REFERENCES

(1) Hashiba, T. and Whitman, R. V. (1968): "Soil-structure interaction during earthquakes," Soils and Foundations, Vol. 8, No. 2, pp. 1-12.
(2) Whitman, R. V. (1968): "Equivalent lumped system for structure founded upon stratum of soil," Technical Paper No. T68-6, M. I. T.
(3) Davis, E. H. and Taylor, H. (1961): "The surface displacement of an elastic layer due to horizontal and vertical surface loading," Proc. 5th. ICSMFE, Vol. I, pp. 621-627.
(4) Arnold, R. N. et al (1955): "Forced vibration of a body on an infinite elastic solid," J. Appl. Mech., Trans. ASME, Vol. 77, pp. 391-400.
(5) Jacobsen, L. S. and Ayre, R. S. (1958): Engineering Vibrations, McGraw-Hill Book Co., pp. 113-120.

(Received November 25, 1971)

STIFFNESS MATRIX FOR BEAMS ON ELASTIC FOUNDATION BY VIRTUAL WORK PRINCIPLE

SHIH-FANG CHEN*

ABSTRACT

This technical note provides a theoretical analysis of finite length beams on soil subjected to general loadings and given end support conditions. The principle of virtual work was employed to formulate the stiffness matrix of the soil-structure interaction problem. Numerical methods such as finite element analyses were employed, and the results from the examples indicated that the proposed method gave results close to those solved using Harr's approach.

Key words: static, elasticity, settlement, computer application, mat foundation

IGC: EO

INTRODUCTION

The theory of beams on elastic foundation is commonly used in soil-structure interaction and occupies a very important place in soil mechanics. The methods for the solution due to Winkler's assumption were used (Winkler (7)) for more than one century. Winkler's hypothesis states that the reactions of the foundation are proportional at every point to the deflection of the beam at that point; this simplified "independent spring" model of the elastic foundation leads frequently to incorrect results.

Until recently, a large number of studies and many numerical methods have been proposed by many researchers (Iyengar (2), Malter (3) and Matlock et al (4)) to solve this problem. Some of these methods may be modified to take into account different elastic foundations

* Ph. D., Bridge Engineer, Taiwan Railway Administration, Taipei, Taiwan; formerly Graduate Student, Civil Engineering Department, University of Washington, Seattle, Washington, U. S. A.
Written discussions on this paper should be submitted before April 1, 1973.

with different moduli k values. Based on the theory of Vlasov's general variational method (Vlasov et al (6)), instead of using k, a digital computer program has been developed for analyzing beams on elastic foundation by Harr et al (1). This method is more accurate than the well-known theory of Winkler, and is simpler than the theory of the elastic semi-infinite space. However, analysis by a sophisticated finite element method procedure may contribute greatly to the understanding of the prototype performance. In this paper, the author's technique and Harr's are similar, and attack essentially the same problem, but a new approach based on the virtual work principle is outlined herein, and a stiffness matrix is developed which may be useful in engineering practice.

DERIVATION OF STIFFNESS MATRIX

Basic Assumptions

Consider that a finite element beam rests on a homogeneous, elastic, compressible layer of soils of the thickness H which produces conditions of plane strain (Fig. 1). The finite

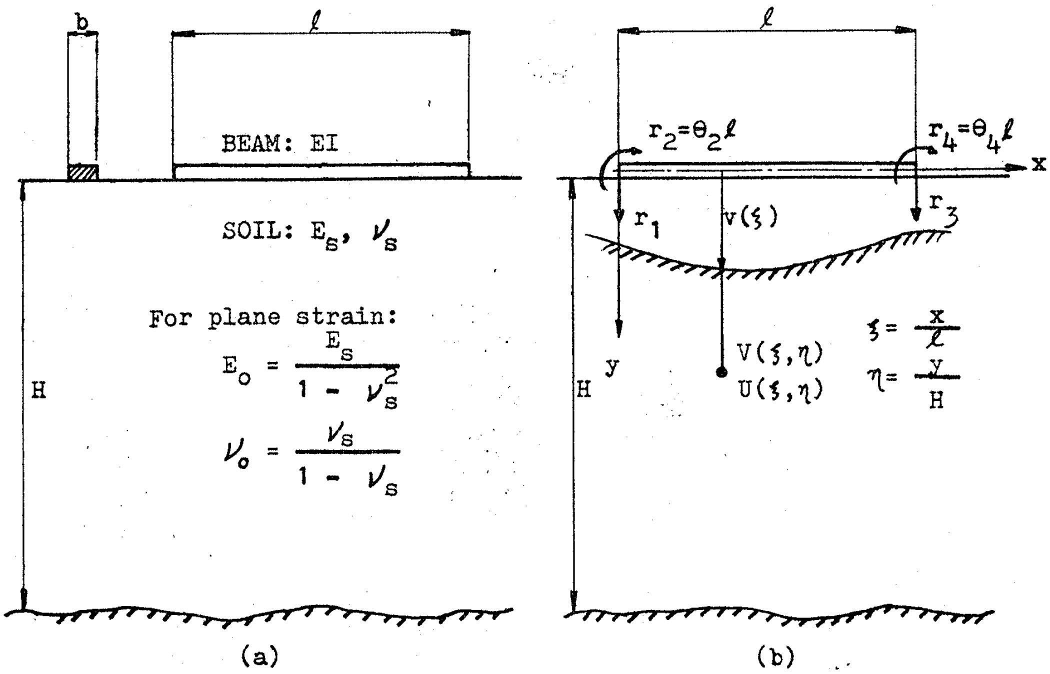

Fig. 1. Finite element beam on soil medium

element beam model is subjected to the same assumptions as the classical equations for bending of a beam:

(1) Axial and shear deformations and secondary effects due to bending are neglected.
(2) Consideration is limited to straight beams of symmetrical cross section.
(3) Plane sections are assumed to remain plane both during and after bending.
(4) Lateral deflections are assumed to be small compared to the length of the beam.
(5) Torsional effects are neglected.
(6) The material of the beam is assumed to behave in a linearly elastic manner.

The degree of freedom is so selected (see Fig. 1(b)) that the vertical displacement function for a segment of length l can be represented by a third-degree polynominal such as (Przemieniecki (5)):

$$v(\xi)=<1-3\xi^2+2\xi^3,\ \xi-2\xi^2+\xi^3,\ 3\xi^2-2\xi^3,\ -\xi^2+\xi^3>\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \tag{1}$$

which may be written symbolically as:

$$v(\xi)=\underset{\sim}{N}(\xi)\underset{\sim}{r} \tag{2}$$

where the wiggle sign ($\sim$) is to denote that the quantities are matrices.

In a soil medium, in order to obtain a simple approximate solution, the unknown displacement functions are expanded in finite series (Vlasov (6)):

$$U(\xi,\eta)=\sum_{i=1}^{m} u_i(\xi)\ \phi_i(\eta) \tag{3}$$

$$V(\xi,\eta)=\sum_{j=1}^{n} v_j(\xi)\ \psi_j(\eta) \tag{4}$$

The shape functions $\phi_i(\eta)$ and $\psi_i(\eta)$ are assumed to be known, and the functions $u_i(\xi)$ and $v_j(\xi)$ are assumed to be unknown. It is convenient to introduce dimensionless functions for $\phi_i(\eta)$ and $\psi_j(\eta)$; and the dimensional displacement functions for $u_i(\xi)$ and $v_j(\xi)$.

In our problem described herein, the horizontal displacements $U(\xi,\eta)$ in the soil medium may be considered to be of negligible magnitude in comparison with the vertical displacements $V(\xi,\eta)$. It is further assumed that no slip occurs at the interface between the beam and soils. Thus, we have the simple equation to represent the displacements in the soil medium. This is of the form using the first term approximation (Ref. (1)):

$$V(\xi,\eta)=v(\xi)\ \psi(\eta) \tag{5}$$

where $v(\xi)$=the vertical deflections of the beam at the interface as expressed by Equation (1).

$\psi(\eta)$=a selected or known dimensionless function.

According to Vlasov (6), we can choose the following expression for $\psi(\eta)$:

(1) For a relatively thin compressible layer of soils (i.e. H=fairly small):

$$\psi(\eta)=1-\eta \tag{6}$$

(2) For a thick layer of soils (H=very large):

$$\psi(\eta)=\frac{\sinh \gamma \dfrac{H}{L}(1-\eta)}{\sinh \gamma \dfrac{H}{L}} \tag{7}$$

where

$L=\sqrt[3]{\dfrac{2EI(1-\nu_0^2)}{E_0 b}}$, the elastic characteristics of the beam.

γ=some coefficient which is dependent on the elastic properties of the foundation.

Strain-Displacement Relations

Employing the previous assumptions in the soil medium, we may have the following equations:

$$U(\xi,\eta)\doteqdot 0 \tag{8}$$

$$V(\xi,\eta)=v(\xi)\ \phi(\eta) \tag{9}$$

Substituting Equation (2) into Equation (9) to obtain:

$$V(\xi,\eta)=\underset{\sim}{N}(\xi)\ \phi(\eta)\ \underset{\sim}{r} \tag{10}$$

By conventional strain-displacement relations,

$$e_x=\frac{\partial U}{\partial x}=0 \tag{11}$$

$$e_y=\frac{\partial V}{\partial y}=\frac{\partial V}{\partial \eta}\frac{d\eta}{dy}=\frac{\phi'(\eta)}{H}\underset{\sim}{N}(\xi)\ \underset{\sim}{r} \tag{12}$$

$$\gamma_{xy}=\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}=\frac{\partial V}{\partial \xi}\frac{d\xi}{dx}=\frac{\phi(\eta)}{l}\underset{\sim}{N}'(\xi)\ \underset{\sim}{r} \tag{13}$$

in which

$$\phi'(\eta)=\frac{\partial\phi(\eta)}{\partial\eta} \tag{14}$$

$$N'(\xi)=\frac{\partial N(\xi)}{\partial\xi} \tag{15}$$

Combining Equations (11), (12) and (13), and writing in matrix form:

$$\begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\phi'(\eta)}{H}(1-3\xi^2+2\xi^3), & \frac{\phi'(\eta)}{H}(\xi-2\xi^2+\xi^3), & \frac{\phi'(\eta)}{H}(3\xi^2-2\xi^3), & \frac{\phi'(\eta)}{H}(-\xi^2+\xi^3) \\ \frac{\phi(\eta)}{l}(-6\xi+6\xi^2), & \frac{\phi(\eta)}{l}(1-4\xi+3\xi^2), & \frac{\phi(\eta)}{l}(6\xi-6\xi^2), & \frac{\phi(\eta)}{l}(-2\xi+3\xi^2) \end{bmatrix}$$

$$\cdot\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \tag{16}$$

or symbolically as:

$$\underset{\sim}{e}=\underset{\sim}{B}(\xi,\eta)\ \underset{\sim}{r} \tag{17}$$

Stress-Strain Relations

In the soil medium, in the present case, the strains are related to the stresses by a linearly elastic, isotropic rule. Treating the problem as a plane strain case, so

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E_0}{1-\nu_0^2} \begin{bmatrix} 1 & \nu_0 & 0 \\ \nu_0 & 1 & 0 \\ 0 & 0 & \frac{1-\nu_0}{2} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix} \tag{18}$$

in which

$$E_0 = E_s/(1-\nu_s^2) \tag{19}$$

$$\nu_0 = \nu_s/(1-\nu_s) \tag{20}$$

Equation (18) may be written symbolically as:

$$\underset{\sim}{\sigma} = \underset{\sim}{D}\, \underset{\sim}{e} \tag{21}$$

Application of Virtual Work Principle

According to the principle of virtual work from virtual displacements:

$$\underbrace{VW_I}_{\text{Internal virtual work}} = \underbrace{VW_E}_{\text{External virtual work}} \tag{22}$$

$$\int_v \delta \underset{\sim}{e}^T \underset{\sim}{\sigma}\, dv = \int_S \delta \underset{\sim}{v}^T p\, ds + \int_v \delta v^T \underset{\sim}{b}\, dv \tag{23}$$

Assume the absence of body forces, and from Equation (23) we have:

$$\underbrace{\frac{1}{l^4}\int_0^1 \delta v''(\xi)\, EI_v''(\xi) l\, d\xi}_{VW_I \text{ due to beam}} + \underbrace{\int_v \delta \underset{\sim}{e}^T \underset{\sim}{\sigma}\, dv}_{VW_I \text{ due to soil}} = \underbrace{\int_S \delta \underset{\sim}{v}^T \underset{\sim}{p}\, ds}_{VW_E \text{ due to surface traction}} \tag{24}$$

Substituting Equations (2), (17) and (21) into Equation (24) yields:

$$\delta r^T \int_0^1 \frac{EI}{l^3} \underset{\sim}{N}''(\xi)^T \underset{\sim}{N}''(\xi)\, d\xi\, \underset{\sim}{r} + \int_0^1\int_0^1 \delta \underset{\sim}{r}^T \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, lHb\, d\xi\, d\eta\, \underset{\sim}{r} = \delta \underset{\sim}{r}^T \underset{\sim}{R} \tag{25}$$

After doing simple algebraic calculations, we may obtain:

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$+\frac{E_0}{1-\nu_0^2}\,\frac{bl}{420H}\int_0^1(\psi'(\eta))^2\,d\eta\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$+\frac{E_0}{2(1+\nu_0)}\,\frac{bH}{30l}\int_0^1(\psi(\eta))^2\,d\eta\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$=\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \tag{26}$$

The terms $\int_0^1(\psi'(\eta))^2\,d\eta$ and $\int_0^1(\psi(\eta))^2\,d\eta$ in Equation (26) depend upon the conditions which are expressed in either Equation (6) or Equation (7).

(1) For the case of $\psi(\eta)=1-\eta$

$$\int_0^1(\psi(\eta))^2\,\mathrm{d}\eta=1/3 \tag{27}$$

$$\int_0^1(\psi'(\eta))^2\,d\eta=1 \tag{28}$$

Substituting Equations (27) and (28) into Equation (26), we may obtain:

$$\left\{\frac{EI}{l^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}+\frac{E_0}{1-\nu_0^2}\,\frac{bl}{420H}\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\right.$$

$$\left.+\frac{E_0}{2(1+\nu_0)}\,\frac{bH}{90l}\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}\right\}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}=\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \tag{29}$$

(2) For the case of $\phi(\eta)=\dfrac{\sinh\gamma\dfrac{H}{L}(1-\eta)}{\sinh\gamma\dfrac{H}{L}}$

$$\int_0^1(\phi(\eta))^2\,d\eta=\frac{1}{2(\gamma H/L)\sinh^2(\gamma H/L)}\left(-(\gamma H/L)+\frac{1}{2}\sinh 2(\gamma H/L)\right) \tag{30}$$

$$\int_0^1(\phi'(\eta))^2\,d\eta=\frac{1}{2}(\gamma H/L)\frac{1}{\sinh^2(\gamma H/L)}\left((\gamma H/L)+\frac{1}{2}\sinh 2(\gamma H/L)\right) \tag{31}$$

Substituting Equations (30) and (31) into Equation (26), we may obtain:

$$\left\{\frac{EI}{l^3}\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\right.$$

$$+\frac{E_0}{1-\nu_0^2}\,\frac{bl}{420H}\,\frac{1}{2}(\gamma H/L)\,\frac{(\gamma H/L)+1/2\sinh 2(\gamma H/L)}{\sinh^2(\gamma H/L)}\begin{bmatrix}156 & 22 & 54 & -13\\ 22 & 4 & 13 & -3\\ 54 & 13 & 156 & -22\\ -13 & -3 & -22 & 4\end{bmatrix}$$

$$\left.+\frac{E_0}{2(1+\nu_0)}\,\frac{bH}{30l}\,\frac{1}{2(\gamma H/L)}\,\frac{-(\gamma H/L)+1/2\sinh 2(\gamma H/L)}{\sinh^2(\gamma H/L)}\begin{bmatrix}36 & 3 & -36 & 3\\ 3 & 4 & -3 & -1\\ -36 & -3 & 36 & -3\\ 3 & -1 & -3 & 4\end{bmatrix}\right\}$$

$$\cdot\begin{bmatrix}r_1\\ r_2\\ r_3\\ r_4\end{bmatrix}=\begin{bmatrix}R_1\\ R_2\\ R_3\\ R_4\end{bmatrix}. \tag{32·a}$$

If the elastic foundation is on a semi-infinite plane, i.e., $H\to\infty$, Equation (32·a) reduces to:

$$\left\{\frac{EI}{l^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} + \frac{E_0}{1-\nu_0^2}\,\frac{bl}{840}\left(\frac{\gamma}{L}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\right.$$

$$\left. + \frac{E_0}{2(1+\nu_0)}\,\frac{b}{30l}\,\frac{1}{2(\gamma/L)}\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}\right\}\cdot\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \tag{32·b}$$

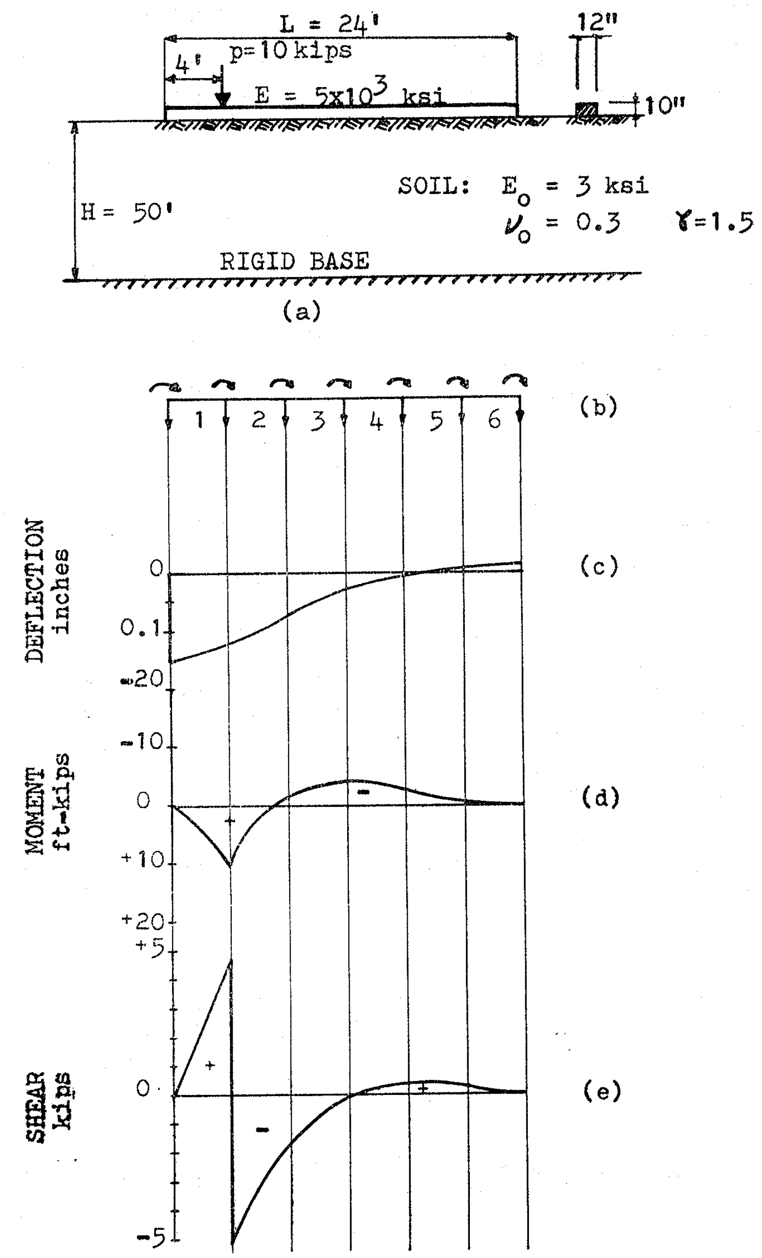

Fig. 2. Numerical example (1)

Note that the equilibrium relations of Equations (29), (32·a) and (32·b) can be written in the compact matrix form such as:

$$\underset{\sim}{K}\,\underset{\sim}{r} = \underset{\sim}{R} \tag{33}$$

in which:

$\underset{\sim}{r}$ = a vector composed of the nodal displacements.

$\underset{\sim}{R}$ = a vector of the nodal forces.

$\underset{\sim}{K}$ = all the terms in the left side of Equations (29), (32·a) and (32·b), i.e., the stiffness matrix relating the nodal displacements and the nodal forces.

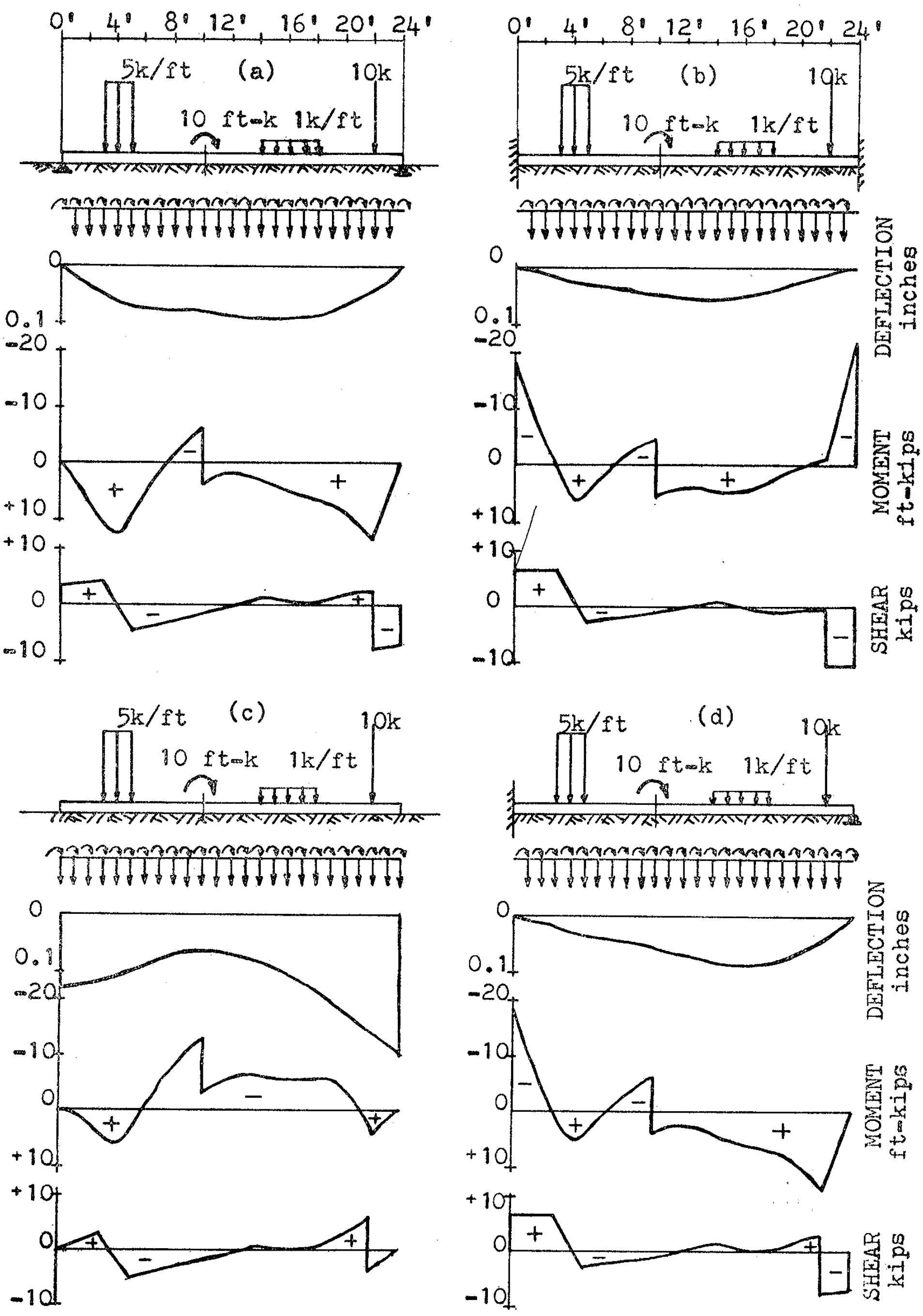

Fig. 3. Numerical example (2)

NUMERICAL EXAMPLES

Example (1)—From Harr et al (Ref. (1))

The problem described in Fig. 2 is that of a continuous beam with given beam and soil parameters, subject to a concentrated load 10 kips at the distance with 4 ft from the left end. It is desired to determine the deflection curve, the shear curve and the corresponding bending moments for computation of stresses.

The model which is used for calculation is devided into 6 elements and is shown in Fig. 2(b). By using the stiffness equation represented by Equation (32·a), the results of deflections, bending moments and shears are shown in Fig. 2(c) to 2(e).

Example (2)—From Harr et al (Ref. (1))

The problems described in Fig. 3 are those for a beam under a given load system for the indicated end conditions. The beam and soil parameters are the same as those given in the previous example.

The beam was idealized with 24 elements. The total stiffness matrix can be assembled from the element stiffness matrix (Equation (32·a)). However, the size of the total stiffness matrix can be reduced by eliminating rows and columns corresponding to zero displacements according to the given end conditions. The results are also shown in Fig. 3.

The results of both Example (1) and Example (2) are nearly identical to those solved by Harr's approach.

CONCLUSIONS

The method presented provides a matrix formulation for beams on elastic supports, which is based on the two parameters E_s and ν of the soil medium. The beam may be of varying cross-section or the soil parameters may also be varying.

The proposed method is easy to apply and can be extended to the analysis of soil-supported structural problems such as anchored bulkhead problems. The method can also be extended to include the axial forces for the analysis of buckling problems of beams on the elastic foundation.

ACKNOWLEDGEMENTS

Thanks are extended to the personnel of the University of Washington Computer Center for their excellent facilities and services.

NOTATION

The following symbols are used in this paper:

$\underset{\sim}{b}$=body forces

$\underset{\sim}{B}$=element strain-displacement matrix

$\underset{\sim}{D}$=elastic matrix containing the appropriate material properties

$\underset{\sim}{e}=\langle e_x,\ e_y,\ \gamma_{xy}\rangle$, strain vector

E=the elastic modulus of the beam

E_s=the elastic modulus of soils

I=the moment of inertia of the beam

$\underset{\sim}{K}$=the stiffness matrix relating the nodal displacements and the nodal forces

l=the length of the finite element beam

L=the elastic characteristics of the beam

$\underset{\sim}{N}$=the shape function of the beam
$\underset{\sim}{r}$=a vector composed of the nodal displacements
$\underset{\sim}{R}$=a vector of the nodal forces
γ=some coefficient which is dependent on the elastic properties of the foundation
U=the horizontal displacement in the soil medium
v=the vertical displacement of the beam
V=the vertical displacement in the soil medium
ν_s=Poisson's ratio of soils
$\underset{\sim}{\sigma}=\langle\sigma_x, \sigma_y, \tau_{xy}\rangle$, stress vector
ϕ_i=shape function
ψ_i=shape function
ξ, η=non-dimensional parameters, see Fig. 1(b)

REFERENCES

(1) Harr, M. E., et al. (1969): "Euler beams on a two parameter foundation model," Journal of Soil Mechanics and Foundations Division, ASCE, Vol. 95, No. SM4, pp. 933-948.
(2) Iyengar, K. T. S. (1965): "Matrix analysis of finite beam on elastic foundation," Institution of Engineers Journal, India, Vol. 45, pp. 837-855.
(3) Malter, M. (1960): "Numerical solutions for beams on elastic foundation," Transactions, ASCE, Vol. 125, pp. 757-791.
(4) Matlock, H., et al. (1963): "Bending and buckling of soil supported structural elements," Proceedings, Paper No. 32, 2nd Panamerican Conference on Soil Mechanics and Foundation Engineering, Brazil.
(5) Przemieniecki, J. S. (1968): Theory of Matrix Structural Analysis, McGraw-Hill, Inc.
(6) Vlasov, V. Z. and Leont'ev, N. N. (1966): Beams, Plates, and Shells on Elastic Foundation (Translated from Russian), Israel Program for Scientific Translations, Jerusalem.
(7) Winkler, E. (1867): Die Lehe Von der Elasticitaet und Festigkeit, Prag, Dominicus.

(Received February 15, 1972)