is required to have risk free asset, $\gamma^i > 1$. When leverage is allowed, $0 < \gamma^i < 1$. Because it can be assumed that there is an upper limit of leverage due to margin (collateral), γ^i is determined to a finite value. The rate of margin is supposed to be the same for all securities.

The investor's problem is equivalent to determining the weights of portfolios as the same number of securities, which have no correlation to each other, because investors can compose the portfolio with arbitrary weight of each security. Therefore, we decided to seek the price of these portfolios[6]. Since then, securities refer to portfolios uncorrelated with each other, and price, the expected value of the dividend, etc. are the ones of the portfolio. The symbols are incorporated as a symbol of the original securities.

Some investors receive the signal θ_s for payoff of the security s. Proportion of investors

[5] In this paper, the investment period is one and the supply of securities is given. Frazzini and Pedersen [12] use an over-lapping generation model where the securities are supplied by previous generation.
[6] We assume the price is not zero, to define the rate of return.

receiving the signal is ϕ_s, who receive the same signal. The signal is

$$\theta_s = \tilde{\delta}_s + e_s - \delta_s \tag{4}$$

where $\tilde{\delta}_s$ is a true dividend of security s. The signal is the true dividend plus noise e_s minus the expectation of dividend δ_s, to simplify the subsequent description. The noise has no correlation to other random variables, whose variance is V_s^R. However, investors are overconfident about the received signal, taking its variance as a less V_s^C than true value. We define the precision $v_s^k = 1/V_s^k$ (k=R or C). The knowledge about the payoffs and investors are common.

There is no noise trader or shock of security supply. The investor who does not receive any signal can observe the information from market prices, correctly for the uncertainty unlike investors who receive signals directly. Let v_s be the precision before receiving the signal, the expectation and variance of dividend of security s after receiving the signal are

$$E^k[\tilde{\delta}_s] = \delta_s^k = \delta_s + \frac{v_s^k}{v_s + v_s^k}\theta_s \tag{5}$$

$$E^k\left[\left(\tilde{\delta}_s - \delta_s^k\right)^2\right] = \mathrm{var}^k(\tilde{\delta}_s) = \frac{1}{v_s + v_s^k}. \tag{6}$$

3.2. *The price and return of the securities*

We can analyze securities one by one because the covariance matrix is diagonal[7]. In the following, we think of the competitive equilibrium. According to the standard procedures for exponential utility and payoff from normal distribution, the first order condition of investor i,

$$0 = \delta_s^k - (1 + r_f)P_s - \frac{A}{v_s + v_s^k}x_s^i - \gamma^i\Psi^i P_s, s = 1, \cdots, S \tag{7}$$

Ψ^i is Lagrange multiplier for constraint (3). If the constraint is bind, Ψ^i is positive. The second order condition of optimality is met from the setting. By solving these for x_s^i and summing them, from equilibrium condition $\frac{1}{I}\sum_i x_s^i = 1$, we get

$$1 = \frac{v_s + v_s^A}{A}\delta_s + \frac{v_s^A}{A}\theta_s - \frac{v_s + v_s^A}{A}(1 + r_f + \Psi)P_s \tag{8}$$

v_s^A is the 'consensus precision', $v_s^A = \phi_s v_s^C + (1 - \phi_s)v_s^R$. $\Psi = \frac{1}{I}\sum_i \frac{v_s + v_s^k}{v_s + v_s^A}\gamma^i\Psi^i$, which is the effect of leverage constraint averaged by the weight of investor i's relative precision to the

[7] Without overconfident investors, one by one analysis can be made even if the correlation is not zero.

consensus precision.

Thus, we can express the equilibrium price,

$$\begin{aligned} P_s &= \frac{1}{1+r_f+\Psi}\left(\delta_s + \frac{v_s^A}{v_s+v_s^A}\theta_s - \frac{A}{v_s+v_s^A}\right) \\ &= \frac{1}{1+r_f+\Psi}\left(\delta_s^R + (\lambda_s-1)(\delta_s^R-\delta_s) - \mathrm{Avar}^R(\tilde{\delta}_s)\eta_s\right) \end{aligned} \tag{9}$$

The second line is expressed from the point of view of the investor who doesn't become overconfident, where $\lambda_s = \frac{v_s^A(v_s+v_s^R)}{v_s^R(v_s+v_s^A)}$ is the coefficient of sensitivity to the signal, and $\eta_s = \frac{v_s+v_s^R}{v_s+v_s^A} < 1$ is the correction factor to the incorrect small estimate of the risk. $\lambda_s > 1$ means overreaction to the signal that is amplifying the change of the expected dividend more than the one estimated rationally, $\delta_s^R - \delta_s$.

The expected return in this equilibrium, from the point of view of the investor who doesn't become overconfident, is $E^R[R_s] = \delta_s^R/P_s - 1$, which is substituted to (9), then we get

$$\begin{aligned} E^R[R_s] &= r_f + \Psi - \frac{(\lambda_s-1)(\delta_s^R-\delta_s)}{P_s} + \mathrm{Avar}^R(R_s)P_s\eta_s \\ &= r_f + \Psi - \frac{(\lambda_s-1)(\delta_s^R-\delta_s)}{P_s} + \mathrm{Acov}^R(R_s,R_m)\sum_s P_s\eta_s \end{aligned} \tag{10}$$

$\mathrm{cov}^R(\cdot,\cdot)$ means covariance from the point of view of the investor who doesn't become overconfident. To get the second line it is used that there is no correlation among securities.

The portfolio constituted by η_s is called as a modified market portfolio, expressed by m, and its return is R_m. The market portfolio is M. Summing (10) with weight $w_s = \frac{P_s\eta_s}{\sum_s P_s\eta_s}$, we get

$$E^R[R_m] = r_f + \Psi - \textstyle\sum_s w_s \frac{(\lambda_s-1)(\delta_s^R-\delta_s)}{P_s} + \mathrm{Avar}^R(R_m)\sum_s P_s\eta_s. \tag{11}$$

We define μ as

$$E^R[R_m] - (r_f+\Psi) + \sum_s w_s \frac{(\lambda_s-1)(\delta_s^R-\delta_s)}{P_s} = \mu. \tag{12}$$

From (10) and (11),

$$E^R[R_s] - r_f = \Psi - \frac{(\lambda_s-1)(\delta_s^R-\delta_s)}{P_s} + \mu\beta_s \tag{13}$$

β_s is the sensitivity of the s security's return R_s to the modified market portfolio's return R_m.

It is similar to standard CAPM that the excess return to risk free rate is proportional to the sensitivity to a specified portfolio. But there are some differences. The first is the rewards to be investing in capital markets Ψ, arising uniformly from the leverage constraint. The second is the adjustment to the signal in the opposite direction due to overreaction. Third, the portfolio which is a measure of the risk premium is, rather than the market portfolio, the modified market portfolio by incorrect assessment of risk. Moreover, the separation theorem does not hold because there is no common portfolio for all investors due to leverage constraint and overconfidence.

At the end of this section, we'll show the risk allocation of the modified market portfolio. Let the wealth allocation to the security s be x_s^m, and the dividend be $\tilde{\delta}_m$, then the risk allocation[8] is,

$$\frac{RC_m}{\sqrt{var^R(R_m)}} = x_s^m \beta_s = \frac{1}{v_s + v_s^A} \cdot \frac{\eta_s}{var^R(\tilde{\delta}_m)}. \tag{14}$$

4. Characteristics of the equilibrium

Under leverage constraints and overconfidence, even if the other assumptions are normal, equilibrium return becomes different from the CAPM as described in the previous section. While limiting the circumstances here, we analyze the implications.

4.1. No overconfident investor

First, suppose that there was no investor overconfidence, the impact of leverage constraints are described. Under this circumstance, $v_s^C = v_s^R = v_s^A, \lambda_s = \eta_s = 1$, and the modified market portfolio is identical to the market portfolio. Then,

$$E^R[R_s] = r_f + \Psi + \mu\beta_s. \tag{15}$$

The return of the securities are uniformly increased with Ψ by the leverage constraint, depressing the prices at the common ratio, $\frac{1+r_f}{1+r_f+\Psi}$. We have the following.

Proposition 1
Considering the excess return is proportional to β_s like CAPM, there is $\alpha_s = \Psi(1-\beta_s)$.

The smaller β_s, the greater excess premium to market risk exists.This is the same result as Frazzini and Pedersen [12].

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8 $x_s^m = \frac{P_s \eta_s}{\sum_s P_s \eta_s}$ and $\beta_s = \dfrac{\frac{var^R(\tilde{\delta}_s)}{P_s^2} x_s^m}{\sum_s \frac{var^R(\tilde{\delta}_s)}{P_s^2}(x_s^m)^2}$ due to zero correlation. Substituting these and arranging, then the expression in the text is obtained.

The tangent portfolio, which is the optimal portfolio of investors *FP* who are not constrained for leverage,

$$\mathrm{x}_{\mathrm{s}}^{\mathrm{FP}} = \frac{1}{1 + \mathrm{r}_{\mathrm{f}} + \Psi}\left(\Psi\frac{(\mathrm{v}_{\mathrm{s}} + \mathrm{v}_{\mathrm{s}}^{\mathrm{R}})\delta_{\mathrm{s}}^{\mathrm{R}}}{\mathrm{A}} + 1 + \mathrm{r}_{\mathrm{f}}\right) \tag{16}$$

We have the following about the market portfolio from (16).

Proposition 2
The market portfolio is on the mean-variance frontier, but it is not the tangent portfolio.

Although the tangent portfolio has a greater ratio of low risk asset than the market, we cannot identify it, because it depends on the availability of each investor's leverage. This result does not approve that RP portfolio is closer to the tangent portfolio, as Asness et al. [1] wrote. We claim this as the next proposition and prove it showing the counter numerical example in a later section

Proposition 3
The risk allocations of the tangent portfolio can have greater dispersion than the market portfolio.

In such cases, the RP portfolio is not justified. The tangent portfolio which has the highest Sharpe ratio in this market is the portfolio (16) that the investor with no leverage constraint holds. According to (16), the deviation of the tangent portfolio from the market portfolio depends on $(\mathrm{v}_{\mathrm{s}} + \mathrm{v}_{\mathrm{s}}^{\mathrm{R}})\delta_{\mathrm{s}}^{\mathrm{R}}$. $\mathrm{v}_{\mathrm{s}} + \mathrm{v}_{\mathrm{s}}^{\mathrm{R}}$ which is the inverse of the asset s variance is large when the risk is small. However, the deviation also tends to be large if the expected value of payoff δ_{s} is greater. As a result, the allocation of risk is not necessarily modified towards the equalization.

4.2. No leverage constraint

Next, we analyze the market where there are overconfident investors, but no leverage constraints. To simplify the discussion, in the following, considering the properties under the expected value of θ_{s}, then we get the next proposition.

Proposition 4
The modified market portfolio m is the tangent portfolio under the expected value of the signal θ_s from the point of view of the investors who do not become overconfident.

With respect to the signal, the investors who do not become overconfident have asset s additionally from m in the amount of their own suppressed reaction $(2 - \lambda_{\mathrm{s}} < 1)$, because of the price overreaction by investors' overconfidence. The price depends on the realized value of the signal θ_s. From proposition 4, the following is true.

Corollary 1

Under the expected value of the signal θ_s, the modified market portfolio m has the greater Sharpe ratio than the one of the market portfolio M.

In addition, we get the following.

Proposition 5

The differences among risk allocations of the modified market portfolio m is less than those of the market portfolio M if the conditions except the prior risk V_s are equal.

This is because the low risk assets are less affected by investor overconfidence due to the high reliability of the payoff. Considering the safe asset as an extreme example, it does not become overvalued by overconfidence. The less the asset's payoff risk V_s is, the more the asset is held in the modified market portfolio. The risk allocation is modified in the direction of smoothing.

This result is important. Allocationg the risk more evenly than the market portfolio, a more efficient portfolio is reached. This correction is approaching the RP portfolio.

4.3. Compound effects

As (10) shows, the effect of the leverage constraint and the effect of overconfidence is separated. Suppose the investor who does not become overconfident and whose leverage is constrained to the degree of average investors, we get the following in the same manner as in Proposition 2.

Proposition 6

Under the expected value of the signal θ_s, the modified market portfolio m is on the mean-variance frontier, from the point of view of the investors who do not become overconfident.

In this way, we know the position of portfolio m, but we do not know about market portfolio M. The modified market portfolio m has smaller payoff risk because $\eta_s < 1$, that means it includes lower quantity of assets. But it may have bigger risk of rate of return than M, in the case of being biased to assets whose risk of rate of return is large.

4.4. Numerical example

Here, the preference for low-risk assets arising from the leverage constraints and overconfidence are shown by numerical examples, comparing the RP portfolio and market portfolios, to understand each of the implications well. First, the counter example against

Asness et al. [1] such as following is illustrated in the equilibrium under leverage constraint.

1. The tangent portfolio allocates risk to low risk asset more than the market portfolio, but the RP portfolio does not.
2. In the tangent portfolio, the differences among the risk allocations of assets are greater than those in the market portfolio.
3. The Sharpe ratio of the RP portfolio is less than the one of the market portfolio.

The first and the second say that the market portfolio is closer to the tangent portfolio than the PR portfolio. The third claims that the RP portfolio is inferior to the market portfolio in the expected utility.

Second, it is described that the modifications in the direction to smooth risk allocation from the market portfolio is desirable when overconfidence about a signal's precision exists. The RP portfolio is in the right direction. But in the optimal portfolio, the risk allocation is not necessarily even.

4.4.1. No overconfident investor

Suppose there are two assets (s=1,2), two investors (i=1,2) who have the same wealth W, only investor 1 is constrained, and γ^2 is sufficiently small. Then, $\Psi = \frac{\Psi^1}{2}$, (P_s, Ψ) satisfies the following in the equilibrium.

$$P_s = \frac{\delta_s^R - A/(v_s + v_s^R)}{1 + r_f + \Psi}, s = 1,2 \tag{17}$$

$$W = \gamma^1 \sum_{s=1}^{2} \frac{v_s + v_s^R}{A} (\delta_s^R - (1 + r_f + 2\Psi)P_s)P_s \tag{18}$$

$$W < \sum_{s=1}^{2} \frac{v_s + v_s^R}{A} (\delta_s^R - (1 + r_f)P_s)P_s \tag{19}$$

In addition, we set $A = 2.0$, $W = 1.0$, $\gamma^1 = 1$, $r_f = 0.05$, and the payoff as,

	Dividend	
	Expected value (δ_s)	Variance (V_s)
Security 1	1	0.1^2
Security 2	5	0.15^2

The correlation between two securities is 0.

Although security 2 has greater risk of the payoff, which also has greater expected value of dividend, it is a low risk asset in the sense of rate of return.

We do not treat the signal here, assuming there is no overconfident investor. The above

Table 1: The equilibrium without overconfident investor

Characteristics of securities

	Price (P_s)	Expected Return	Risk	Sharpe ratio	Market β	α
Security 1	0.9257	0.0803	0.1080	0.2805	1.8634	-0.0075
Security 2	4.6803	0.0683	0.0320	0.5714	0.8292	0.0015

Portfolios

	Holdings	Money allocation	Risk allocation	Expected return	Risk	Sharpe ratio
Constrained Investor (i=1)						
Security 1	0.5975	0.5531	0.9457	0.0749	0.0614	0.4060
Security 2	0.0955	0.4469	0.0543			
Unconstrained Investor (i=2)						
Security 1	1.4025	0.1271	0.1942	0.2526	0.3182	0.6365
Security 2	1.9045	0.8729	0.8058			
RP portfolio						
Security 1	0.2472	0.2288	0.5000	0.0711	0.0350	0.6023
Security 2	0.1648	0.7712	0.5000			
Market portfolio						
Security 1	0.1784	0.1651	0.3077	0.0703	0.0322	0.6310
Security 2	0.1784	0.8349	0.6923			

Ψ=0.008697, μ=0.0203. Money allocation is the ratio of wealth invested among the risk assets. Risk allocation is the ratio of risk contribution of the security. The three columns on the right show the characteristics of the portfolio. The leverage of an unconstrained investor is 10.42, while the others are 1.0.

setting of the payoff is seen as the ex post value, $\delta_s^R = \delta_s$, $(v_s + v_s^R)^{-1} = V_s$. Then we get the price, the portfolio of each investor, etc., as in **Table 1**.

Security 2 is relatively low risk, having small market β. The α of security 1 is negative and the α of security 2 is positive as Frazzini and Pedersen [12] claimed. The portfolio held by the unconstrained investor is the tangent portfolio which has the largest Sharpe ratio, including a lot of security 2. In spite of the risk allocation of security 2 being large in the market portfolio, the tangent portfolio is further biased to security 2.

The investor who is not allowed to leverage is taking higher risk than the market portfolio, expecting higher return but lower Sharpe ratio. However, under leverage constraint, it is optimal for the investor.

From these, we can confirm the previous three claims. The money allocation to security 2 is smaller in the RP portfolio than in the market portfolio, which means the market portfolio is closer to the tangent portfolio than the RP portfolio. The same holds true for the risk allocation. As a result, the Sharpe ratio of the market portfolio is larger than the one of the RP portfolio.

In this market, the price has decreased uniformly and the return has increased across the board. Therefore, because the asset whose expected dividend is large and risk is small in the sense of the rate of return is advantageous, its allocation in the tangent portfolio is increasing, even if its risk allocation is large in the market portfolio. Such deviation from the market portfolio is a reverse direction toward the RP portfolio.

The risk allocation of the market portfolio is the product of the money allocation and risk as in (14). Advocate of the RP portfolio had emphasized that the risk allocation is not diversified when the diversified investment is performed based on the money allocation without considering the magnitude of risk. On the contrary, it can be said that Asness et al. [1] discussed the allocation of risk regarding only risk and ignoring the money allocation.

4.4.2. No leverage constraint

The market portfolio modified by investor overconfidence is shown in this section, when leverage is not constrained. Basic settings are the same as the previous section. The standard deviation of the payoff in the previous section is the one before obtaining signal. We set signal and configuration of investor as

payoff and signal		overconfident investor	
prior risk (V_s)	error of signal (V_s^R)	mistaken belief (V_s^C)	ratio (ϕ_s)
0.1^2	0.5^2	0.1^2	0.3
0.15^2	0.5^2	0.1^2	0.3

Suppose the realized signal is zero equal to the expected value. Then the asset's prices and portfolios are derived as in **Table 2**.

The results such as the market portfolio are similar to the previous section. However, there is a positive α in security 1, because its payoff risk is small while security 1 has high market β. The smaller the asset's payoff risk is, the less overvalued the price is in the market. Since it is recognized on average that the risks are small by overconfidence, the prices rise and the level of Sharpe ratio is down.

The modified market portfolio holds security 1 more and allocates risk to it rather than the market portfolio. Therefore, it can be said that the modified market smoothes the risk allocation. Deviating from the market portfolio in the same direction as the modified market portfolio, the efficiency of RP portfolio is higher than the market portfolio.

But, RP portfolio is different from the modified market portfolio. Moreover, it is not always better than the market portfolio. It depends on the degree of overconfidence. In this example, changing the ratio of overconfident investor ϕ_s to below 0.24, the Sharpe ratio of the market portfolio become bigger than the one of the RP portfolio. Also, we can make a case that the risk allocations of the tangent portfolio are even, setting $\phi_s = 0.7$ in this example. When there is an irrational overconfidence of investors against the risk, it can be

Table 2: The equilibrium without leverage constraint

Characteristics of securities

	Price (P_s)	Expected Return	Risk	Sharpe ratio	Market β	α
Security 1	0.9380	0.0661	0.1045	0.1536	1.9226	0.0022
Security 2	4.7372	0.0555	0.0303	0.1802	0.8173	-0.0004

Portfolios

	Holdings	Money allocation	Risk allocation	Expected return	Risk	Sharpe ratio
RP portfolio						
Security 1	0.2397	0.2249	0.5000	0.0578	0.0332	0.2360
Security 2	0.1636	0.7751	0.5000			
Market portfolio						
Security 1	0.1762	0.1653	0.3178	0.0572	0.0307	0.2354
Security 2	0.1762	0.8347	0.6822			
Modified Market portfolio						
Security 1	0.2113	0.1982	0.4207	0.0576	0.0319	0.2368
Security 2	0.1692	0.8018	0.5793			

μ=0.0072. Money allocation is ratio of wealth invested among the risk assets. Risk allocation is ratio of risk contribution of the security. Three columns on the right show the characteristics of the portfolio.

claimed that the market portfolio should be modified in the direction of leveling the risk allocation. However, we want to note that it cannot be said that the risk allocation should be completely leveled.

The effect of the leverage constraint was that there is α for assets with a low risk of return, since the return is increased uniformly. However, the risk and the risk allocation are not one-to-one correspondence. On the other hand, with the effects of overconfidence, we will be able to link directly the magnitude of the risk of the payoff to the undervalued or overvalued . From the fact that the risk of the payoff is the allocation of risk in the market, it can be obtained that the allocation of risk should be leveled compared to the market portfolio.

5. Empirical Analysis of the bond market

Here, we examine the market where the characteristics of the RP portfolio and those of the desired portfolio under leverage constraint do not match as shown in the previous section. We are concerned with the market where there are a large amount of low-risk assets.

If the RP portfolio was inferior to the market portfolio in such markets, it could be said that favorable performance of the RP portfolio seen in other markets is due to the weights of the low β assets being greater by chance, and that there is no point smoothing risk allocations.

It would be desirable to have a portfolio that incorporates a lot of low β assets directly. On the other hand, if the performance of the RP portfolio was excellent, the results could not be explained by the leverage constraint and a description provided by different factors would be required.

To simplify the interpretation of the results, the markets where there are not many assets and the supply of low-risk assets accounts for the majority are suitable. As such a market, there is a bond market.

In the bond market the amount of issues of government or public institutions which is low-risk compared to the corporate bond is large. Then, the bond market will be looked at here.

5.1. Data

Because of the availability of data, we use the indices that are organized as World Broad Investment-Grade Bond Index, from among the many indices that are published by Citigroup[9]. Sub-indices by a multi-stage grouping of issuers as well as the entire bond market have also been published. We can regard a group that combines the sub-indices as a single market. **Table 3** shows the market value and classification of the index of the bond that was issued with respect to the four major currencies.

According to **Table 3**, the bonds that dominate each market in market value are 10 in total: Sovereign/Sovereign-Guaranteed (Sovereign/Sovereign-Guaranteed/Supranational for US dollar) in each currency market, Domestic Sovereign in each Sovereign/Sovereign-Guaranteed (Sovereign/Sovereign-Guaranteed/Supranational for US dollar) market, Supranational in Government-Sponsored/Regional Government market of Japanese yen and UK sterling. In the following, we analyze these 10 indices considering securities that are much supplied.

5.2. Empirical results

Next, we check the risk characteristics of these indices. **Table 4** shows the minimum and the number of samples which are less than 0.98 with respect to the sensitivity (β) to the index of the upper group regarded as the market. Sovereign/Sovereign-Guaranteed/Supranational in the US dollar market and Supranational in the Government-Sponsored/Regional Government market of UK sterling bonds have small sensitivities. The risk allocation of these in the market are beyond the amount divided equally well due to their abundant supply[10].

In the following, we will focus on these two markets. **Table 5** shows the performance of RP portfolio and market portfolio of these market. In the US market, not shown in the table,

[9] For dollar-denominated bonds, we use US Broad Investment-Grade Bond Index because its long data is available.

[10] Trying to alter the conditions of β or data period used to estimate the risk from 12 months to 48 months, the result does not change significantly.

Table 3: The market value of World Broad Investment Grade (WorldBIG) Bond Index for the four largest currency sectors

US dollar, Available from Jan.1980, millions of US dollars	5,210,642	
Sovereign/Sovereign-Guaranteed/Supranational	2,265,384	
Domestic Sovereign		1,787,460
Agency		431,709
Supranational		48,443
Collateralized	1,728,952	
Credit	1,216,305	
Euro, Available from Apr.2007, millions of euros	6,267,524	
Sovereign/Sovereign-Guaranteed	3,999,454	
Domestic Sovereign		3,667,296
Sovereign-Guaranteed		266,859
Foreign Sovereign		65,299
Government-Sponsored/Regional Government	469,591	
Collateralized	805,259	
Corporate	993,220	
Japanese yen, Available from Mar.2007, millions of yen	424,156,519	
Sovereign/Sovereign-Guaranteed	422,265,976	
Domestic Sovereign		420,174,286
Foreign Sovereign		496,523
Sovereign-Guaranteed		1,595,167
Government-Sponsored/Regional Government	984,832	
Agency		146,884
Supranational		558,824
Other Sovereign-Sponsored		94,010
Regional Government		87,063
Regional Government-Guaranteed		121,433
Collateralized	210,387	
Corporate	695,323	
UK sterling, Available from Apr.2007, millions of pounds	736,298	
Sovereign/Sovereign-Guaranteed	550,448	
Domestic Sovereign		520,101
Foreign Sovereign		3,573
Sovereign-Guaranteed		26,774
Government-Sponsored/Regional Government	41,215	
Agency		1,580
Supranational		36,851
Other Sovereign-Sponsored		2,370
Regional Government/-Guaranteed		498
Collateralized	19,668	
Corporate	124,967	

The table is created by the author from Citigroup Global Fixed Income Index Catalog-2012 Edition, January 17, 2012. The Category written down to the right indicates that it is the sub-classification of the upper one. Market value is the average of the time period for which data exists, that there are cases where the total is not aligned. Data is up to July 2012 in all series. For more information, refer to the above-mentioned document.

Table 4: The market risk of bonds issued in large numbers

	Minimum of β	Number of β < 0.98 samples
US dollar (Number of all samples)		(354)
Sovereign/Sovereign-Guaranteed/Supranational	0.769	65
Domestic Sovereign	0.991	0
Euro（number of all samples）		(27)
Sovereign/Sovereign-Guaranteed	1.114	0
Domestic Sovereign	1.036	0
Japanese yen（number of all samples）		(28)
Sovereign/Sovereign-Guaranteed	1.000	0
Domestic Sovereign	1.001	0
Government-Sponsored/Regional Government		0
Supranational	1.073	0
UK sterling（number of all samples）		(27)
Sovereign/Sovereign-Guaranteed	1.023	0
Domestic Sovereign	1.011	0
Government-Sponsored/Regional Government		0
Supranational	0.897	27

The βs are estimated from the data of the past 36 month returns at that time.

Table 5: The performance conditioned by β < 0.98

US dollar

			average of β	average of risk allocation	
Sovereign/Sovereign-Guaranteed/Supranational			0.871	50.43%	
	average	excess return standard deviation	Sharpe ratio	Regressed to the market coefficient	intercept (t value)
RP portfolio	4.13%	6.11%	0.676	1.023	0.00016
market portfolio	3.84%	5.96%	0.645		(1.045)

Government-Sponsored/ Regional Government market of UK sterling bond

			average of β	average of risk allocation	
Supranational			0.938	83.09%	
	average	excess return standard deviation	Sharpe ratio	Regressed to the market coefficient	intercept (t value)
RP portfolio	8.93%	6.51%	1.372	1.191	0.00024
market portfolio	7.26%	5.38%	1.350		(0.331)

The estimation was done from the data of the past 36 month returns at that time.

the RP portfolio outperformed the market portfolio, not putting a condition to the β of Sovereign/Sovereign-Guaranteed/Supranational. In the **Table 5**, the result when a condition $\beta < 0.98$ was put is shown. According to the pricing model with leverage constraint, $\alpha > 0$ is a null hypothesis. The p-value calculated from the regression results is 0.850. Thus, leverage constraint will not be considered appropriate as a description of the performance of the RP portfolio.

Supranational in the Government-Sponsored/Regional Government market of UK sterling bond has small β constantly. Thus, we can use all sample and find that null hypothesis is not rejected. We don't get the result such that the RP portfolio which reduced holding of low β asset is inferior to the market portfolio significantly.

In the RP portfolio, the weight of the assets which account for a large portion of the market, has been reduced because the risk allocation is greater. In these bond markets, β of such asset is below one, the β of RP portfolio is greater than one and has more risk than the market portfolio. This portfolio selection is opposite to what is desirable under leverage constraints.

Still, the result here is that RP portfolio shows a good performance. Because the degree of investor overconfidence cannot be observed from the data of this paper, we cannot claim that it was a factor empirically. However, the fact that the performance of the RP portfolio is good makes an explanation by factors other than leverage constraint plausible.

6. Conclusion

Recently a portfolio management method called risk parity is attracting attention for its high performances. One of the reasons is that even if there is no special information or skills, it shows a performance that exceeds the market portfolio using only the risk; another is that smoothing the impact on portfolio risk of each asset reminds us the risk diversification intuitively. But very few explanations have been provided for high performances.

One of the exceptions is Asness et al. [1]. Asness et al. [1] links the discounting of low-risk assets by the leverage aversion to risk parity portfolios. But risks and risk allocations should not be regarded as the same. For example, when there is abundant low-risk asset, RP portfolio is a portfolio that reduces the low-risk assets contrary to their suggestion.

On the other hand, it is also possible to show that investor overconfidence is a discounting factor for low-risk assets. In this case, since the size of risk allocations in the market are directly linked to either overvaluation or undervaluation, we could get a result supporting risk parity portfolios, which suggests that it is desirable to level out risk allocations by constantly comparing them with the market portfolio.

Here we seek the more appropriate explanation for risk parity portfolios between the leverage aversion and overconfidence. As a result, we choose the latter as the more plausible explanation. We summarize the relationship between the respective implications of the two theories and risk parity portfolios. Then we conduct research on bond markets, where we could detect some difference between the leverage aversion and overconfidence. Our empirical study shows that risk parity portfolios demand explanations other than the leverage aversion.

Considering the factors of good performance of RP portfolio from the point of view of risk allocation in this way, we realized that we cannot avoid discussion of how to classify securities. The empirical result here might as well change if the taxonomy of the bond index is altered. It seems that considering the risk factors behind the risk of payoff is an effective approach because they don't depend on the classification of securities.

Appendix

A.1. The optimal portfolio under uniform Sharpe ratios and correlation coefficients

Let the risk (standard deviation) of asset s be σ_s and vector lining them be Q. Diagonal matrix lining variance σ_s^2 is Σ. All correlation coefficients are the same as ρ. Then, the covariance matrix Ω is expressed as $\Omega = (1-\rho)\Sigma + \rho QQ'$. Since Sharpe ratio is ξ evenly, vector of expected excess returns is $\nu = \xi Q$. The optimal portfolio of risky assets is $x^* = \frac{1}{A}\Omega^{-1}\nu$, where A is the coefficient of risk aversion. According to the formula of the Sherman-Morrison,

$$\Omega^{-1} = \frac{1}{1-\rho}\Sigma^{-1} - \frac{\rho}{1-\rho+\rho Q'\ \Sigma^{-1}Q}\Sigma^{-1}QQ'\ \Sigma. \tag{20}$$

Expressing it for each element,

$$(\Omega^{-1})_{st} = \begin{cases} \dfrac{a-b}{\sigma_s^2} & \text{if } s = t \\ -\dfrac{b}{\sigma_s\sigma_t} & \text{if } s \neq t \end{cases} \tag{21}$$

$$\text{where } a = \frac{1}{1-\rho}, b = \frac{\rho}{(1-\rho)(1-\rho+\rho S)}.$$

The optimal portfolio is

$$x_s^* = \frac{\xi}{A}\left(\frac{a}{\sigma_s^2}\sigma_s - \sum_{t=1}^{S}\frac{b}{\sigma_s\sigma_t}\sigma_t\right) = \frac{\xi}{A\sigma_s}(a - bS) \tag{22}$$

Risk contributions of each asset are constant as

$$RC_s = \frac{\hat{x}_s^{*'}\ \Omega x^*}{\sigma_p} = \frac{1}{\sigma_p}x_s^*\frac{\xi\ \sigma_s}{A} = \frac{\xi^2}{\sigma_p A^2}(a - bS). \tag{23}$$

We can see that the optimal portfolio is RP portfolio.

A.2. Proof of proposition 1

The risk premium of the market portfolio M=m is Ψ+μ from (15). (15) can be rearranged as

$$E^R[R_s] - r_f = (\mu + \Psi)\beta_s + \Psi(1-\beta_s). \tag{24}$$

$\Psi(1-\beta_s)$ is α that is not proportional to the market β_s.

A.3. Proof of proposition 2

When the leverage is constrained, Ψ is positive and (16) does not match the market portfolio. For the second half, suppose investor L for whom the impact of leverage constraints matches the average among investors Ψ, then L's holding is $x_s^L = 1$, which is equal to the market portfolio configuration. Since the portfolio that might be held by investors must be located on the frontier, the market portfolio locates on the frontier.

A.4. Proof of proposition 4

The demand of investor R who is not overconfident is

$$\begin{aligned} x_s^R &= \frac{v_s + v_s^R}{A}\delta_s + \frac{v_s^R}{A}\theta_s - \frac{v_s + v_s^R}{A}(1 + r_f)P_s \\ &= \frac{v_s + v_s^R}{A}\delta_s + \frac{v_s^R}{A}\theta_s \\ &\quad - \frac{v_s + v_s^R}{A}\left(\delta_s^R + (\lambda_s - 1)\frac{v_s^R}{v_s + v_s^R}\theta_s - \frac{A}{v_s + v_s^R}\eta_s\right) \\ &= (2 - \lambda_s)\frac{v_s^R}{A}\theta_s + \eta_s \end{aligned} \tag{25}$$

from (7). Taking expectation with respect to θ_s, the first term disappears and only the term which indicates the modified market portfolio is left. Thus m is the tangency portfolio for investor R.

A.5. Proof of proposition 5

The risk allocation of the market portfolio M is proportional to $\frac{1}{v_s+v_s^A}$ from (14), substituting $\eta_s = 1$ and replacing $\mathrm{var}^R(\tilde{\delta}_m)$ and $\mathrm{var}^R(\tilde{\delta}_M)$. The risk allocation of the modified market portfolio m which is proportional to $\frac{\eta_s}{v_s+v_s^A}$ increases relatively as V_s is small, because $\frac{\partial\eta_s}{\partial V_s} = \frac{v_s^A - v_s^R}{(v_s+v_s^A)^2}\frac{\partial v_s}{\partial V_s} = -\frac{v_s^A - v_s^R}{(v_s+v_s^A)^2 V_s^2} < 0.$

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