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ON THE REPLACEMENT OF MATERIAL-TIME DERIVATIVE TO COROTATIONAL RATE OF YIELD FUNCTION: MATHEMATICAL PROOF

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ABSTRACT

Constitutive equations have to be formulated in an indifferent form independent of the frame (i.e. coordinate systems) by which they are described or so as to be independent of the superposition of rigid body rotation. This fact is required by the *principle of material-frame indifference* (Oldroyd, 1950) and is attained conveniently by describing rate variables in terms of rate tensors with objectivity in constitutive equations in rate form. A plastic strain rate is derived by substituting the plastic flow rule into the consistency condition given as the material-time derivative of yield condition. In this note the mathematical process demonstrates the fact that rate variables involved in the material-time derivative of yield function can be directly replaced with their objective rate tensors. Here, the yield function involves arbitrary tensors, whilst the special case that the yield function involves only a single tensor or second-order tensors has been discussed in the past.

Key words: constitutive equation, (corotational rate), elastoplasticity, (material-time derivative), (objectivity), yield (**IGC**: D6/E2)

INTRODUCTION

Mechanical properties of materials are observed to be identical by different observers and thus constitutive equations have to be formulated in an identical form independent of the frame (i.e. coordinate system) by which they are described. In other words, they have to be independent of the superposition of rigid body rotation. This fact is advocated and called the *principle of material-frame indifference* by Oldroyd (1950).

Elastoplastic constitutive equations are formulated in rate forms since no unique relation exists between stress and deformation in the elastoplastic deformation process. They can be formulated conveniently by using corotational rate tensors which obey the coordinate transformation of the objective tensors even if a relative movement exists between the coordinate systems. The plastic strain rate is derived by substituting the plastic flow rule into the consistency condition given by the *material-time derivative* of yield condition. Therefore, rate variables in the consistency condition are required to be transformed to their corotational rates. The fact that a rate variable involved in the material-time derivative of yield function can be directly replaced with the corotational rate has been verified for isotropic hardening models by Hashiguchi et al. (2002), for isotropic-kinematic hardening models (Edelman and Drucker, 1951; Ishlinsky, 1954; Prager, 1955) by Papamichos and Vardoulakis (1995) and for isotropic-rotational hardening models (Sekiguchi and Ohta, 1977; Hashiguchi, 1977, 1994, 2001; Hashiguchi and Chen, 1998) by Asaoka et al. (2002). It was also verified for a general yield function of a single tensor by Dafalias (1985, 1998) without a detailed discussion.

A scalar quantity does not need to be observed to be identical by different observers or to be independent of the superposition of rigid body motion. For instance, the kinetic energy depends on observers and that superposition. On the other hand, the rate of yield function which is a scalar quantity describing a property of material has to be observed to be identical by different observers. Therefore, it would be expected that rate variables involved in the material-time derivative of yield function can be replaced directly with their objective rate tensors. In this note, a mathematical proof of this fact is given. Here, the yield function includes arbitrary tensors with objectivity, whilst up to now proofs are only for yield functions with scalars or second-order tensors.

The sign of a stress (rate) is taken to be positive for tension, and the Einstein's summation convention is used throughout this note.

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TRANSFORMATION OF RATE OF YIELD FUNCTION

Consider two different coordinate systems with bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ $(i=1, 2, 3)$, whilst let the orthogonal tensor between these coordinate systems be denoted as $\boldsymbol{Q}$, i.e. $Q_{ij}=\boldsymbol{e}'_i\cdot\boldsymbol{e}_j$, fulfilling $\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}}=\boldsymbol{I}$. The notation $(\ )^{\mathrm{T}}$ stands for the transpose and $\boldsymbol{I}$ is the second-order identity tensor. The bases $\{\boldsymbol{e}_i\}$, $\{\boldsymbol{e}'_i\}$ and thus the orthogonal tensor $\boldsymbol{Q}$ depend on time in general. Let the position vector of material point observed by the coordinate systems with bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ be denoted $\boldsymbol{x}(\boldsymbol{X}, t)$ and $\boldsymbol{x}'(\boldsymbol{X}, t)$, respectively, where $\boldsymbol{X}$ is position vector of material point in the reference configuration and t is a time. Then, it hold that

$$\boldsymbol{x}'(\boldsymbol{X}, t)=\boldsymbol{Q}(t)\boldsymbol{x}(\boldsymbol{X}, t)+\boldsymbol{c}(t), \tag{1}$$

where $\boldsymbol{c}(t)$ is the position vector of the origin of the coordinate system with bases $\{\boldsymbol{e}_i\}$ observed in the coordinate system with bases $\{\boldsymbol{e}'_i\}$.

If the components of a quantity $\boldsymbol{T}$ observed by these coordinate systems are related by the following equations, $\boldsymbol{T}$ is defined to be the m-th order tensor having *objectivity*.

$$T'_{p_1p_2\cdots p_m}=Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m}, \tag{2}$$

denoting the components of $\boldsymbol{T}$ in the coordinate systems with bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ $(i=1, 2, 3)$ as $T_{p_1p_2\cdots p_m}$ and $T'_{p_1p_2\cdots p_m}$, respectively. Equation (2) can be written for the second-order tensor as

$$\boldsymbol{T}'=\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^{\mathrm{T}} \tag{3}$$

by the symbolic notation.

The material-time derivative of Eq. (2) for the objective tensor is given as

$$\begin{aligned}\dot{\boldsymbol{T}}'=\dot{T}'_{p_1p_2\cdots p_m}&=\dot{Q}_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m}\\&+Q_{p_1q_1}\dot{Q}_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m}+\cdots\\&+Q_{p_1q_1}Q_{p_2q_2}\cdots \dot{Q}_{p_mq_m}T_{q_1q_2\cdots q_m}\\&+Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}\dot{T}_{q_1q_2\cdots q_m}\end{aligned} \tag{4}$$

which does not have the property of the objective tensor when the relative rate of rotation exists between bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$, i.e. $\dot{\boldsymbol{Q}}\neq\boldsymbol{0}$.

Now, consider the following *corotational rate tensor*

$$\begin{aligned}\mathring{T}_{p_1p_2\cdots p_m}&\equiv R^{\mathrm{m}}_{p_1q_1}R^{\mathrm{m}}_{p_2q_2}\cdots R^{\mathrm{m}}_{p_mq_m}(R^{\mathrm{m}}_{s_1q_1}R^{\mathrm{m}}_{s_2q_2}\cdots R^{\mathrm{m}}_{s_mq_m}T_{s_1s_2\cdots s_m})^{\boldsymbol{\cdot}}\\&=R^{\mathrm{m}}_{p_1q_1}R^{\mathrm{m}}_{p_2q_2}\cdots R^{\mathrm{m}}_{p_mq_m}(\dot{R}^{\mathrm{m}}_{s_1q_1}R^{\mathrm{m}}_{s_2q_2}\cdots R^{\mathrm{m}}_{s_mq_m}T_{s_1s_2\cdots s_m}\\&\quad+R^{\mathrm{m}}_{s_1q_1}\dot{R}^{\mathrm{m}}_{s_2q_2}\cdots R^{\mathrm{m}}_{s_mq_m}T_{s_1s_2\cdots s_m}+\cdots\\&\quad+R^{\mathrm{m}}_{s_1q_1}R^{\mathrm{m}}_{s_2q_2}\cdots \dot{R}^{\mathrm{m}}_{s_mq_m}T_{s_1s_2\cdots s_m}\\&\quad+R^{\mathrm{m}}_{s_1q_1}R^{\mathrm{m}}_{s_2q_2}\cdots R^{\mathrm{m}}_{s_mq_m}\dot{T}_{s_1s_2\cdots s_m})\\&=\dot{T}_{p_1p_2\cdots p_m}-\Omega^{\mathrm{m}}_{p_1s_1}T_{s_1p_2\cdots p_m}-\Omega^{\mathrm{m}}_{p_2s_2}T_{p_1s_2\cdots p_m}\\&\quad-\cdots-\Omega^{\mathrm{m}}_{p_ms_m}T_{p_1p_2\cdots s_m},\end{aligned} \tag{5}$$

where $\boldsymbol{\Omega}^{\mathrm{m}}$ is the spin of *material-substructure* and is generally related to the rotation tensor $\boldsymbol{R}^{\mathrm{m}}$ of material-substructure as follows:

$$\boldsymbol{\Omega}^{\mathrm{m}}=\dot{\boldsymbol{R}}^{\mathrm{m}}\boldsymbol{R}^{\mathrm{mT}}, \tag{6}$$

provided that $\boldsymbol{R}^{\mathrm{m}}$ is related to the motion $\boldsymbol{x}(\boldsymbol{X}, t)$ of material-substructure and fulfills the following transformation rule

$$\boldsymbol{R}^{\mathrm{m}'}=\boldsymbol{Q}\boldsymbol{R}^{\mathrm{m}}, \tag{7}$$

and thus $\boldsymbol{\Omega}^{\mathrm{m}}$ obeys the transformation rule

$$\boldsymbol{\Omega}^{\mathrm{m}'}=\boldsymbol{Q}\boldsymbol{\Omega}^{\mathrm{m}}\boldsymbol{Q}^{\mathrm{T}}+\boldsymbol{\Omega}, \tag{8}$$

where $\boldsymbol{\Omega}$ is the spin of bases $\{\boldsymbol{e}_i\}$ observed from bases $\{\boldsymbol{e}'_i\}$ as known from

$$\begin{aligned}\boldsymbol{\Omega}&\equiv(\dot{\boldsymbol{e}}_r\otimes\boldsymbol{e}_r)_{ij}\boldsymbol{e}'_i\otimes\boldsymbol{e}'_j=(\dot{\boldsymbol{e}}_r)'_i(\boldsymbol{e}_r)'_j\boldsymbol{e}'_i\otimes\boldsymbol{e}'_j\\&=(\dot{\boldsymbol{e}}_r\cdot\boldsymbol{e}'_i)(\boldsymbol{e}_r\cdot\boldsymbol{e}'_j)\boldsymbol{e}'_i\otimes\boldsymbol{e}'_j=\dot{Q}_{ir}Q_{jr}\boldsymbol{e}'_i\otimes\boldsymbol{e}'_j\\&=(\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}})_{ij}\boldsymbol{e}'_i\otimes\boldsymbol{e}'_j=\dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}},\end{aligned} \tag{9}$$

noting $\dot{\boldsymbol{e}}_i=\boldsymbol{\Omega}\boldsymbol{e}_i$ $(\dot{\boldsymbol{e}}_i\otimes\boldsymbol{e}_i=\boldsymbol{\Omega}\boldsymbol{e}_i\otimes\boldsymbol{e}_i=\boldsymbol{\Omega})$ and that the magnitudes of $\boldsymbol{e}_i$ are constant as unit. The corotational rate tensor $\mathring{\boldsymbol{T}}$ is interpreted to be generated from $(R^{\mathrm{m}}_{s_1q_1}R^{\mathrm{m}}_{s_2q_2}\cdots R^{\mathrm{m}}_{s_mq_m}T_{s_1s_2\cdots s_m})^{\boldsymbol{\cdot}}$, which is the material-time derivative of $\boldsymbol{T}$ observed in the coordinate system rotating with the material-substructure, i.e. $R^{\mathrm{m}}_{s_1q_1}R^{\mathrm{m}}_{s_2q_2}\cdots R^{\mathrm{m}}_{s_mq_m}T_{s_1s_2\cdots s_m}$, by the inverse transformation rule of objective quantity. Equation (5) is written in case of the second-order tensor as

$$\begin{aligned}\mathring{\boldsymbol{T}}&=\boldsymbol{R}^{\mathrm{m}}(\boldsymbol{R}^{\mathrm{mT}}\boldsymbol{T}\boldsymbol{R}^{\mathrm{m}})^{\boldsymbol{\cdot}}\boldsymbol{R}^{\mathrm{mT}}\\&=\dot{\boldsymbol{T}}+\boldsymbol{R}^{\mathrm{m}}\dot{\boldsymbol{R}}^{\mathrm{mT}}\boldsymbol{T}+\boldsymbol{T}\dot{\boldsymbol{R}}^{\mathrm{m}}\boldsymbol{R}^{\mathrm{mT}}\\&=\dot{\boldsymbol{T}}-\boldsymbol{\Omega}^{\mathrm{m}}\boldsymbol{T}+\boldsymbol{T}\boldsymbol{\Omega}^{\mathrm{m}}\end{aligned} \tag{10}$$

by the symbolic notation.

$\mathring{\boldsymbol{T}}$ has the objectivity as verified as follows:

$$\begin{aligned}\mathring{T}'_{p_1p_2\cdots p_m}&=R^{\mathrm{m}'}_{p_1q_1}R^{\mathrm{m}'}_{p_2q_2}\cdots R^{\mathrm{m}'}_{p_mq_m}(R^{\mathrm{m}'}_{s_1q_1}R^{\mathrm{m}'}_{s_2q_2}\cdots R^{\mathrm{m}'}_{s_mq_m}T'_{s_1s_2\cdots s_m})^{\boldsymbol{\cdot}}\\&=Q_{p_1t_1}R^{\mathrm{m}}_{t_1q_1}Q_{p_2t_2}R^{\mathrm{m}}_{t_2q_2}\cdots Q_{p_mt_m}R^{\mathrm{m}}_{t_mq_m}\\&\quad\times\{Q_{s_1r_1}R^{\mathrm{m}}_{r_1q_1}Q_{s_2r_2}R^{\mathrm{m}}_{r_2q_2}\cdots Q_{s_mr_m}R^{\mathrm{m}}_{r_mq_m}(Q_{s_1u_1}Q_{s_2u_2}\cdots\\&\quad\times Q_{s_mu_m}T_{u_1u_2\cdots u_m})\}^{\boldsymbol{\cdot}}\\&=Q_{p_1t_1}Q_{p_2t_2}\cdots Q_{p_mt_m}R^{\mathrm{m}}_{t_1q_1}R^{\mathrm{m}}_{t_2q_2}\cdots R^{\mathrm{m}}_{t_mq_m}(R^{\mathrm{m}}_{r_1q_1}R^{\mathrm{m}}_{r_2q_2}\cdots\\&\quad\times R^{\mathrm{m}}_{r_mq_m}T_{r_1r_2\cdots r_m})^{\boldsymbol{\cdot}}\\&=Q_{p_1t_1}Q_{p_2t_2}\cdots Q_{p_mt_m}\mathring{T}_{t_1t_2\cdots t_m}\end{aligned} \tag{11}$$

for the first term of Eq. (5) or

$$\begin{aligned}\mathring{T}'_{p_1p_2\cdots p_m}&=\dot{T}'_{p_1p_2\cdots p_m}-\Omega^{\mathrm{m}}_{p_1s_1}T'_{s_1p_2\cdots p_m}-\Omega^{\mathrm{m}}_{p_2s_2}T'_{p_1s_2\cdots p_m}\\&\quad-\cdots-\Omega^{\mathrm{m}}_{p_ms_m}T'_{p_1p_2\cdots s_m}\\&=(Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m})^{\boldsymbol{\cdot}}-(Q_{p_1r_1}\Omega^{\mathrm{m}}_{r_1t_1}Q_{s_1t_1}\\&\quad+\dot{Q}_{p_1r_1}Q_{s_1r_1})Q_{s_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m}\\&\quad-(Q_{p_2r_2}\Omega^{\mathrm{m}}_{r_2t_2}Q_{s_2t_2}+\dot{Q}_{p_2r_2}Q_{s_2r_2})Q_{p_1q_1}Q_{s_2q_2}\cdots\\&\quad\times Q_{p_mq_m}T_{q_1q_2\cdots q_m}-\cdots-(Q_{p_mr_m}\Omega^{\mathrm{m}}_{r_mt_m}Q_{s_mt_m}\\&\quad+\dot{Q}_{p_mr_m}Q_{s_mr_m})Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{s_mq_m}T_{q_1q_2\cdots q_m}\\&=(\dot{Q}_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}T_{q_1q_2\cdots q_m}+Q_{p_1q_1}\dot{Q}_{p_2q_2}\cdots\\&\quad\times Q_{p_mq_m}T_{q_1q_2\cdots q_m}+\cdots+Q_{p_1q_1}Q_{p_2q_2}\cdots\\&\quad\times\dot{Q}_{p_mq_m}T_{q_1q_2\cdots q_m}+Q_{p_1q_1}Q_{q_2q_2}\cdots Q_{p_mq_m}\dot{T}_{q_1q_2\cdots q_m})\\&\quad-(Q_{p_1r_1}\Omega^{\mathrm{m}}_{r_1t_1}Q_{s_1t_1}+\dot{Q}_{p_1r_1}Q_{s_1r_1})Q_{s_1q_1}Q_{p_2q_2}\cdots\end{aligned}$$

$$
\begin{aligned}
&\times Q_{p_m q_m} T_{q_1 q_2 \cdots q_m} - (Q_{p_2 r_2} \Omega^{m}_{r_2 t_2} Q_{s_2 t_2} \\
&+ \dot{Q}_{p_2 r_2} Q_{s_2 r_2}) Q_{p_1 q_1} Q_{s_2 q_2} \cdots Q_{p_m q_m} T_{q_1 q_2 \cdots q_m} - \cdots \\
&- (Q_{p_m r_m} \Omega^{m}_{r_m t_m} Q_{s_m t_m} + \dot{Q}_{p_m r_m} Q_{s_m r_m}) Q_{p_1 q_1} Q_{p_2 q_2} \cdots \\
&\times Q_{s_m q_m} T_{q_1 q_2 \cdots q_m} \\
= &Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_m q_m} (\dot{T}_{q_1 q_2 \cdots q_m} - \Omega^{m}_{q_1 s_1} T_{s_1 q_2 \cdots q_m} \\
&- \Omega^{m}_{q_2 s_2} T_{q_1 s_2 \cdots q_m} - \cdots - \Omega^{m}_{q_m s_m} T_{q_1 q_2 \cdots s_m}) \\
= &Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_m q_m} \mathring{T}_{q_1 q_2 \cdots q_m}. \qquad (11)'
\end{aligned}
$$

for the last term in Eq. (5), noting Eq. (8).

Arbitrary n-th order *tensor-valued isotropic tensor function* f of arbitrary scalar- or tensor-valued variables $\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)}$ of m_1-, m_2-, $\cdots$, m_n-th order, respectively, has to fulfill the relation

$$
\begin{aligned}
&Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_n q_n} f_{q_1 q_2 \cdots q_n}(T^{(1)}_{r_1 r_2 \cdots r m_1}, \\
&\quad T^{(2)}_{r_1 r_2 \cdots r m_2}, \cdots, T^{(n)}_{r_1 r_2 \cdots r m_n}) \\
&= f_{p_1 p_2 \cdots p_n}(T^{(1)\prime}_{r_1 r_2 \cdots r m_1}, T^{(2)\prime}_{r_1 r_2 \cdots r m_2}, \cdots, T^{(n)\prime}_{r_1 r_2 \cdots r m_n}) \\
&= f_{p_1 p_2 \cdots p_n}(Q_{r_1 s_1} Q_{r_2 s_2} \cdots Q_{r m_1 s m_1} T^{(1)}_{s_1 s_2 \cdots s m_1}, Q_{r_1 s_1} Q_{r_2 s_2} \cdots \\
&\quad \times Q_{r m_2 s m_2} T^{(2)}_{s_1 s_2 \cdots s m_2}, \cdots, Q_{r_1 s_1} Q_{r_2 s_2} \cdots Q_{r m_n s m_n} T^{(n)}_{s_1 s_2 \cdots s m_n}).
\end{aligned} \qquad (12)
$$

Yield condition is generally described as

$$
f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)}) = 0, \qquad (13)
$$

where $\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)}$ stand for stress and internal variables describing alteration of mechanical response due to plastic deformation. The yield function f as a *scalar-valued isotropic tensor function* has to be invariant for the orthogonal transformation, i.e.

$$
f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)}) = f(\boldsymbol{T}^{(1)\prime}, \boldsymbol{T}^{(2)\prime}, \cdots, \boldsymbol{T}^{(n)\prime}) \qquad (14)
$$

which is the special case of Eq. (12).

It follows from Eqs. (2) and (14) that

$$
\begin{aligned}
&\frac{\partial f(T^{(1)}_{r_1 r_2 \cdots r m_1}, T^{(2)}_{r_1 r_2 \cdots r m_2}, \cdots, T^{(n)}_{r_1 r_2 \cdots r m_n})}{\partial T^{(i)}_{p_1 p_2 \cdots p m_i}} \dot{T}^{(i)}_{p_1 p_2 \cdots p m_i} \quad \text{(no sum over (i))} \\
&= \frac{\partial f(T^{(1)\prime}_{r_1 r_2 \cdots r m_1}, T^{(2)\prime}_{r_1 r_2 \cdots r m_2}, \cdots, T^{(n)\prime}_{r_1 r_2 \cdots r m_n})}{\partial T^{(i)\prime}_{p_1 p_2 \cdots p m_i}} \dot{T}^{(i)\prime}_{p_1 p_2 \cdots p m_i} \\
&= \frac{\partial f(Q_{r_1 s_1} Q_{r_2 s_2} \cdots Q_{r m_1 s m_1} T^{(1)}_{s_1 s_2 \cdots s m_1}, Q_{r_1 s_1} Q_{r_2 s_2} \cdots Q_{r m_2 s m_2} T^{(2)}_{s_1 s_2 \cdots s m_2}, \cdots, Q_{r_1 s_1} Q_{r_2 s_2} \cdots Q_{r m_n s m_n} T^{(n)}_{s_1 s_2 \cdots s m_n}}{\partial Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p m_i q m_i} T^{(i)}_{q_1 q_2 \cdots q m_i}} \\
&\quad \times (Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p_m q_m} T_{q_1 q_2 \cdots q_m})^{\cdot} \\
&= \frac{\partial f(T^{(1)}_{s_1 s_2 \cdots s m_1}, T^{(2)}_{s_1 s_2 \cdots s m_2}, \cdots, T^{(n)}_{s_1 s_2 \cdots s m_n})}{\partial T^{(i)}_{q_1 q_2 \cdots q m_i}} Q_{p_1 q_1} Q_{p_2 q_2} \cdots Q_{p m_i q m_i} (Q_{p_1 r_1} Q_{p_2 r_2} \cdots Q_{p_m r_m} T_{r_1 r_2 \cdots r_m})^{\cdot}.
\end{aligned} \qquad (15)
$$

The selection $\boldsymbol{Q} = \boldsymbol{R}^{m^{T}}$ in Eq. (15) leads to

$$
\mathrm{tr}\left\{\frac{\partial f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)})}{\partial \boldsymbol{T}^{(i)}} \dot{\boldsymbol{T}}^{(i)}\right\} = \mathrm{tr}\left\{\frac{\partial f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)})}{\partial \boldsymbol{T}^{(i)}} \mathring{\boldsymbol{T}}^{(i)}\right\} \quad \text{(no sum over (i))}. \qquad (15)'
$$

Thus, one obtains

$$
\sum_{i=1}^{n} \mathrm{tr}\left\{\frac{\partial f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)})}{\partial \boldsymbol{T}^{(i)}} \dot{\boldsymbol{T}}^{(i)}\right\} = \sum_{i=1}^{n} \mathrm{tr}\left\{\frac{\partial f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)})}{\partial \boldsymbol{T}^{(i)}} \mathring{\boldsymbol{T}}^{(i)}\right\}. \qquad (16)
$$

That is to say, the rate tensors involved in the material-time derivative of the yield function f can be replaced directly with their objective rate tensors.

Dafalias (1985a, b) introduced the following symbolic notations $\boldsymbol{N}[\boldsymbol{T}]$ for arbitrary orthogonal tensor $\boldsymbol{N}(\boldsymbol{N}\boldsymbol{N}^{T} = \boldsymbol{I})$. Here, add the symbolic notation $\boldsymbol{N}^{T}[\boldsymbol{T}]$ for the inverse transformation, i.e.

$$
\left.
\begin{aligned}
(\boldsymbol{N}[\boldsymbol{T}])_{p_1 p_2 \cdots p_m} &= N_{p_1 q_1} N_{p_2 q_2} \cdots N_{p_m q_m} T_{q_1 q_2 \cdots q_m}, \\
(\boldsymbol{N}^{T}[\boldsymbol{T}])_{p_1 p_2 \cdots p_m} &= N_{q_1 p_1} N_{q_2 p_2} \cdots N_{q_m p_m} T_{q_1 q_2 \cdots q_m}.
\end{aligned}
\right\} \qquad (17)
$$

The corotational rate tensor $\mathring{\boldsymbol{T}}$ in Eq. (5) and the objective transformation in Eq. (11) can be expressed concisely using the notation defined by Eq. (17).

$$
\mathring{\boldsymbol{T}} = \boldsymbol{R}^{m}[(\boldsymbol{R}^{m^{T}}[\boldsymbol{T}])^{\cdot}], \qquad (18)
$$

$$
\begin{aligned}
\mathring{\boldsymbol{T}}' &= \boldsymbol{R}^{m\prime}[(\boldsymbol{R}^{m\prime T}[\boldsymbol{T}'])^{\cdot}] = \boldsymbol{Q}\boldsymbol{R}^{m}[\{(\boldsymbol{Q}\boldsymbol{R}^{m})^{T}[\boldsymbol{Q}[\boldsymbol{T}]]\}^{\cdot}] \\
&= \boldsymbol{Q}[\boldsymbol{R}^{m}[(\boldsymbol{R}^{m^{T}}[\boldsymbol{T}])^{\cdot}]] = \boldsymbol{Q}[\mathring{\boldsymbol{T}}].
\end{aligned} \qquad (19)
$$

Equations (12) and (14) for the isotropic tensor function are expressed as

$$
\boldsymbol{Q}[f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)})] = f(\boldsymbol{Q}[\boldsymbol{T}^{(1)}], \boldsymbol{Q}[\boldsymbol{T}^{(2)}], \cdots, \boldsymbol{Q}[\boldsymbol{T}^{(n)}]) \qquad (20)
$$

$$
f(\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \cdots, \boldsymbol{T}^{(n)}) = f(\boldsymbol{Q}[\boldsymbol{T}^{(1)}], \boldsymbol{Q}[\boldsymbol{T}^{(2)}], \cdots, \boldsymbol{Q}[\boldsymbol{T}^{(n)}]), \qquad (21)
$$

using the symbolic notation (17).

Here, let the following symbolic notations $\dot{\boldsymbol{N}}[\boldsymbol{T}]$ and $\dot{\boldsymbol{N}}^{T}[\boldsymbol{T}]$ be newly defined in accordance with the notation (17) that

$$\left.\begin{aligned}(\dot{N}[T]_{p_1p_2\cdots p_m}&=(\dot{N}_{p_1q_1}N_{p_2q_2}\cdots N_{p_mq_m}+N_{p_1q_1}\dot{N}_{p_2q_2}\cdots N_{p_mq_m}+\cdots+N_{p_1q_1}N_{p_2q_2}\cdots\dot{N}_{p_mq_m})T_{q_1q_2\cdots q_m},\\(\dot{N}^{\mathrm{T}}[T])_{p_1p_2\cdots p_m}&=(\dot{N}_{q_1p_1}N_{q_2p_2}\cdots N_{q_mp_m}+N_{q_1p_1}\dot{N}_{q_2p_2}\cdots N_{q_mp_m}+\cdots+N_{q_1p_1}N_{q_2p_2}\cdots\dot{N}_{q_mp_m})T_{q_1q_2\cdots q_m}.\end{aligned}\right\} \tag{22}$$

Further, define the novel symbolic notation $\boldsymbol{\Omega}^{\mathrm{N}}[\boldsymbol{T}]$:

$$\begin{aligned}(\boldsymbol{\Omega}^{\mathrm{N}}[\boldsymbol{T}])_{p_1p_2\cdots p_m}&\equiv(\dot{N}^{\mathrm{T}}[N[T]])_{p_1p_2\cdots p_m}=(\dot{N}_{q_1p_1}N_{q_2p_2}\cdots N_{q_mp_m}+N_{q_1p_1}\dot{N}_{q_2p_2}\cdots N_{q_mp_m}+\cdots\\&\quad+N_{q_1p_1}N_{q_2p_2}\cdots\dot{N}_{q_mp_m})N_{q_1s_1}N_{q_2s_2}\cdots N_{q_ms_m}T_{s_1s_2\cdots s_m}\\&=(\dot{N}_{q_1p_1}N_{q_1s_1}\delta_{p_2s_2}\cdots\delta_{p_ms_m}+\delta_{p_1s_1}\dot{N}_{q_2p_2}N_{q_2s_2}\cdots\delta_{p_ms_m}+\cdots+\delta_{p_1s_1}\delta_{p_2s_2}\cdots\dot{N}_{q_mp_m}N_{q_ms_m})T_{s_1s_2\cdots s_m}\\&=\Omega^{N}_{p_1s_1}T_{s_1p_2\cdots p_m}+\Omega^{N}_{p_2s_2}T_{p_1s_2\cdots p_m}+\cdots+\Omega^{N}_{p_ms_m}T_{p_1p_2\cdots s_m},\end{aligned} \tag{23}$$

noting

$$\Omega^{\mathrm{N}}\equiv\dot{N}^{\mathrm{T}}N. \tag{24}$$

Then, for choosing $N=\boldsymbol{R}^{\mathrm{m}^{\mathrm{T}}}$ the corotational rate tensor $\mathring{T}$ in Eq. (5) is expressed as follows:

$$\mathring{T}=\dot{T}-\Omega^{\mathrm{m}}[T] \tag{25}$$

by the symbolic notation (23). Further, the mathematical process (11)′ is expressed as follows:

$$\begin{aligned}\mathring{T}'&=\dot{T}'-\Omega^{\mathrm{m}'}[T']\\&=(Q[T])^{\cdot}-(Q\Omega^{\mathrm{m}}Q^{\mathrm{T}}+\dot{Q}Q^{\mathrm{T}})[Q[T]]\\&=\dot{Q}[T]+Q[\dot{T}]-(Q\Omega^{\mathrm{m}}Q^{\mathrm{T}}+\dot{Q}Q^{\mathrm{T}})[Q[T]]\\&=Q[\dot{T}-\Omega^{\mathrm{m}}[T]]=Q[\mathring{T}].\end{aligned} \tag{26}$$

Furthermore, the algebraic manipulation of Eq. (15) can be expressed concisely as follows:

$$\begin{aligned}&\mathrm{tr}\left\{\frac{\partial f(T^{(1)},T^{(2)},\cdots,T^{(n)})}{\partial T^{(i)}}(T^{(i)})^{\cdot}\right\}\quad\text{(no sum over (i))}\\&=\mathrm{tr}\left\{\frac{\partial f(Q[T^{(1)}],Q[T^{(2)}],\cdots,Q[T^{(n)}])}{\partial Q[T^{(i)}]}(Q[T^{(i)}])^{\cdot}\right\}\\&=\mathrm{tr}\left\{Q\left[\frac{\partial f(T^{(1)},T^{(2)},\cdots,T^{(n)})}{\partial T^{(i)}}\right](Q[T^{(i)}])^{\cdot}\right\}\\&=\mathrm{tr}\left\{\frac{\partial f(T^{(1)},T^{(2)},\cdots,T^{(n)})}{\partial T^{(i)}}\{Q^{\mathrm{T}}[(Q[T^{(i)}])^{\cdot}]\}\right\},\end{aligned} \tag{27}$$

noting

$$\mathrm{tr}\,\{(Q[A])B\}=\mathrm{tr}\,\{(A(Q^{\mathrm{T}}[B])\} \tag{28}$$

due to

$$Q_{p_1q_1}Q_{p_2q_2}\cdots Q_{p_mq_m}A_{q_1q_2\cdots q_m}B_{p_1p_2\cdots p_m}=A_{q_1q_2\cdots q_m}Q_{p_1q_1}Q_{p_2q_2}\cdots\times Q_{p_mq_m}B_{p_1p_2\cdots p_m} \tag{29}$$

for arbitrary same order tensors A and B. Besides, Eq. (20) is used for deriving the third equation from the second one in Eq. (27).

From the first and the sixth equations of Eq. (27) it results that

$$\mathrm{tr}\left\{\frac{\partial f(T^{(1)},T^{(2)},\cdots,T^{(n)})}{\partial T^{(i)}}\Omega[T^{(i)}]\right\}=0\quad\text{(no sum over (i))}. \tag{30}$$

For the special case that $T^{(i)}$ is the second-order tensor, denoted as α, it follows from the first and the last equations of Eq. (27) with Eq. (10) that

$$\alpha\frac{\partial f}{\partial\alpha}=\frac{\partial f}{\partial\alpha}\alpha \tag{31}$$

by the arbitrariness of $\boldsymbol{\Omega}$, noting $\mathrm{tr}\,(AB)=\mathrm{tr}\,(BA)$. Equation (31) was derived by Dafalias (1993).

The transformation rule of Eq. (16) can be also derived by the following method in the special case that f is a scalar-valued isotropic function of the second-order symmetric tensor α, which has three independent invariants. The function f can be described by the invariants of them, i.e.

$$f(\alpha)=f(I,II,III), \tag{31}$$

where I, II, III are the first, the second and third invariants of α, respectively, i.e.

$$I\equiv\mathrm{tr}\,\alpha,\quad II\equiv\mathrm{tr}\,\alpha^2,\quad III\equiv\mathrm{tr}\,\alpha^3. \tag{32}$$

It holds that

$$\begin{aligned}\mathrm{tr}\left(\frac{\partial f}{\partial\alpha}\dot{\alpha}\right)&=\mathrm{tr}\left\{\left(\frac{\partial f}{\partial I}\frac{\partial I}{\partial\alpha}+\frac{\partial f}{\partial II}\frac{\partial II}{\partial\alpha}+\frac{\partial f}{\partial III}\frac{\partial III}{\partial\alpha}\right)\dot{\alpha}\right\}\\&=\mathrm{tr}\,\{(\mathrm{a}_0I+\mathrm{a}_1\alpha+\mathrm{a}_2\alpha^2)\dot{\alpha}\}\\&=\mathrm{tr}\,\{(\mathrm{a}_0I+\mathrm{a}_1\alpha^{(i)}+\mathrm{a}_2\alpha^2)(\mathring{\alpha}+\Omega\alpha-\alpha\Omega)\}\\&=\mathrm{tr}\,\{(\mathrm{a}_0I+\mathrm{a}_1\alpha+\mathrm{a}_2\alpha^2)\mathring{\alpha}\}\\&=\mathrm{tr}\left(\frac{\partial f}{\partial\alpha}\mathring{\alpha}\right),\end{aligned} \tag{33}$$

where

$$\mathrm{a}_0\equiv\frac{\partial f}{\partial I},\quad\mathrm{a}_1\equiv\frac{\partial f}{\partial II},\quad\mathrm{a}_2\equiv\frac{\partial f}{\partial III} \tag{34}$$

which are scalar functions of I, II, III, noting $(S^{\mathrm{n}})^{\mathrm{T}}=S^{\mathrm{n}}$ and $\mathrm{tr}\,(ST^{\mathrm{a}})=0$, S and T^{a} being symmetric and skew-symmetric second-order tensors, respectively.

COROTATIONAL RATE TENSOR

In the previous section, it is verified that the corotational rate tensor defined by Eq. (5) has the objectivity and the rate variables involved in the material-time derivative of yield function and can be directly replaced by the corotational rate tensors of Eq. (5). However, one has to select the rotation tensor $\boldsymbol{R}^{\mathrm{m}}$ and the spin tensor $\boldsymbol{\Omega}^{\mathrm{m}}$ describing appropriately the rotation of the substructure of the material. For instance, the *Zaremba-Jaumann rate* (Zaremba, 1903a, b; Jaumann, 1911) with the spin $\boldsymbol{W}(\equiv\boldsymbol{L}-\boldsymbol{L}^{\mathrm{T}})/2$, $\boldsymbol{L}$: velocity gradient) and the *Green-Naghdi* or *Dienes rate* (Green-Naghdi, 1965; Dienes, 1977) with the spin $\boldsymbol{W}_{\mathrm{R}}\equiv\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}$, where $\boldsymbol{R}$ is the rotational part in the polar decomposition of the *deformation gradient tensor* $\boldsymbol{F}$, fulfill Eq. (8) and have been adopted as corotational rate tensors. $\boldsymbol{W}$ is the spin of principal directions of strain rate $\boldsymbol{D}(\equiv(\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}})/2)$ and is called the *continuum spin* or *vorticity tensor*, and $\boldsymbol{W}_{\mathrm{R}}\equiv\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}$ is the spin of principal stretches and is called the *polar* or *body* or *relative* or *material spin*. The rigid body spin of material is exactly given by $\boldsymbol{W}_{\mathrm{R}}$ in general, as known also for the continuum spin $\boldsymbol{W}=\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}+\boldsymbol{R}\{(\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}+\boldsymbol{U}^{-1}\dot{\boldsymbol{U}})/2\}\boldsymbol{R}^{\mathrm{T}}$ ($\boldsymbol{U}$: right stretch tensor).

Mandel (1972) and Kratochvil (1972) revealed that the spin of material-substructure is suppressed from the continuum spin when plastic deformation proceeds. Dafalias (1983, 1985a) and Loret (1983) proposed that the substructure spin is not so large as the continuum spin but has to be given as the spin where the *plastic spin* due to plastic deformation is subtracted from the continuum spin. The pertinent form of plastic spin was given by Dafalias (1985b) and Zbib and Aifantis (1988). However, in order to use this spin tensor, one has to formulate the constitutive equation for the plastic spin tensor, and thus a concrete form of the constitutive equation becomes complicated and difficulty arises in determining the plastic spin since one needs test data for a large shear strain more than 150%. Below this level, the influence of the plastic spin is negligible, but it would be impossible to perform an element test for such a large deformation due to the occurrence of localized deformation. Here, it would be practical to use the Jaumann rate for usual problems in engineering. On the other hand, the Green-Naghdi or Dienes rate is not appropriate for the corotational rate tensor in the formulation of elastoplastic constitutive equations since $\boldsymbol{W}_{\mathrm{R}}$ depends on $\boldsymbol{R}$ describing the whole rotation from the initial state. However, a current response of material for the plastic deformation is hardly dependent on a total strain and/or rotation from the initial state as has been reviewed by Dafalias (1983) and Hashiguchi (2001).

CONCLUDING REMARKS

The general form of a rate tensor with objectivity is shown and it is verified that the rate variables involved in the material-time derivative of yield function can be directly replaced by the objective rate tensors. Here, it should be noted that the verification is applicable not only to yield function but also to arbitrary scalar-valued isotropic functions.

The objectivity of tensors has been interpreted from the following two methods: Method 1) Equation (2) holds between components of the tensor described by two different coordinated systems where a relative rate of rotation exists between these coordinate systems and Method 2) The components of the tensor observed by a fixed coordinate system changes as in Eq. (2) when the rigid body motion $\boldsymbol{W}_{\mathrm{R}}(\boldsymbol{R}^{\mathrm{m}}=\boldsymbol{R})$ is superposed to the material motion. Method 1) adopted in this note would be more natural than Method 2), which reflects directly the concept of the *principle of material-frame indifference* (Oldroyd, 1950). Further, it can avoid the use of a concrete corotational spin tensor of material which remains to unsolved still now.

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