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ON STRESS-DILATANCY RELATION OF SAND IN SIMPLE SHEAR TEST

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ABSTRACT

A stress-dilatancy relation of sand is proposed on the basis of the experimental fact that the principal stress axes and the principal strain increment axes rotate without complete coincidence with each other during monotonous increace of horizontal shear stress acting on a sample in a simple shear test. The inclination angle ψ of the major principal stress axis to the vertical direction is deduced from a simple relationship ; $\tau/\sigma_N=\kappa\tan\psi$, which is derived from theoretical consideration (Oda and Konishi, 1974) and verified by Cole's experimental results. The value of κ is a material constant for a given sand which is dependent only on the interparticle friction angle. By using this relation, the principal stress parameters such as mean stress $(\sigma_1+\sigma_3)/2$ and maximum shear stress $(\sigma_1-\sigma_3)/2$ are easily deduced from the shear stress τ and the normal stress σ_N acting on the horizontal shear plane in a simple shear test or even in a direct shear test.

Key words: dilatancy, progressive failure, sand, simple shear, soil structure
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INTRODUCTION

Some experimental studies on the stress-strain behaviour of sand sheared in a simple shear apparatus have been carried out with the following conclusions (Oda and Konishi, 1974):

1) As a first approximation, sand is considered to be a plastic material which undergoes anisotropic strain hardening and is composed of rigid, cohesionless particles.

2) Both of principal axes of stress and of strain increment rotate gradually during monotonous increase of shear stress applied parallel to the horizontal plane of a simple shear apparatus.

3) The principal axes of stress and of strain increment do not generally coincide, especially at an early stage of the test.

From these experimental facts, a stress-dilatancy relation of sand will be proposed in this paper.

STRAIN INCREMENTS IN SIMPLE SHEAR TEST

The principal stresses acting on a sample in a simple shear apparatus are usually estimated on making some assumption about their orientation with respect to the horizontal shear plane. An assumption that the principal axes of stress and of strain increment coincide was used with success in interpreting the results of drained tests on Leighton

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Buzzard sand (Arthur, James and Roscoe, 1964). From the extensive studies on the orientation of principal stress by the improved simple shear apparatus, Cole (1967) and Roscoe, Bassett and Cole (1967) also pointed out that this assumption should be a basis to interpret the experimental facts obtained from the simple shear test, whatever the state path, and that the other assumptions such as the horizontal plane = a plane of maximum shear stress and the horizontal plane = a plane of maximum obliquity did not predict accurately the directions of the principal stress axes.

Bassett (1967), however, found that the principal axes of stress and of strain increment diverged when the sample was sheared beyond the peak stress ratio in drained tests. Oda and Konishi (1974) also pointed out that both of these principal axes did not always coincide at least up to the peak stress ratio in the two-dimensional simple shear tests on circular cylinder assemblies packed at random. Therefore, the discrepancy between these two kinds of principal axes which occurs at least at an early stage of test should be considered to discuss the stress-dilatancy relation of sand.

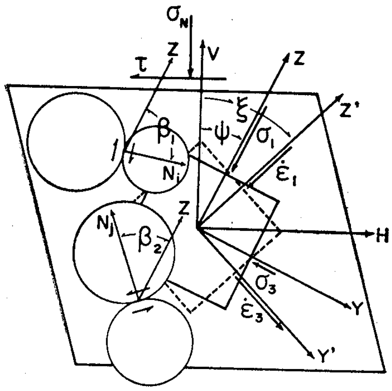

Fig. 1. Axes Y and Z of principal stresses and axes Y' and Z' of principal strain increments. In order to satisfy the condition of zero strain increment in the horizontal direction H, sliding movements which decide the strain increment of mass occur at contacts whose normals incline to the major principal stress axes Z at $\beta_1=\frac{\pi}{4}+\frac{1}{2}\phi_\mu+\bar{\theta}$ or at $\beta_2=\left|-\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+\bar{\theta}\right|$. If deviation angle $\bar{\theta}$ is zero (i.e., $\beta_1=\beta_2$), the principal stress axes (Y,Z) coincide with the principal strain increment axes (Y', Z').

Consider* the reason why the principal stress axes diverge from the principal strain increment axes during the simple shear of sand.

When shear stress τ is applied parallel to the horizontal (Fig. 1), the axis Z of major principal stress inclines to the vertical V at the angle ψ. Oda and Konishi (1974) inferred that there are some critical contacts which slid under the minimum principal stress ratio for a given granular fabric of sand and that these critical contacts should have the inclination angles β_i of their normals N_i to the major principal stress axis Z at $\pm\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)$ (clockwise being positive as shown). However, the condition of zero strain increment in the horizontal direction in the simple shear tests leads to obstruct full sliding at these critical contacts even when the necessary condition of forces for sliding is satisfied, because the full sliding at these contacts results in extensional or compressional strain increment in the horizontal direction. Therefore, it is reasonable to consider that the slidings must occur at some contacts deviating slightly from the critical contacts so as to satisfy the condition of the zero strain increment in the horizontal direction. This is the reason why the principal axes of stresses do not coincide with the principal axes of strain increments. Let the average deviation angle be equal to the angle $\bar{\theta}$. Then the sliding movements which give decisive influence on the strain increments of the assembly must occur at the contacts whose normals N_i incline to the major principal stress direction at $\beta_1=\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+\bar{\theta}$ or at $\beta_2=\left|-\left(\frac{\pi}{4}\right.\right.$

* Two-dimensional model composed of right circular cylinders will be discussed to avoid the intricate description of theory in this paper. The theoretical equations derived on the basis of the two-dimensional model, however, can be applied to the behaviour of sand composed of non-spherical grains.

$+\frac{1}{2}\phi_\mu\Big)+\bar{\theta}\Big|$ as shown in Fig. 1.

Now consider two solid paths passing entirely through the solid particles in the directions of Y and Z (Fig. 2). These paths were called "solid path" by Horne (1965) and Oda (1974). Since the principal axes of anisotropy with respect to the distribution of N_t nearly coincide with the principal axes of stresses (Biarez and Wiendieck (1963), Oda (1972), Oda and Konishi (1974)), the number m_Z of contacts traversed in proceeding a unit length parallel to the major principal stress direction Z from the contacts C_1 to C_{m_z} can be estimated by the mean radius $\bar{r}$ and the function $E(\beta)$. The function $E(\beta)$ is a probability density function to show the two-dimensional distribution of N_t (Oda, 1974).

$$m_Z=\frac{1}{2\bar{r}\int_{-\pi/2}^{\pi/2}2E(\beta)\cos\beta d\beta} \tag{1}$$

where the angle β is the inclination angle of N_t to the major principal stress axis (clockwise being positive). From the same consideration on the minor principal direction Y, we get

$$m_Y=\frac{1}{2\bar{r}\int_0^{\pi}2E(\beta)\sin\beta d\beta} \tag{2}$$

In proceeding entirely the solid path from a contact C_1 to another contact C_{m_z} in Fig. 2, the number of contacts whose normals lie within the angle ranges of $\beta_1-\frac{\Delta\beta}{2}$ to $\beta_1+\frac{\Delta\beta}{2}$ and of $-\beta_2-\frac{\Delta\beta}{2}$ to $-\beta_2+\frac{\Delta\beta}{2}$ are given by

$$2m_Z\{E(\beta_1)+E(-\beta_2)\}\Delta\beta \tag{3}$$

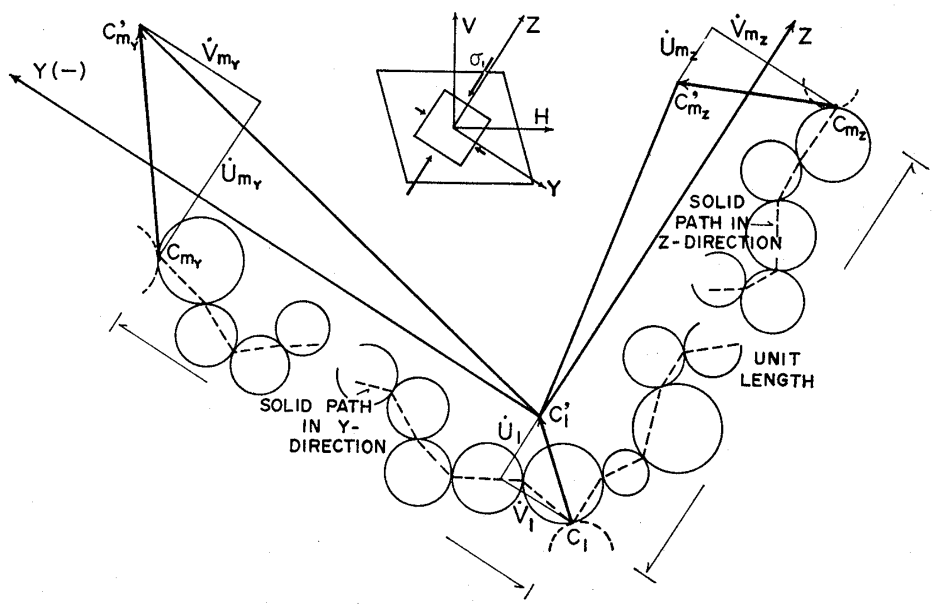

Fig. 2. Solid paths ($C_1\sim C_{mY}$ and $C_1\sim C_{mZ}$) in the principal stress directions Y and Z. When granular assembly undergoes deformation increment due to the sliding movements at contacts, the solid paths ($C_1\sim C_{mY}$ and $C_1\sim C_{mZ}$) are displaced into new positions ($C_1'\sim C_{mY}'$ and $C_1'\sim C_{mZ}'$).

where the angles β_1 and β_2 are equal to $\frac{\pi}{4}+\frac{1}{2}\phi_\mu+\bar{\theta}$ and $\left|-\left(\frac{\pi}{4}+\frac{\phi_\mu}{2}\right)+\bar{\theta}\right|$ respectively. It has been said that the sliding movements to decide the strain increments of the assembly occur at the contacts whose normals incline to the major principal stress axis at β_1 and $-\beta_2$. Thus, Eq. (3) gives the number of sliding contacts along the solid path from the contact C_1 to the contact C_{mZ}.

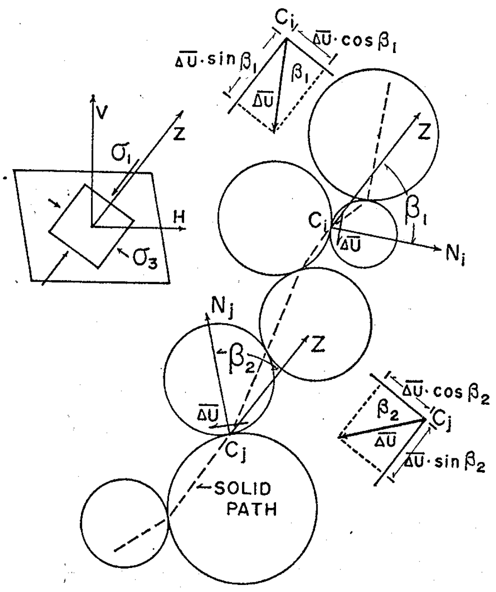

Fig. 3. Component of shortening in the major principal stress axis Z due to sliding at a contact whose normal N_i inclines to the axis Z at β_1 or β_2. Shortening of solid path ($C_1 \sim C_{mZ}$) in the direction of Z is equal to the summation of components of shortening (i.e., $\Sigma\overline{\Delta U}\sin\beta_{1\text{ or }2}$).

The granular assembly undergoes nonrecoverable deformation due to the sliding movements at all of these sliding contacts in the entirely solid paths. Then, $(\dot{U}_1, \dot{V}_1)$ are the components of displacement increment of the contact C_1 in the principal stress directions Z and Y, and $(\dot{U}_{mY}, \dot{V}_{mY})$ and $(\dot{U}_{mZ}, \dot{V}_{mZ})$ give the components of displacements of the contacts C_{mY} and C_{mZ} respectively as shown in Fig. 2.

Let the mean increment of sliding lengths at these sliding contacts be $\overline{\Delta U}$ as shown in Fig. 3. The component of shortening in the direction Z due to sliding at a contact whose normal N_i inclines to the axis Z at β_1 or $-\beta_2$ can be given by $\overline{\Delta U}\sin\beta_1$ or $\overline{\Delta U}\sin\beta_2$. Since the number of sliding contacts within the solid path from the contact C_1 to the contact C_{mZ} is given by Eq. (3), the summation of component of shortening in the direction Z can be represented by

$$2\overline{\Delta U}m_Z\{E(\beta_1)\sin\beta_1+E(-\beta_2)\sin\beta_2\}\Delta\beta \quad (4)$$

Eq. (4) gives the value of shortening in length of $\overline{C_1C_{mZ}}$; that is, $(\dot{U}_{mZ}-\dot{U}_1)$. As the length of $\overline{C_1C_{mZ}}$ in the direction Z is unit, we get

$$\dot{\varepsilon}_Z=2\overline{\Delta U}m_Z\{E(\beta_1)\sin\beta_1+E(-\beta_2)\sin\beta_2\}\Delta\beta \quad (5)$$

where $\dot{\varepsilon}_Z$=compresssional strain increment in the direction Z (compressional strain increment being positive). In the same manner, it can be shown that the extensional strain increments $\dot{\varepsilon}_Y$ and the shear strain increment $\dot{\gamma}_{ZY}$ are also given by $(\dot{V}_{mY}-\dot{V}_1)$ and $(\dot{V}_{mZ}-\dot{V}_1)+(\dot{U}_{mY}-\dot{U}_1)$ respectively, as follows:

$$\dot{\varepsilon}_Y=-2\overline{\Delta U}m_Y\{E(\beta_1)\cos\beta_1+E(-\beta_2)\cos\beta_2\}\Delta\beta \quad (6)$$

$$\begin{aligned}\dot{\gamma}_{ZY}&=\dot{\gamma}_{YZ}\\&=2\overline{\Delta U}m_Z\{-E(\beta_1)\cos\beta_1+E(-\beta_2)\cos\beta_2\}\Delta\beta\\&\quad+2\overline{\Delta U}m_Y\{E(\beta_1)\sin\beta_1-E(-\beta_2)\sin\beta_2\}\Delta\beta\end{aligned} \quad (7)$$

The probability density function $E(\beta)$ in Eqs. (5) to (7) is nearly symmetrical about the axis Z (Oda and Konishi, 1974). Therefore, the assumption that $E(\beta_1)=E(-\beta_2)$, strictly speaking, is not correct. We, however, consider that the relation $E(\beta_1)\fallingdotseq E(-\beta_2)$ is nearly hold in the granular assembly when the deviation angle $\bar{\theta}$ is not so large*.

* It is reasonable to consider that the deviation angle $\bar{\theta}$ is less than 20° in a usual assembly of particles because the discrepancy between the principal axes of stress and of strain increment is also less than 20° (according to the Cole's experiment). The experimental evidence given by Oda and Konishi (1974) shows that the relation $E(\beta_1)\fallingdotseq E(-\beta_2)$ is nearly satisfied when the deviation angle is less than 20°.

MOHR'S CIRCLE OF STRAIN INCREMENT

During a simple shear test, the boundary displacements δ and h are measured as shown in Fig. 4-b. Because the strain distribution within the sample is considered to be uniform, the strain increments in the whole sample are easily given by the measurements $\dot{\delta}$ and $\dot{h}$ as follows:

$$\dot{\varepsilon}_H=0,\ \ \dot{\varepsilon}_V=\dot{h}/D,\ \ \dot{\gamma}_{VH}=\dot{\gamma}_{HV}=\dot{\delta}/D \tag{8}$$

where D is the thickness of sample.

Fig. 4-a shows the Mohr's circle for the strain increments. The angles ψ and ξ in this figure give the inclination angles of the major principal axes of stress and of strain increment to the vertical V respectively. The values of strain increments in the directions V, Z, H and Y are presented by the points A, B, C and D on the Mohr's circle respectively.

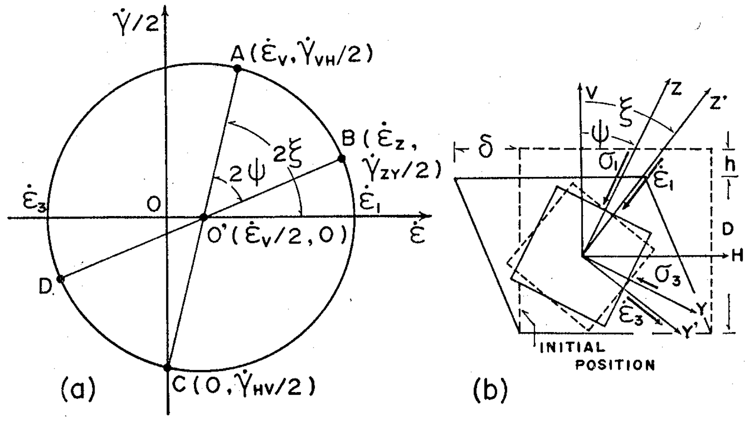

Fig. 4. Mohr's circle for strain increment. The coordinates represented by the points A, B, C and D correspond to the value of strain increments in the directions V, Z, H and Y respectively.

On drawing the Mohr's circle, the following facts must be taken into consideration or must be satisfied:

1) Since the axis H corresponds to the horizontal which is parallel to the no-extension line, the point C must be at the coordinate of $(0,\ \dot{\gamma}_{HV}/2)$.

2) The axis V is perpendicular to the axis H.

3) The major principal stress axis Z inclines to the vertical V and the major principal strain increment axis at the angles ψ and $(\xi-\psi)$ respectively (see, Fig. 4-b).

4) The strain increments $\dot{\varepsilon}_Z$ and $\dot{\gamma}_{ZY}$ in the major principal stress axis (given by the point B on the Mohr's circle) must be equal to the strain increments given by Eqs. (5) and (7).

5) The angle ξ which is the inclination angle of the major principal strain increment axis to the vertical must be determined by the condition of no-extension parallel to the axis H. That is, from the Mohr's circle of strain increment,

$$\tan 2\xi=\frac{\dot{\gamma}_{VH}}{\dot{\varepsilon}_V}=\frac{\dot{\delta}}{\dot{h}} \tag{9}$$

6) There must be a unique relationship between the stress ratio τ/σ_N acting on the horizontal and the inclination angle ψ during a simple shear test, as pointed out by Cole (1967). Oda and Konishi (1974) also got the following relationship;

$$\frac{\tau}{\sigma_N}=\kappa\cdot\tan\psi \tag{10}$$

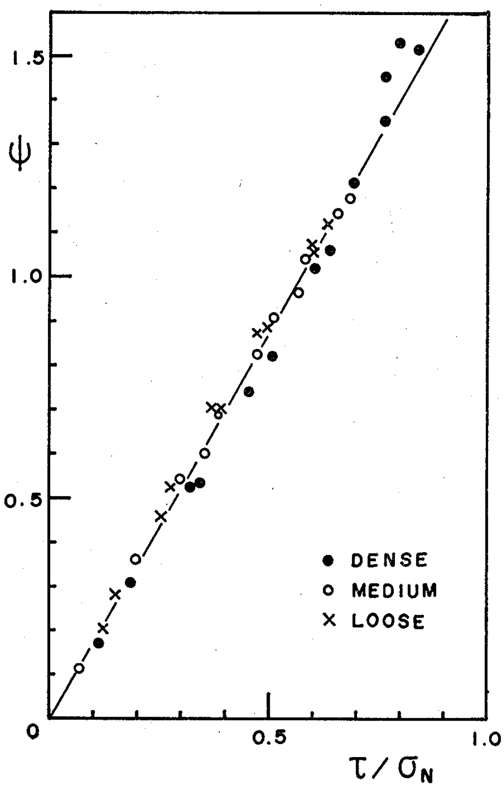

Fig. 5. Linear relationship between τ/σ_N and $\tan\psi$ for drained simple shear tests on Leighton Buzzard sand

where the constant κ is determined only by the interparticle friction angle ϕ_μ for a given material, not by other experimental conditions such as the initial void ratio and the normal stress. The relationship between τ/σ_N and ψ experimentally determined by Cole (1967) is redrafted in Fig. 5 as the relationship between τ/σ_N and $\tan\psi$. According to Fig. 5, the constant κ for Leighton Buzzard sand is equal to 0.58. Therefore, the angle ψ in the Mohr's circle of Fig. 4 must satisfy the relationship given by Eq. (10).

7) The inclination angle of the major principal stress axis to the major principal strain increment axis (i. e., $(\xi-\psi)$) must decrease with the increase of shear distortion during drained simple shear tests (see Figs. 6 and 7 quoted from Cole (1967)).

From the geometrical considerations on the Mohr's circle in Fig. 4, the principal strain increments which are represented by the notations $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_3$ in Fig. 4 are given by the following equations:

$$\left.\begin{aligned}\dot{\varepsilon}_1&=\frac{1}{2}(\dot{\varepsilon}_Z+\dot{\varepsilon}_Y)+\frac{1}{2\cdot\cos2(\xi-\psi)}(\dot{\varepsilon}_Z-\dot{\varepsilon}_Y)\\ \dot{\varepsilon}_3&=\frac{1}{2}(\dot{\varepsilon}_Z+\dot{\varepsilon}_Y)-\frac{1}{2\cdot\cos2(\xi-\psi)}(\dot{\varepsilon}_Z-\dot{\varepsilon}_Y)\end{aligned}\right\} \tag{11}$$

In the same way, the strain increments $\dot{\varepsilon}_V$ and $\dot{\gamma}_{VH}$ in the vertical direction are also given by the following equations:

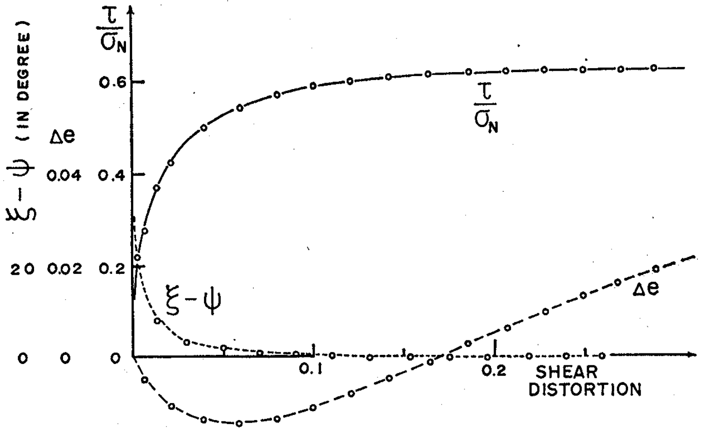

Fig. 6. Change of $(\xi-\psi)$ with the increase of shear distortion during drained simple shear test on loosely compacted Leighton Buzzard sand (after Cole, 1967)

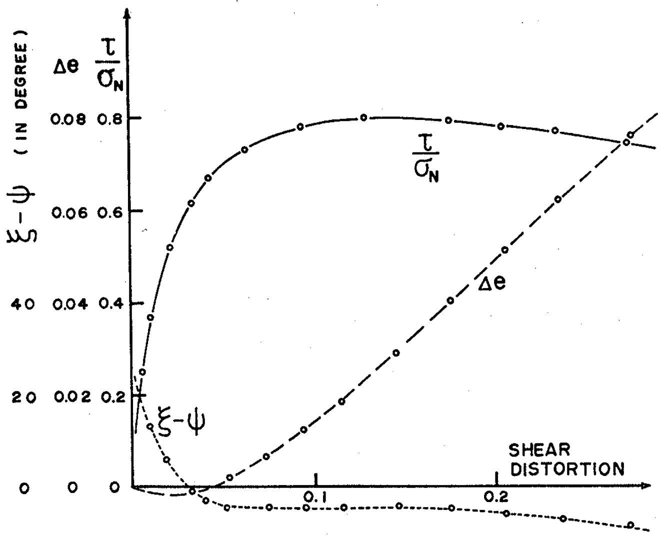

Fig. 7. Change of $(\xi-\psi)$ with the increase of shear distortion during simple shear test on densely compacted Leighton Buzzard sand (after Cole, 1967)

$$\left.\begin{aligned}\dot{\varepsilon}_V&=\dot{\varepsilon}_Z+\dot{\varepsilon}_Y\\ \dot{\gamma}_{VH}&=\frac{\sin 2\xi}{\cos 2(\xi-\psi)}(\dot{\varepsilon}_Z-\dot{\varepsilon}_Y)\end{aligned}\right\}\tag{12}$$

Of course, Eqs. (11) and (12) satisfy the requirements described above.

STRESS-DILATANCY RELATION IN SIMPLE SHEAR TEST

The volumetric strain increment $\dot{v}$ and the maximum shear strain increment $\dot{\gamma}$ are given by

$$\left.\begin{aligned}\dot{v}&=\dot{\varepsilon}_1+\dot{\varepsilon}_3=\dot{\varepsilon}_Z+\dot{\varepsilon}_Y\\ \dot{\gamma}&=(\dot{\varepsilon}_1-\dot{\varepsilon}_3)=\frac{1}{\cos 2(\xi-\psi)}(\dot{\varepsilon}_Z-\dot{\varepsilon}_Y)\end{aligned}\right\}\tag{13}$$

The strain increment ratio of $\dot{v}$ to $\dot{\gamma}$ is given by

$$\frac{\dot{v}}{\dot{\gamma}}=\cos 2(\xi-\psi)\frac{(\dot{\varepsilon}_Z+\dot{\varepsilon}_Y)}{(\dot{\varepsilon}_Z-\dot{\varepsilon}_Y)}\tag{14}$$

Inserting Eqs. (5) and (6) into Eq. (14) and rearranging on the assumption $E(\beta_1)=E(-\beta_2)$:

$$\begin{aligned}\frac{\dot{v}}{\dot{\gamma}}&=\cos 2(\xi-\psi)\cdot\frac{m_Z(\sin\beta_1+\sin\beta_2)-m_Y(\cos\beta_1+\cos\beta_2)}{m_Z(\sin\beta_1+\sin\beta_2)+m_Y(\cos\beta_1+\cos\beta_2)}\\ &=\cos 2(\xi-\psi)\cdot\frac{\tan\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-m_Y/m_Z}{\tan\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+m_Y/m_Z}\end{aligned}\tag{15}$$

where

$$\frac{m_Z}{m_Y}=\frac{\int_{-\pi/2}^{\pi/2}2E(\beta)\cos\beta d\beta}{\int_0^{\pi}2E(\beta)\sin\beta d\beta}\tag{16}$$

Oda and Konishi (1974) have obtained the following basic relationship between principal stress ratio σ_1/σ_3 and granular fabric represented by $E(\beta)$:

$$\frac{\sigma_1}{\sigma_3}=\frac{\int_{-\pi/2}^{\pi/2}2E(\beta)\cos\beta d\beta}{\int_0^{\pi}2E(\beta)\sin\beta d\beta}\cdot\tan^2\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right) \tag{17}$$

where σ_1 and σ_3 are major and minor principal stresses acting on the sample in the directions Z and Y respectively (see Fig. 1). Inserting Eq. (17) into Eq. (15), we get the following equation:

$$\frac{\dot{v}}{\dot{\gamma}}=\cos 2(\xi-\psi)\cdot\frac{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-\frac{\sigma_1}{\sigma_3}}{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+\frac{\sigma_1}{\sigma_3}} \tag{18}$$

By the notations that the mean normal stress $s=(\sigma_1+\sigma_3)/2$ and the maximum shear stress $t=(\sigma_1-\sigma_3)/2$, Eq. (18) gives the stress-dilatancy relation in terms of s, t, $\dot{v}$ and $\dot{\gamma}$:

$$\frac{t}{s}=\frac{\cos 2(\xi-\psi)\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-1\right\}-\frac{\dot{v}}{\dot{\gamma}}\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+1\right\}}{\cos 2(\xi-\psi)\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+1\right\}-\frac{\dot{v}}{\dot{\gamma}}\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-1\right\}} \tag{19}$$

Cole (1967) showed the experimental relationship between the rate of contraction $\dot{v}/\dot{\gamma}$ and the stress ratio t/s for the drained tests on Leighton Buzzard sand as shown by Fig. 8 for loose and medium samples and by Fig. 9 for dense samples. According to these figures, there is a unique relationship between $\dot{v}/\dot{\gamma}$ and t/s for a given range of void ratio.

The solid lines in Figs. 8 and 9 show the theoretical results derived from Eq. (19) on the assumptions that $\xi=\psi$ and $\phi_\mu=23°$. The broken lines A and B in Fig. 8 are also obtained on the assumption that the values of $(\xi-\psi)$ are 15° and 20° respectively which are very probable values of $(\xi-\psi)$ at an early stage of shear distortion as shown in Figs. (6) and (7). These broken lines are drafted only when $\dot{v}/\dot{\gamma}\geq 0$, because ξ is nearly equal

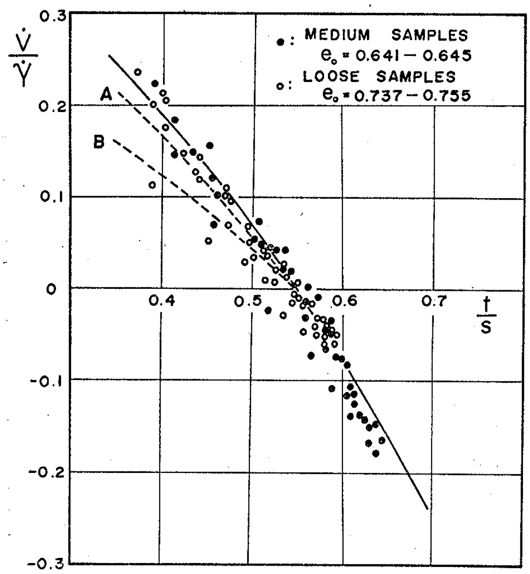

Fig. 8. **Relationship between the rate of contraction $\dot{v}/\dot{\gamma}$ and stress ratio t/s for drained tests on loose and medium Leighton Buzzard sand (after Cole, 1967)**

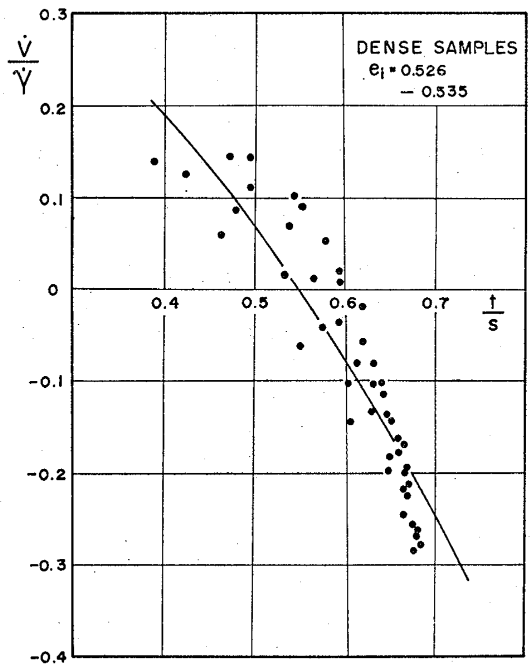

Fig. 9. Diagram similar to Fig. 8 for drained tests on dense Leighton Buzzard sand (after Cole, 1967)

to ψ for Leighton Buzzard sand when $\dot{v}/\dot{\gamma}<0$.

The following facts can be detected in Figs. 8 and 9:

1) Experimental data for dense sample (Fig. 9) slightly deviate from the theoretical line, especially at the condition of large stress ratio t/s. This reason can not be clearly explained because the following two problems were left unsolved in the preparations of Figs. 8 and 9 (see Cole, 1967):

a) non-homogeneous strain distribution in dense samples, and

b) elastic strain in dense samples which is not negligible as compared with plastic strain.

It is worthy of note that the stress-dilatancy relation by Eq. (19) is in good agreement with the experimental results on loose and medium samples in which strain distributions are homogeneous and elastic strains are negligible (Fig. 8).

2) When taking account of the scatter in the experimental results, it is reasonable to consider that there is only slight influence of the discrepancy between the principal axes of stress and of strain increment on the stress-dilatancy relation of sand.

The stress-dilatancy theory orginally proposed by Rowe (1962) which has been shown to hold for granular media in triaxial compression tests (Rowe, 1962) and extension tests (Barden and Khayatt, 1966) was further extended so as to be applied to the simple shear tests of sand by Rowe (1969). In the case of plane strain, the principal stress ratio R and the dilatancy D are given by:

$$\left.\begin{aligned} R&=\frac{\sigma_1}{\sigma_3}=\frac{1+t/s}{1-t/s} \\ D&=-\frac{\dot{\varepsilon}_3}{\dot{\varepsilon}_1}=\frac{1-\dot{v}/\dot{\gamma}}{1+\dot{v}/\dot{\gamma}} \end{aligned}\right\} \qquad (20)$$

Substituting Eq. (20) into the relation $R=DK$ and rearranging;

$$\frac{t}{s}=\frac{(K-1)-\dot{v}/\dot{\gamma}(K+1)}{(K+1)-\dot{v}/\dot{\gamma}(K-1)} \qquad (21)$$

This equation is quite similar to Eq. (19) except for the following two fundamental points:

1) The value of $\cos 2(\xi-\psi)$ in Eq. (19) presents the effect of the discrepancy between the principal axes of stress and of strain increment on the stress-dilatancy relation of sand.

2) The value of K in Eq. (21) is equal to $\tan^2\left(\frac{\pi}{4}+\frac{1}{2}\phi_{cv}\right)$ where ϕ_{cv} is equal to the internal friction angle at the critical void ratio state of sand. The term corresponding to K in Eq. (19) is $\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)$. It seems to the present author that the angle of ϕ_{cv} is a semi-empirical friction angle with no physically distinct definition. When the angle of ϕ_{cv} for Leighton Buzzard sand is 35°, the value, 3.72, of $\tan^2\left(\frac{\pi}{4}+\frac{1}{2}\phi_{cv}\right)$ is nearly equal to the value, 3.65, of $\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)$, although this near agreement may be of accidental.

DETERMINATION OF STRESS RATIO τ/σ_N AT $\dot{v}/\dot{\gamma}=0$

The stress ratio t/s at the rate of contraction $\dot{v}/\dot{\gamma}$ being zero is determined by Eq. (19) as follows:

$$\left(\frac{t}{s}\right)_0=\frac{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-1}{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+1} \qquad (22)$$

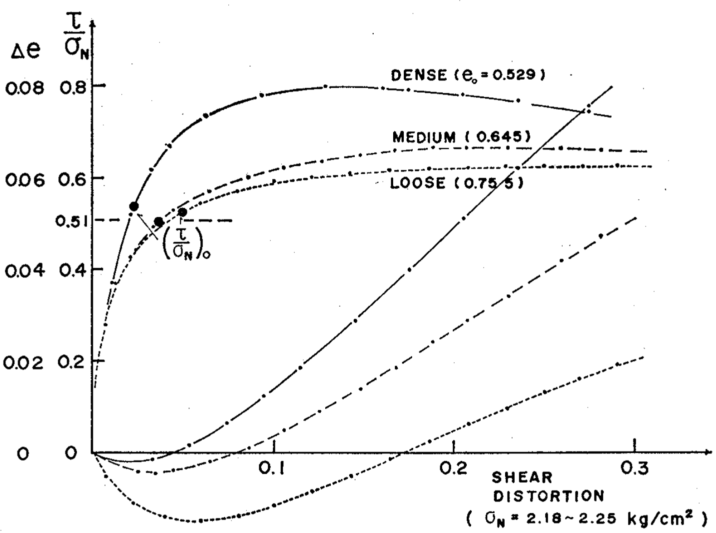

Fig. 10. Effect of initial void ratio on the value of $(\tau/\sigma_N)_0$ for a given normal stress

It is worthy of note that the stress ratio t/s at $\dot{v}/\dot{\gamma}=0$ is determined not by the inclination angles ξ and ψ, but only by the interparticle friction angle ϕ_μ.

Since $\left(\frac{t}{s}\right)_0=\frac{(\sigma_1/\sigma_3)_0-1}{(\sigma_1/\sigma_3)_0+1}$, Eq. (22) can be rewritten as follows:

$$\left(\frac{\sigma_1}{\sigma_3}\right)_0=\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right) \tag{23}$$

Eq. (23) can also be rewritten as follows:

$$\left(\frac{\sigma_1}{\sigma_3}\right)_0=\frac{\left(\frac{\tau}{\sigma_N}\right)_0(\cos 2\psi_0-1)-\sin 2\psi_0}{\left(\frac{\tau}{\sigma_N}\right)_0(\cos 2\psi_0+1)-\sin 2\psi_0}=\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right) \tag{24}$$

where $(\tau/\sigma_N)_0$ and ψ_0 represent the stress ratio acting on the horizontal and the inclination angle of major principal stress at $\dot{v}/\dot{\gamma}=0$ respectively.

Rearranging the relation given by Eq. (10), we get

$$\left.\begin{aligned}\sin\psi_0&=\frac{(\tau/\sigma_N)_0}{\sqrt{(\tau/\sigma_N)_0{}^2+\kappa^2}}\\ \cos\psi_0&=\frac{\kappa}{\sqrt{(\tau/\sigma_N)_0{}^2+\kappa^2}}\end{aligned}\right\} \tag{25}$$

Inserting Eq. (25) into Eq. (24) and rearranging;

$$\left(\frac{\tau}{\sigma_N}\right)_0^2=\kappa\left\{(1-\kappa)\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-1\right\} \tag{26}$$

The value of κ for Leighton Buzzard sand is 0.58 as shown in Fig. 5. The theoretically estimated value of $(\tau/\sigma_N)_0$ obtained when we insert this value of κ into Eq. (26) is 0.51.

Figs. 10 and 11 show a portion of drained tests carried out by Cole (1967) on loose, medium and dense samples of Lei-

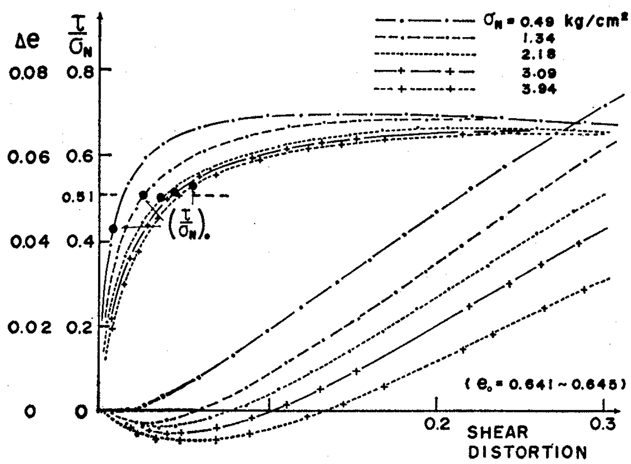

Fig. 11. Effect of average normal stress on the value of $(\tau/\sigma_N)_0$ for a given initial void ratio (after Cole, 1967)

ghton Buzzard sand at the vertical stresses ranging from 0.49 to 3.94kg/cm^2. From these Cole's experimental results, it is found that except for the test at the low vertical normal stress the stress ratio τ/σ_N at $\dot{v}/\dot{\gamma}=0$ is nearly constant, irrespective of a given initial void ratio and also a given normal stress. Besides, the values of $(\tau/\sigma_N)_0$ are nearly equal to the theoretically estimated value.

APPLICATION OF THE THEORY TO DIRECT SHEAR TEST

If it can be assumed that there is a narrow shear zone with a homogeneous stress and strain distribution as shown in Fig. 12, the theoretical equations described above are all valid to predict the stress-dilatancy relation of sand in a direct shear test.

The data of direct shear tests on loose and dense samples of Ottawa sand are quoted from Taylor (1948) as shown in Fig. 13. In order to apply Eq. (19) to Ottawa sand, the inclination angle ψ must be determined.

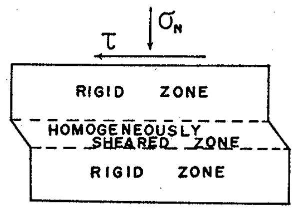

Fig. 12. Shear zone with homogeneous stress and strain distribution in direct shear test

Because of the mineral composition (quartz) and the rounded shape of grains of Ottawa sand, it is reasonable to use 22° as its interparticle friction angle ϕ_μ. The change of inclination angle ψ during shear test can be determined if the value of κ is estimated from Eq. (26) by the value of ϕ_μ (22°) and the value of $(\tau/\sigma_N)_0$ obtained from the test on the loose sample. (Even though Eq. (26) gives two values of κ for Ottawa sand, $\kappa=0.58$ is used in the following calculation.) The inclination angle ψ thus determined are plotted in Fig. 13 against the shear displacement up to the peak stress ratio. The gradual rotation of major principal axis during the shear is quite similar to those of simple shear tests on Leighton Buzzard sand by Cole (1967).

Fig. 14 shows the stress-dilatancy relation of Ottawa sand in terms of t/s and $\dot{v}/\dot{\gamma}$.

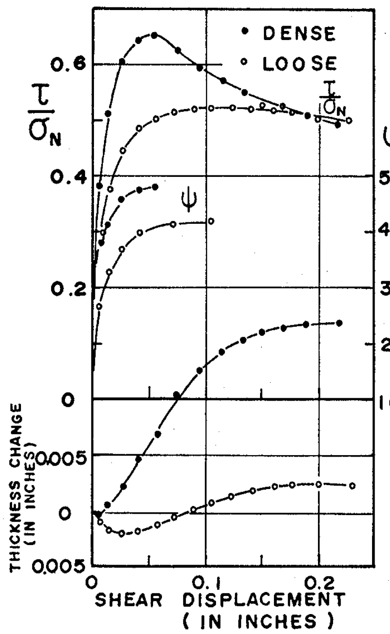

Fig. 13. Direct shear tests on Ottawa standard sand (after Taylor) and change of the inclination angle ψ of major principal stress axis during shear test

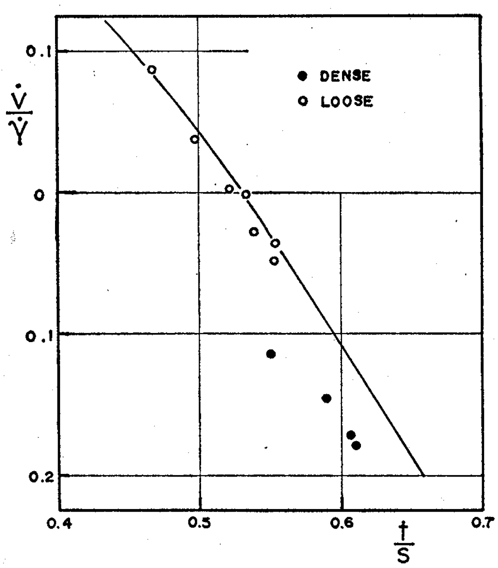

Fig. 14. Stress-dilatancy relation for drained direct shear tests on loose and dense samples of Ottawa sand

When calculating the value of $\dot{v}/\dot{\gamma}$, it was assumed that the thickness of homogeneously sheared zone was nearly constant during shear test. A solid line in this figure shows the result of the application of Eq. (19) to Ottawa sand on the basis that $\phi_\mu=22°$ and $\xi=\psi$. Although there is a very nice agreement between theoretical data and experimentally obtained ones for the loose sample, experimental data for the dense sample show fairly lower $\dot{v}/\dot{\gamma}$ values than theoretical estimation. This must be chiefly due to the erroneous assumption that there should be a narrow shear zone with a homogeneous stress and strain distribution even in a direct shear test on dense sample.

CONCLUSIONS

A simple shear or direct shear test of sand is characterized by the fact that the principal axes of stress and of strain increment rotate during the test without complete coincidence between them. On the other hand, the complete coincidence between these two types of principal axes is held during the deformation in triaxial test without any rotation. Therefore, the stress-dilatancy relation derived from the triaxial tests of sand cannot always be applicable to the simple or direct shear test. In this paper, the author proposed a stress-dilatancy rule for the simple shear test of sand in terms of principal stresses and of principal strain increment by treating sand as an assembly of rigid, cohesionless particles. The following conclusions were obtained:

1) The stress ratio τ/σ_N acting on the horizontal plane in a simple shear test has the following relation to the inclination angle ψ of the major principal stress axis to the vertical:

$$\frac{\tau}{\sigma_N}=\kappa\cdot\tan\psi$$

where the constant κ depends only on the interparticle friction angle.

2) The value of κ in the above equation is considered to be an important material constant. This value can be estimated by inserting the value of interparticle friction angle ϕ_μ and the stress ratio $(\tau/\sigma_N)_0$ at dilatancy rate being zero into the following equation:

$$\left(\frac{\tau}{\sigma_N}\right)_0^2=\kappa\left\{(1-\kappa)\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-1\right\}$$

3) A flow rule, taking into account the discrepancy between principal axes of stress and of strain increment, is represented in terms of the rate of contraction $\dot{v}/\dot{\gamma}$ and the stress ratio t/s as follows:

$$\frac{t}{s}=\frac{\cos 2(\xi-\psi)\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-1\right\}-\frac{\dot{v}}{\dot{\gamma}}\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+1\right\}}{\cos 2(\xi-\psi)\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)+1\right\}-\frac{\dot{v}}{\dot{\gamma}}\left\{\tan^3\left(\frac{\pi}{4}+\frac{1}{2}\phi_\mu\right)-1\right\}}$$

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NOTATION

D = thickness of sample sheared in a simple shear apparatus

$E(\beta)$ = probability density function to show two-dimensional distribution of N_t

h = thickness change during a simple shear test

N_i = normals to tangential planes at contacts

s = mean normal stress $\left(=\frac{\sigma_1+\sigma_3}{2}\right)$

t = maximum shear stress $\left(=\frac{\sigma_1-\sigma_3}{2}\right)$

$\dot{U}$, $\dot{V}$ = components of displacement increment in the principal stress directions

$\dot{v}$ = volumetric strain increment

X, Y, Z = reference axes to show the principal stress directions in a simple shear test

β = inclination angle of N_i to the major principal stress axis

$\dot{\gamma}$ = maximum shear strain increment

$\dot{\varepsilon}$ = principal strain increment

κ = constant for a given material

ξ = inclination angle of the major principal strain increment axis to the vertical direction

$\frac{\tau}{\sigma_N}$ = stress ratio acting on the horizontal shear plane in a simple shear test

σ_1, σ_3 = principal stresses

ϕ_μ = interparticle friction angle

ϕ_{cv} = internal friction angle at critical void ratio state

ψ = inclination angle of the major principal stress axis to the vertical direction

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