GS1603 非線形写像の逆写像の積分表現
Integral representations of inverse functions of nonlinear mappings

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1. Introduction

The 2nd author of this paper considered that for any mapping ϕ from an arbitrary abstract set into an arbitrary set, he tried to consider the representation of the inversion ϕ^{-1} in terms of the direct mapping ϕ and he obtained some simple concrete formulas from some general ideas in ([1]). In this paper, from its general ideas, we shall give practical representation formulas of some general functions. Here, we shall give furthermore some general methods and ideas for the inversion formulas for some general non-linear mappings. We shall first state the principles for our methods for the representations of inverses of non-linear mappings based on ([1]): We shall consider some representation of the inversion ϕ^{-1} in terms of some integral form - at this moment, we shall need a natural assumption for the mapping ϕ . Then, we shall transform the integral representation by the mapping ϕ to the original space that is the defined domain of the mapping ϕ. Then, we will be able to obtain the representation of the inverse ϕ^{-1} in terms of the direct mapping ϕ. In [1], we considered the representation of the inverse ϕ^{-1} in some reproducing kernel Hilbert spaces, and in [4], we considered the representations of the inverse ϕ^{-1} for a very concrete situation and we gave a very fundamental representation of the inverse for some general functions on 1 dimensional spaces.

Indeed, note that

$$K(y_1, y_2) = \frac{1}{2}e^{-|y_1-y_2|} \qquad y_1, y_2 \in [A, B] \tag{1}$$

is the reproducing kernel for the Sobolev Hilbert space H_K whose members are real-valued and absolutely continuous functions on [A,B] and whose inner product is given by

$$(f_1, f_2)_{H_K} = \int_A^B (f_1'(y)f_2'(y) + f_1(y)f_2(y))dy + f_1(A)f_2(A) + f_1(B)f_2(B) \tag{2}$$

For a function $y = f(x)$ that is of C^1 class and a strictly increasing function and $f'(x)$ is not vanishing on $[a, b]$ $(f(a) = A,\ f(b) = B)$, of course, the inverse function $f^{-1}(y)$ is a single-valued function and it belongs to the space H_K and from the reproducing property, we obtain the representation, for any $y_0 \in [f(a), f(b)]$

$$\begin{aligned} f^{-1}(y_0) &= (f^{-1}(\cdot), K(\cdot, y_0))_{H_K} \\ &= \int_{f(a)}^{f(b)} ((f^{-1})'(y)K_y(y, y_0) + f^{-1}(y)K(y, y_0))dy \\ &\quad + aK(f(a), y_0) + bK(f(b), y_0). \end{aligned} \tag{3}$$

From this representation, we obtained in ([4]) the very simple representation

$$f^{-1}(y_0) = \frac{a+b}{2} + \frac{1}{2}\int_a^b \mathrm{sign}(y_0 - f(x))dx. \tag{4}$$

Furthermore, by using the several reproducing kernel Hilbert spaces from [2] as in (3), we calculated similarly with the related assumptions, however, surprisingly enough, we obtain the same formula (4). For the formula (4), we note directly that we do not need any smoothness assumptions for the function $f(x)$, indeed, we need only the strictly increasing assumption. The assumption of integrability does not, even, need for the formula (4). Now, we would like to obtain some multi-dimensional versions. At this moment, it seems that we can not find some simple representations as in (3) by some concrete known reproducing kernels for some general domains, and indeed, we know the reproducing kernels only for special domains and for special reproducing kernel Hilbert spaces. In order to consider some general integral representations for some general functions, we shall recall the fundamental facts: We can represent a function f in terms of the delta function δ in the form

$$f(q) = \int_D f(p)\delta(p-q)dq \tag{5}$$

in some domain, symbolically. Meanwhile, a fundamental solution $G(p-q)$ for some linear differential operator L is given by the equation, symbolically

$$LG(p-q) = \delta(p-q). \tag{6}$$

So, from (5) we obtain the representation

$$f(q) = \int_D f(p)LG(p-q)dp. \tag{7}$$

Then, we can obtain the representation symbolically, by using the Green- Stokes formula, for some adjoint operator L^* for L,

$$f(q) = \int_D L^*f(p)G(p-q)dp + (\text{boundary integrals}). \tag{8}$$

2. 2-dimensional formula

We are interested in some very concrete results that may be realized by computers. So, we considered very concrete cases in the 2 dimensional spaces. It seems that the results will depend on dimensions, domains and functions spaces dealing with. In [5], we considered the following typical problem: Let $D \subset \mathbf{R}^2$ be a bounded domain with a finite number of piecewise C^1 class boundary components. Let f be a one-to-one C^1 class mapping from $\bar{D}$ into $\mathbf{R}^2$ and we assume that its Jacobian $J(x)$ is positive on D. We shall represent f as follows:

$$y_1 = f_1(x) = f_1(x_1, x_2) \quad y_2 = f_2(x) = f_2(x_1, x_2) \tag{9}$$

and the inverse mapping f^{-1} of f as follows:

$$\begin{aligned} x_1 &= (f^{-1})_1(y) = (f^{-1})_1(y_1, y_2) \\ x_2 &= (f^{-1})_2(y) = (f^{-1})_2(y_1, y_2). \end{aligned} \tag{10}$$

Then, we represented $((f^{-1})_1(y^*), (f^{-1})_2(y^*))^t$ in terms of the direct mapping (9).

Of course, we are interested in some numerical and practical solutions of the non-linear simultaneous equations (9) and we obtained

Theorem 1 *For the mappings (9) with (10), we obtain the representation, for any $y^* = (y_1^*, y_2^*) \in f(D)$.*

$$\begin{pmatrix}(f_1^{-1}(y^*))\\(f_2^{-1}(y^*))\end{pmatrix} = \frac{1}{2\pi}\oint_{\partial D}\begin{pmatrix}x_1\\x_2\end{pmatrix} dArctan\frac{f_2(x)-y_2^*}{f_1(x)-y_1^*}$$
$$-\frac{1}{2\pi}\int\int_D \frac{1}{|f(x)-y^*|^2} adjJ(x)\begin{pmatrix}f_1(x)-y_1^*\\f_2(x)-y_2^*\end{pmatrix}dx_1dx_2. \tag{11}$$

3. 3-dimensional formula derived from the Poisson integral formula

Let D be a bounded domain in $\mathbf{R}^3$ with a finite number of C^1 boundary components ∂D. Let f be a one to one C^2 class mapping of D onto $f(D)$ in $\mathbf{R}^3$ with sense preserving and we assume that its Jacobian is positive on D. We set

$$y = f(x) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))^t$$

and its inversion f^{-1} as follows:

$$x = ((f^{-1})_1(y_1, y_2, y_3), (f^{-1})_2(y_1, y_2, y_3), (f^{-1})_3(y_1, y_2, y_3))^t$$

Let $\Delta = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}$ and $\frac{\partial y}{\partial x} = J$ be the Jacobian of $y = (y_1, \cdots, y_n)$ with respect to $x = (x_1, \cdots, x_n)$. For a matrix A, let $(A)_i$ be the i low vector of A and $(A)_{ij}$ the i, j element of A.

We set the vector fields

$$S_i(x) = \sum_{j=1}^{3}\frac{\mathrm{adj}(J^t)J_{ij}}{\det(J)}\frac{\partial}{\partial x_j} \tag{12}$$

$$T_i(x) = \frac{x_i}{|y_0 - f(x)|^2}(\mathrm{adj}(J))_i \cdot (y_0 - f(x))\sum_{j=1}^{3}\frac{\partial}{\partial x_j}. \tag{13}$$

Then, we obtain the theorem:

Theorem 2 *For any point $y_0 \in f(D)$, we obtain the representation,*

$$(f^{-1})_i(y_0) = -\frac{1}{4\pi}\int\int\int_D \frac{1}{|y_0 - f(x)|} divS_i(x)dx_1dx_2dx_3$$
$$+\frac{1}{4\pi}\int\int_{\partial D}\frac{1}{|y_0 - f(x)|}(S_i - T_i)(x)\cdot d\boldsymbol{A}_x \tag{14}$$

4. n-dimensional formulas

We shall give the very beautiful representation

Theorem 3 *Let D be a bounded domain in $\mathbf{R}^n$ with a finite number ∂D of C^1 class boundary components. Let f be a C^1 class real-valued function on $\bar{D}$. For any $\hat{x} \in D$ and for any $n \in \mathbf{N}$ we have the representation*

$$f(\hat{x}) = -c_n(df(x), dG_n(x-\hat{x})) + c_n\int_{\partial D} f(x)\star dG_n(x-\hat{x}) \tag{15}$$

Here, for $n \le 2$, $c_n = 1$ and for $n \ge 3$, $c_n = n-2$. $\star$ is the Hodge star operator, G_n the fundamental solution of the Laplacian Δ_n, and $(,)$ the inner product of the vector space $A^k(D)$ comprising of the k order differential forms over D with finite L^2 norms that is

$$(\omega, \eta) = \int_D \omega\wedge\star\eta = \int_D \eta\wedge\star\omega \quad (\omega, \eta \in A^k(D)).$$

Theorem 4 *In the situation of Theorem 3 and we assume furthermore that f is a sense preserving C^1 class function on $\bar{D}$ in $\mathbf{R}^n$ with a single-valued inverse. Then, for $\hat{y} \in f(D)$, we obtain the representation*

$$(f^{-1})_i(y_0) = -\int_D dx_i \wedge f^*\star dG_n(y - y_0)$$
$$+\int_{\partial D} x_i f^*\star dG_n(y - y_0). \tag{16}$$

Here, f_i^{-1} denotes the i component of f^{-1} and f^* means the pull back of the mapping f.

In particular, for $n = 1$, we obtain (4), directly.

For $n = 2$, we obtain (11) and this formula may be represented as follows, from our general formula:
For any $\hat{y} \in f(D)$, we have

$$f_1^{-1}(\hat{y}) = \frac{1}{2\pi}\Big(\int_{\partial D} x_i d\theta_i - \int_D dx_i \wedge d\theta_i\Big)$$

Here, $\theta_1 = \mathrm{Arctan}\frac{f_1(x)-\hat{y}_1}{f_2(x)-\hat{y}_2}$, $\theta_2 = -\mathrm{Arctan}\frac{f_2(x)-\hat{y}_2}{f_1(x)-\hat{y}_1}$.

Some examples and some results of numerical experiments using these our theory will be showed in the talk on September 26.

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