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SEISMIC SHEAR WAVE PROPAGATION THROUGH EARTH DAMS

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ABSTRACT

Two methods are presented to study the one-dimensional propagation of seismic shear waves through tapered cross sections of earth dams with truncated crests. The earth dam material is assumed to be unsaturated and linear viscoelastic, or nonlinear strain-softening. The shear modulus may be considered to vary with depth or to be constant throughout the dam cross section. A closed form solution which involves Hankel functions with complex arguments is developed in case the earth dam material is viscoelastic and the dam base is subject to harmonic excitations. The method is extended to cases of random seismic vibrations by employing Fourier analysis in conjunction with a least squares criterion. Response curves are obtained for viscously damped tapered dam cross sections with truncated crests, by using the analytical solution developed. The method of characteristics is also used to provide a solution to the shear wave propagation problem. The method of characteristics exhibits versatility in handling different descriptions of dynamic material response, such as "strain-softening" material behavior laws. Three examples are presented to illustrate the application of the two methods and to demonstrate the relative simplicity and flexibility of the method of characteristics.

Key words: damping, earthquake, elasticity, viscoelasticity, wave propagation

IGC: G 2/G 6

INTRODUCTION

One of the first dynamic response analyses of earth dams was presented by Mononobe, Takata, and Matsumura (8) in 1936. Their theoretical development was based on the following simplifying assumptions : the earth dam material was homogeneous and viscoelastic having uniform density and shear modulus; the earth dam cross section was wedge-shaped and the foundation was rigid; shear stresses over any horizontal plane were assumed to be uniformly distributed; the dam was infinitely long and its base width was greater than its height so that bending deformations could be considered negligible compared with deformations due to shear; and the water stored in the earth dam was not considered in the analysis.

Hatanaka (5) in 1955 studied the case of a triangular elastic cross section in a rectangular canyon and computed the horizontal response over the length and the height of the dam. Ambraseys (2, 3, 4) studied extensively the dynamic reaction of dams to earthquakes. In

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1960, he investigated the shear response of a two-dimensional symmetric wedge of finite length which had a truncated crest (1). The wedge was considered to be linearly elastic, bounded at its base and its two vertical sides by rigid planes. Internal or Coulomb friction was modeled by a viscous damping term proportional to the particle velocities. The shear modulus was assumed to be constant since Ambraseys' previous work showed that an error of less than 10% occurred in computing natural frequencies when the variable shear modulus was replaced with its mean value.

In 1966, Seed and Martin (16) presented the one-dimensional shear slice theory. The differential equation of motion was solved analytically for a random horizontal base motion. Viscous damping forces were considered to be proportional to the particle velocities. Making use of the orthogonal properties of mode shapes and the principle of mode superposition, the general solution of the equation of motion was obtained as a summation of an infinite number of terms involving Duhammel (Convolution) integrals. The dissipation mechanism was represented in the solution by the fraction of critical damping for each mode. Results obtained by the above analysis were of similar form to measurements of the response of the 100-ft (30.5m) high Sannokai Dam in Japan to several small earthquakes (11).

The one-dimensional shear wave propagation through tapered cross sections of earth dams with truncated crests is the subject of this paper. The earth dam material is assumed to be unsaturated, linear viscoelastic or strain-softening, with either constant or variable shear modulus. A closed form solution is obtained for the governing partial differential equation. This solution involves Hankel functions with complex arguments and is valid in case the material is linear and the dam base is subject to harmonic excitations. The method is extended to cases of random seismic vibrations by employing Fourier analysis and a least squares criterion. Response curves are obtained for viscously damped tapered dam cross sections with truncated crests, by using the analytical solution developed. The method of characteristics is also used to provide a solution to the shear wave propagation problem. The two methods compare favorably. Two examples are presented to illustrate the application of the two methods and to demonstrate the relative simplicity and flexibility of the method of characteristics. A third example illustrates the application of the method of characteristics to a case where soil properties are a function of depth and the soil behaves as a strain-softening material according to the Ramberg-Osgood hysteresis law (13, 17).

ANALYTICAL METHOD

The basic assumption of the one-dimensional shear slice theory, that shearing stress is uniformly distributed over any horizontal plane, is maintained through the following development. The earth dam material is assumed to be viscoelastic behaving as a Voigt solid. The dam cross sections considered are wedge-shaped with bank slopes 1 vertical to α horizontal, with truncated crests.

For one-dimensional shear wave transmission through an earth dam under dynamic conditions, the equation of motion for a thin horizontal slice at depth z below the crest 0 (Fig. 1), is

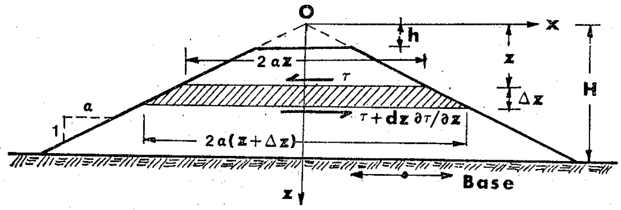

Fig. 1. One-dimensional shear slice

$$-\tau A_0+\left[\tau A_0+\frac{\partial(\tau A_0)}{\partial z}dz\right]=\rho dz\left[A_0+\frac{\partial A_0}{\partial z}\frac{dz}{2}\right]\frac{\partial^2 u}{\partial t^2} \quad (1)$$

where $A_0=2\alpha z$ is the area of the horizontal plane per unit length; τ=shearing stress uniformly distributed over the horizontal plane of area A_0; ρ=mass density of the soil; t=time; and u=absolute horizontal displacement of the slice. Omitting second-order differentials Eq. (1) reduces to:

$$\frac{\partial \tau}{\partial z}+\frac{\tau}{z}-\rho\frac{\partial^2 u}{\partial t^2}=0 \tag{2}$$

It should be noted that the dam face slope, α, cancels and does not appear in Eq. (2). The effect of the tapered cross section is depicted by the term (τ/z) for any degree of taper. For $\alpha \rightarrow 0$ this term vanishes and Eq. (2) reduces to the equation of motion for shear wave propagation through a one-dimensional shear beam. Values of α often encountered in practice are $1.5 \leq \alpha \leq 3$. For the same dam height such differences in the geometry of the cross section lead to differences in the natural modes, a result not obtainable from the shear slice theory. Therefore, the advantage of using two-dimensional analyses becomes apparent. However, the one-dimensional approach is a simple and inexpensive method to examine the reaction of earth dams to earthquakes. Results obtained by this approach are usually satisfactory along the dam's axis of symmetry.

The equation of state relating shearing stress-shearing strain in a viscoelastic material, with viscous damping proportional to the rate of change of strain, is:

$$\tau=G\frac{\partial u}{\partial z}+\mu\frac{\partial^2 u}{\partial z \partial t} \tag{3}$$

where G=shear modulus or modulus of rigidity; $\partial u/\partial z$ is the strain; and μ=soil viscosity. If Eq. (3) is differentiated with respect to z and Eq. (2) is substituted, the following equation is obtained:

$$\rho\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial z}\left(G\frac{\partial u}{\partial z}+\mu\frac{\partial^2 u}{\partial z \partial t}\right)-\frac{1}{z}\left(G\frac{\partial u}{\partial z}+\mu\frac{\partial^2 u}{\partial z \partial t}\right)=0 \tag{4}$$

The shear modulus and the viscosity may vary with depth in any prescribed manner. In this case the dam is divided into horizontal layers and G and μ are considered constant within each layer, having the value calculated at midthickness of the layer. Eq. (4) is solved for each layer in the dam starting from the bottom layer. The total dam response is obtained by combining the individual layer responses with the condition that stresses and displacements should be continuous at all layer interfaces. In order to simplify the presentation, the mean values of μ and G for the whole dam height are used in the following. Then, Eq. (4) reduces to:

$$\rho\frac{\partial^2 u}{\partial t^2}-G\left(\frac{\partial^2 u}{\partial z^2}+\frac{1}{z}\frac{\partial u}{\partial z}\right)-\mu\left(\frac{\partial^3 u}{\partial z^2 \partial t}+\frac{1}{z}\frac{\partial^2 u}{\partial z \partial t}\right)=0 \tag{5}$$

A steady oscillatory solution of angular frequency ω may be obtained by the standard separation of variables. If F is a function of z only:

$$\frac{d^2 F}{dz^2}+\frac{1}{z}\frac{dF}{dz}+\left(\frac{\omega}{v^*}\right)^2 F=0 \tag{6}$$

where

$$v^*=\sqrt{\frac{G}{\rho}+i\frac{\mu\omega}{\rho}} \tag{7}$$

The quantity v^* is called the complex velocity of shear wave propagation. If $\mu=0$, then the soil is elastic, $v^*=\sqrt{G/\rho}$ is the real shear wave velocity and Eq. (6) reduces to an ordinary Bessel equation. A general solution of Eq. (6) in terms of Hankel functions, is:

$$F(z)=AH_0^{(1)}(\omega z/v^*)+BH_0^{(2)}(\omega z/v^*) \tag{8}$$

where A and B are constants to be determined from the boundary conditions. The Hankel functions of order q are:

$$H_q^{(1)}(\omega z/v^*)=J_q(\omega z/v^*)+iY_q(\omega z/v^*) \tag{9}$$

$$H_q^{(2)}(\omega z/v^*)=J_q(\omega z/v^*)-iY_q(\omega z/v^*) \tag{10}$$

where J_q are Bessel functions of the first kind of order q, and Y_q are Bessel functions of the second kind of order q. If the viscous term is not zero, it is evident that the arguments of the Hankel and Bessel functions are complex numbers. In this case, a special technique to evaluate these functions is presented in the Appendix.

The constants in Eq. (8) are determined by use of the boundary conditions of zero shearing stress at $z=h$, and an imposed harmonic excitation of amplitude W and frequency ω at $z=H$. The general solution of Eq. (5), which is valid for $h>0$, is

$$u(z,\ t)=We^{i\omega t}\left(\frac{H_0^{(1)}(\omega z/v^*)-RH_0^{(2)}(\omega z/v^*)}{H_0^{(1)}(\omega H/v^*)-RH_0^{(2)}(\omega H/v^*)}\right) \tag{11}$$

where

$$R=H_1^{(1)}(\omega h/v^*)/H_1^{(2)}(\omega h/v^*) \tag{12}$$

The shearing stress, uniformly distributed over a horizontal cross section at depth z, can be found from the equation of state by substituting u from Eq. (11):

$$\tau(z,\ t)=\omega\rho v^* We^{i\omega t}\left(\frac{H_1^{(1)}(\omega z/v^*)-RH_1^{(2)}(\omega z/v^*)}{H_0^{(1)}(\omega H/v^*)-RH_0^{(2)}(\omega H/v^*)}\right) \tag{13}$$

The above formula apply only to steady oscillatory motions. If n equidistant displacement values are obtained from a digitized seismogram, the transient motion of the dam base can be analyzed to a series of harmonics by applying Fourier techniques and a least squares criterion (13). This procedure allows the user to select a number of harmonics less than the number of equidistant tabulated data points but sufficient to accurately represent the transient motion. The frequencies obtained from the harmonic analysis can be maintained without the need of introducing a constant critical damping ratio into Eq. (7). Each one of the harmonic components of the Fourier transform generates a solution of the form of Eq. (11). The superposition of these solutions by an Inverse Fourier Transform provides the transient response of the dam to the applied excitation. A computer program was written in FORTRAN IV to perform all the necessary calculations (13).

NATURAL FREQUENCIES OF TRUNCATED DAMS

Considering the undamped vibration of a wedge-shaped dam of height H, the natural frequencies of oscillation of the dam are obtained from the zero values of the frequency equation $J_0(\omega H\sqrt{\rho/G})=0$. Thus, for the first and second mode of vibration the natural frequencies are:

$$\omega_n=(2.41/H)\sqrt{G/\rho} \quad \text{and} \quad (5.52/H)\sqrt{G/\rho} \tag{14}$$

The degree of truncation h of the crest of a wedge-shaped dam as well as the amount of damping influence its natural frequency. The term in parenthesis in Eq. (11) is the amplification factor of the response. A plot of the modulus of the amplification factor versus the corresponding frequencies (Fig. 2), for a given degree of damping, reveals the response characteristics of the system. As an example, a dam 295.3 ft (90 m) high is considered having an average $G_0=3,912,000\,\text{lb/ft}^2$ ($19.1\times10^6\,\text{kg/m}^2$), Poisson ratio of 0.45, and $\rho=4.03\,\text{slugs/ft}^3$ ($2,077\,\text{kgm/m}^3$). The natural periods ($T_n=2\pi/\omega_n$) for the first and second mode calculated by different methods are presented in Table 1. For a non-truncated dam cross section, the natural periods computed by using the finite element method (12) which assumes the dam to be a two-dimensional body are somewhat larger than the values obtained by the closed form solution. The difference decreases as the dam sides become steeper. The truncation of the top of a wedge-shaped dam tends to lower the natural

Table 1. Natural periods of earth dams

Method	Natural period (seconds)		Earth dam characteristics
	First	Second	
Wedge-shaped, shear slice method	0.78	0.34	Elastic material $H=90$ m, $h=0$
Truncated wedge-shape, analytical method	0.78	0.34	Elastic material $H=90$ m, $h=3$ m
	0.79	0.35	Viscoelastic material $\mu=0.44\times10^6$ kg·s/m² $H=90$ m, $h=3$ m
	0.75	0.32	Elastic material $H=90$ m, $h=12.5$ m
	0.76	0.35	Viscoelastic material $\mu=0.44\times10^6$ kg·s/m² $H=90$ m, $h=12.5$ m
Method of finite elements (12)	0.81	0.44	Elastic material $H=90$ m, $h=0$, $\alpha=1.5$
	0.88	0.47	Elastic material $H=90$ m, $h=0$, $\alpha=3.0$

periods. However, small truncation influences the natural periods only slightly. The assumption that the earth dam material is viscoelastic, results in slightly higher natural periods (Fig. 2). A viscosity value of $\mu=90,000\,\mathrm{lb\cdot sec/ft^2}$ $(0.44\times10^6\,\mathrm{kg\cdot s/m^2})$ is used in the present example. The influence of damping on the amplitude of the response is apparent in Fig. 2. Response curves of the type of Fig. 2, obtained from Eq. (11), may be very useful in studying resonance effects in dams.

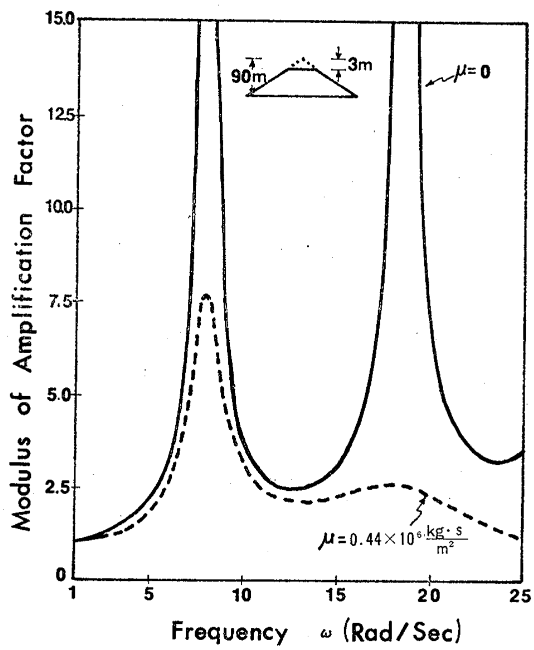

Fig. 2. Response curves for a truncated dam

CHARACTERISTICS METHOD

The same basic partial differential equations governing the propagation of shear waves through tapered cross sections are used in the following analysis; namely, the equation of motion (Eq. (2)) and the equation of state (Eq. (3)). If Eq. (3) is differentiated with respect to time, it can be written in terms of the horizontal particle velocities V:

$$\frac{\partial\tau}{\partial t}-G\frac{\partial V}{\partial z}-\mu\frac{\partial^2 V}{\partial z\partial t}=0 \tag{15}$$

The equation of motion (Eq. (2)) can also be written in terms of the horizontal particle velocities V:

$$\frac{\partial \tau}{\partial z}+\frac{\tau}{z}-\rho\frac{\partial V}{\partial t}=0 \tag{16}$$

The third term of Eq. (15) containing the viscosity μ may be represented by a finite difference approximation as:

$$\mu\frac{\partial^2 V}{\partial z\partial t}=\mu\frac{1}{\Delta t}\left[\frac{\partial V}{\partial z}-\left(\frac{\partial V}{\partial z}\right)_C\right] \tag{17}$$

where Δt is a selected time interval to be kept constant throughout the computations and the subscript C refers to the value determined at point C on the $z-t$ diagram of Fig. 3. At the nodal point C, the quantity: $(\partial V/\partial z)_C$ is considered constant and may be numerically approximated as:

$$\left(\frac{\partial V}{\partial z}\right)_C \simeq V_C{}^*=\frac{V_B-V_A}{z_B-z_A} \tag{18}$$

where the subscripts A and B refer to values at the corresponding nodes of the $z-t$ diagram of Fig. 3. The combination of Eqs. (15), (17), and (18) yields:

$$\frac{\partial \tau}{\partial t}-\left(G+\frac{\mu}{\Delta t}\right)\frac{\partial V}{\partial z}+\frac{\mu}{\Delta t}V_C{}^*=0 \tag{19}$$

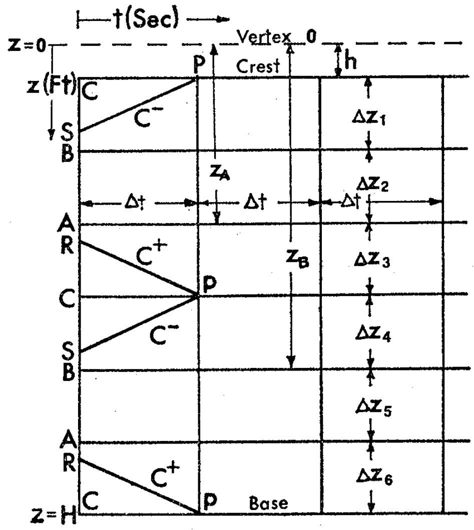

Fig. 3. ***z—t*** **diagram. Method of characteristics.**

Eqs. (16) and (19) are linear hyperbolic partial differential equations and can be transformed into four ordinary differential equations by using the method of characteristics. Eq. (16) is multiplied by an unknown multiplier θ and is added to Eq. (19) to give:

$$\left[\theta\frac{\partial \tau}{\partial z}+\frac{\partial \tau}{\partial t}\right]-\theta\rho\left[\frac{1}{\theta\rho}\left(G+\frac{\mu}{\Delta t}\right)\frac{\partial V}{\partial z}+\frac{\partial V}{\partial t}\right]+\theta\frac{\tau}{z}+\frac{\mu}{\Delta t}V_C{}^*=0 \tag{20}$$

The bracketed terms in Eq. (20) become total derivatives if:

$$\frac{dz}{dt}=\theta=\frac{1}{\theta\rho}\left(G+\frac{\mu}{\Delta t}\right) \tag{21}$$

Eq. (21) solved for θ gives:

$$\theta=\pm\sqrt{\frac{G}{\rho}+\frac{\mu}{\rho\Delta t}}=\pm v_s \tag{22}$$

where v_s is the apparent shear wave velocity in ft/sec, equal to the slope of the characteristic lines in the $z-t$ diagram of Fig. 3. Two pairs of ordinary differential equations originate from Eqs. (20) and (21), one pair with the positive value of $\theta(C^+)$ and the other pair with the negative value of $\theta(C^-)$:

$$C^+\left\{\begin{array}{ll}\dfrac{d\tau}{dt}-\rho v_s\dfrac{dV}{dt}+v_s\dfrac{\tau}{z}+\dfrac{\mu}{\Delta t}V_C{}^*=0 & (23)\\[2ex] \dfrac{dz}{dt}=v_s & (24)\end{array}\right.$$

$$C^-\left\{\begin{array}{ll}\dfrac{d\tau}{dt}+\rho v_s\dfrac{dV}{dt}-v_s\dfrac{\tau}{z}+\dfrac{\mu}{\Delta t}V_C{}^*=0 & (25)\\[2ex] \dfrac{dz}{dt}=-v_s & (26)\end{array}\right.$$

In cases where the dam crest is of finite width ($h>0$), solutions to the C^+ and C^- equations can be obtained after they are placed in a central finite difference form by integration. By choosing a convenient time interval, Δt, to be retained throughout the calculations, the finite difference expressions for Eqs. (23) and (25) are:

$$(C^+):\ \tau_P-\tau_R-\rho v_s(V_P-V_R)+\frac{\tau_P+\tau_R}{2}\ln\frac{z_P}{z_R}+\mu\frac{V_B-V_A}{z_B-z_A}=0 \quad (27)$$

$$(C^-):\ \tau_P-\tau_S+\rho v_s(V_P-V_S)-\frac{\tau_P+\tau_S}{2}\ln\frac{z_S}{z_P}+\mu\frac{V_B-V_A}{z_B-z_A}=0 \quad (28)$$

where the subscripts A, B, R, S, and P refer to values at the corresponding points of the $z-t$ diagram (Fig. 3), and distances z are measured from the vertex of the whole wedge being positive downwards. After a particular time interval, Δt, is selected, the dam height is partitioned into distance intervals (reaches) determined by:

$$\Delta z_j=\Delta t\sqrt{\frac{(G_0)_j}{\rho_j}+\frac{\mu_j}{\rho_j\Delta t}} \quad (29)$$

where G_0 is the initial shear modulus at zero strain level. Although G_0 usually changes with depth, it is considered constant within each reach, j, and the value calculated at midthickness of the reach is used.

In modeling the strain-softening inelastic soil behavior the Ramberg-Osgood stress-strain relationship causes the shear modulus to decrease with increasing strain, but permits the shear modulus to return to its initial low strain amplitude upon strain reversal. For the skeleton curve, which occurs for initial loading the Ramberg-Osgood relationship may be expressed as (13, 17):

$$\frac{\gamma G_0}{\tau_m}=\frac{\tau}{\tau_m}\left\{1+\left|\frac{\tau}{\tau_y}\right|^{R_0-1}\right\} \quad (30)$$

and for the unloading or reloading curves

$$\frac{(\gamma-\gamma_1)G_0}{\tau_m}=\frac{\tau-\tau_1}{\tau_m}\left\{1+\left|\frac{\tau-\tau_1}{2\tau_y}\right|^{R_0-1}\right\} \quad (31)$$

where τ_y is the yield shear stress, τ_m is the maximum shear stress considered equal to $\tau_m=1.25\ \tau_y$, and τ_1, γ_1 represent the last point of reversal of stress. The value of the tangent shear modulus, $d\tau/d\gamma$, obtained from the appropriate equation (30 or 31) is the value used at the corresponding strain level in the solution by the method of characteristics. Therefore, the shear modulus G changes during the transient and so does the apparent shear wave velocity v_s according to Eq. (22). However, $G\leq G_o$ and $v_s\Delta t\leq\Delta z$ at all times in each layer, which is necessary to satisfy stability criteria of the method of characteristics.

The procedure is to determine the tangent shear modulus at the appropriate strain level from the Ramberg-Osgood curves and then calculate the apparent shear wave velocity for each reach from Eq. (22). Since it is necessary to stay on the characteristic lines, values of τ and V at points R and S of the z-t diagram (Fig. 3) can be obtained by linear interpolations as:

$$V_R=V_C-(V_C-V_A)v_{sj}\Delta t/\Delta z_j \quad (32)$$

$$\tau_R=\tau_C-(\tau_C-\tau_A)v_{sj}\Delta t/\Delta z_j \quad (33)$$

$$V_S=V_C-(V_C-V_B)v_{sj+1}\Delta t/\Delta z_{j+1} \quad (34)$$

$$\tau_S=\tau_C-(\tau_C-\tau_B)v_{sj+1}\Delta t/\Delta z_{j+1} \quad (35)$$

where the subscript j refers to the reach containing the C^+ characteristic and the subscript $j+1$ refers to the reach containing the C^-. Shear stresses and velocities are known at points A and B from the previous time step. Eqs. (32) to (35) provide the values of τ and V at points R and S. Then, Eqs. (27) and (28) are solved at each interior node of the z-t diagram to determine the only unknown quantities, τ_P and V_P.

At the crest of the dam where $z_P=h$, the boundary condition is expressed as $\tau_P=0$ (traction free surface). The C^- characteristic equation becomes:

$$-\tau_S+\rho v_s(V_P-V_S)-\frac{\tau_S}{2}\ln\frac{z_S}{h}+\mu\frac{V_B-V_C}{z_B-h}=0 \qquad (36)$$

The particle velocity V_P at the crest of the dam is readily available from Eq. (36). At the base of the dam where $z_P=H$, the boundary condition is that V_P is known as a function of time for the seismic motion under consideration. The C^+ characteristic equation becomes:

$$\tau_P-\tau_R-\rho v_s(V_P-V_R)+\frac{\tau_R+\tau_P}{2}\ln\frac{H}{z_R}+\mu\frac{V_C-V_A}{H-z_A}=0 \qquad (37)$$

The shearing stress τ_P at the base of the dam is obtained by solving Eq. (37). The above procedure is repeated stepwise in the time domain.

If viscoelastic soil behavior is assumed, $G=G_o$ and v_s is constant in each distance interval Δz. Then, points R and S on the z-t diagram of Fig. 3 coincide with points A and B and the characteristic lines do not change slope during the transient. In this case $v_s\Delta t/\Delta z=1$ and the interpolations (Eqs. (32) to (35)) are not needed.

It is interesting to note that for a value of h much larger than the dam height (*i.e.* $h=1155$ ft and $H=1365$ ft), the slope α has a value close to zero and the response of the truncated dam to a base excitation is very similar to that of a one-dimensional soil deposit of the same properties and the same height.

EXAMPLES

Example 1

A 400-ft(122m) high dam with truncated crest [see Fig. 1 with $H=420$ ft (128m), and $h=20$ ft (6.1m)] is considered. The dam, resting on a horizontal rock base, is subjected to the first 15 sec of the S69°E component of the Taft earthquake of 1952. Soil dynamic behavior is assumed viscoelastic. Shear modulus $G_o=9\times10^6$ psf (44×10^6kg/m^2), viscosity $\mu=70,000$ lb·sec/ft^2 (0.34×10^6kg·s/m^2), and soil mass density $\rho=4.0$ slugs/ft^3 (2062kg/m^3) are considered constant throughout the dam. A time increment of 0.01 sec is used in the method of characteristics. The apparent wave velocity is 2000 ft/sec (610 m/s) and the dam is divided into 20 reaches, each 20 ft (6.1m) thick. The computed displacement and velocity responses at the crest of the dam are plotted in Figs. 4 and 5. The computing time required for execution was 80 sec on the IBM 360/67 computer.

The analytical method is also used to solve the above problem. Displacements at the base of the dam are obtained every 0.01 sec by twice integrating numerically the S69°E Taft accelerogram. The result is presented in the lower portion of Fig. 4. This displacement diagram is analyzed into 24 harmonics by using the Fourier transform in conjunction with a least squares criterion (13). The amplification factor at the dam crest for each harmonic is computed from Eq. (11) and an Inverse Fourier Transform reconstitutes the transient response which is plotted in the upper portion of Fig. 4. The comparison with the displacements obtained by the method of characteristics is considered unsatisfactory. The reason that smoother results are obtained by the analytical method is that the 24 harmonics do not model frequencies less than 0.625 sec. Therefore, the problem was solved again by analyzing the base motion into 48 harmonics. The response at the crest found from the analytical solution is substantially the same as that found from the method of characteris-

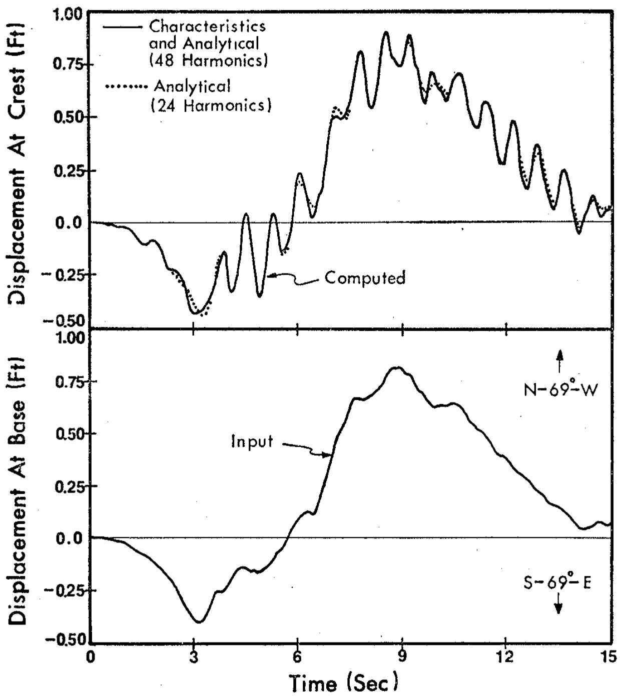

Fig. 4. Displacement response of a 400 ft. (122m) high truncated dam to Taft S69°E earthquake

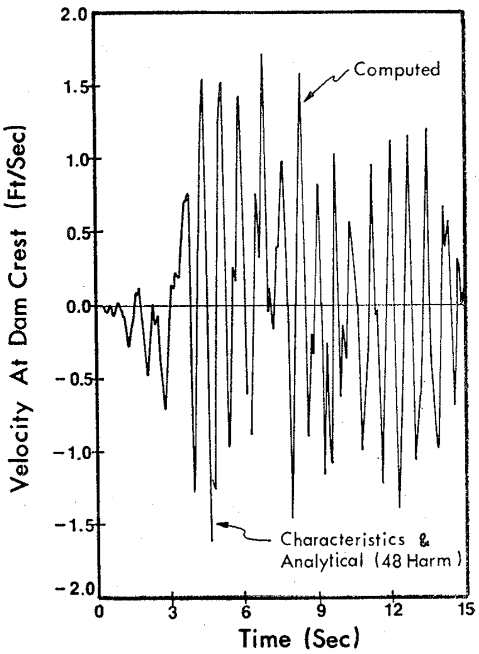

Fig. 5. Velocity response of a 400 ft. (122m) high truncated dam to Taft S69°E earthquake

tics (Fig. 4). The computing time required for execution, using the analytical method, was approximately 110 sec.

Precise agreement of the two methods also was observed in cases where the horizontal base excitation is sinusoidal (13). Starting from static conditions the method of characteristics results in an initial transient which gradually vanishes after a few cycles due to the presence of viscous dissipation.

Example 2

A 100-ft (30.5m) high wedge-shaped earth dam without truncation ($h=0$, $H=100$ ft) is subjected to the first 10 sec of the North-South component of the 1940 El Centro earthquake. The assumption is made that soil behaves as a linear viscoelastic material. Two methods are used to obtain the response accelerations at midheight and at the crest of the dam: the method of characteristics, and Seed and Martin's analytical solution (16). Soil characteristics common to both methods are the average shear modulus $G_0=4\times10^6$ psf (19.5×10^6 kg/m^2) and density of 4.04 slugs/ft^3 (2082 kg/m^3).

In the method of characteristics the dam height is divided into six reaches, each of thickness $\Delta z=16.67$ ft (5.1m). With an assumed constant soil viscosity $\mu=71,250$ lb·sec/ft^2 (0.35×10^6 kg·s/m^2) and a time increment of $\Delta t=0.01$ sec, the apparent shear wave velocity is found to be equal to $v_s=1667.7$ ft/sec (508.3 m/s). The computing time necessary for the execution of this example was 35 sec on the IBM 360/67 computer.

A time increment $\Delta t=0.02$ sec is used in Seed and Martin's solution. A damping factor of 0.2 is assumed, constant for each mode. Eq. (6) of reference (16) is used and the responses for the first six modes of vibration are superimposed.

The results obtained by both methods are plotted in Fig. 6, and exhibit a very similar form. Since the damping mechanisms used by the two methods are different and only the first six modes are considered in the analytical solution, the results obtained by the two

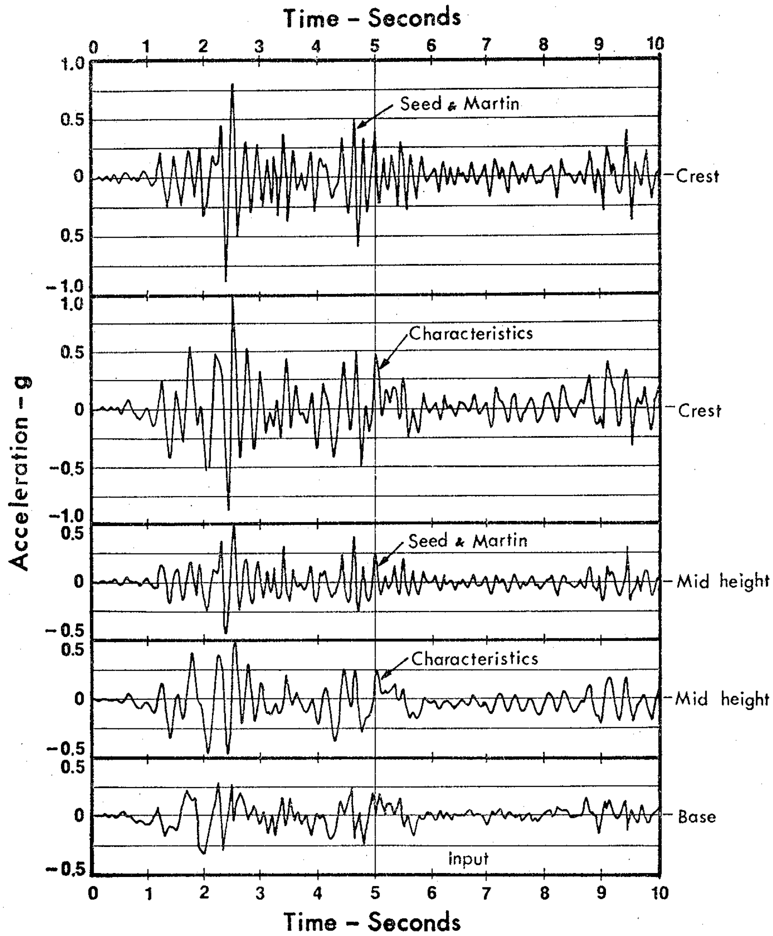

Fig. 6. Response of a 100 ft. (30.5 m) high dam to El Centro, 1940, N-S earthquake. Comparison between Seed and Martin's solution and method of characteristics.

methods are in remarkably good agreement.

Example 3

A 322-ft (98.1 m) high truncated earth dam (h=13 ft, 4 m) with base width of 1340 feet (408 m) and side slopes 2 to 1 ($\alpha=2$) is subjected at its base to the first 12.5 seconds of the North-South component of the 1940 El Centro earthquake. Strain-softening soil behavior is assumed according to the Ramberg-Osgood inelastic hysteresis law. Static shear modulus is considered to be a known function of depth. The method of characteristics is used to find the dynamic response of the earth dam to the given earthquake excitation.

The soil has a unit weight $\gamma_s=134$ lb/ft³ (2147 kg/m³) with void ratio equal to 0.5 and angle of internal friction $\varphi=40°$. The coefficient of earth pressure at rest is $K_0=0.60$. The static shear modulus is determined from the expression (15):

$$G_0=50227\sqrt{\gamma_s z} \quad (\mathrm{lb/ft^2}) \tag{38}$$

where z is measured from the crest downwards in feet. The maximum shear stress is obtained from the expression (15):

$$\tau_m=\left[\left\{\left(\frac{1+K_0}{2}\right)\sigma_z \sin\varphi\right\}^2-\left\{\left(\frac{1-K_0}{2}\right)\sigma_z\right\}^2\right]^{0.5} \tag{39}$$

where σ_z is the effective confining pressure equal to:

$$\sigma_z = \gamma_s z(1+2K_0)/3 \tag{40}$$

hence, for the above assumed soil properties:

$$\tau_m = 0.347\gamma_s z \ (\mathrm{lb/ft^2}) \tag{41}$$

After selecting a time interval $\Delta t = 0.01$ seconds, the dam height is subdivided into 27 distance intervals (reaches) ranging in thickness from $\Delta z_1 = 4.6$ ft (1.4m) to $\Delta z_{27} = 15.7$ ft (4.8m). Soil viscosity is assumed to be zero. To describe the stress-strain relationship for the soil, an exponent $R_0 = 3$ is used in the Ramberg-Osgood equations.

Displacements at the dam crest and at 161.1 feet (49.1m) below the crest, found by the method of characteristics, are displayed in Fig. 7, together with the displacement of the dam base which was obtained by twice integrating the El Centro accelerogram. Displacements at the crest and the mid-height, relative to the base of the dam, at four particular instants of time are shown in Fig. 8. The shearing stress at the base of the dam, computed by the method of characteristics, is plotted in Fig. 9. The Ramberg-Osgood normalized stress-strain diagram at a depth of 154.5 feet (47.1m) below the crest for the 12.5 seconds excitation is shown in Fig. 10. At that particular depth, τ/τ_m never exceeded the value of 1. The most excessive hysteresis loop occured between 2.0 and 2.25 seconds after the beginning of the earthquake. Pronounced shearing

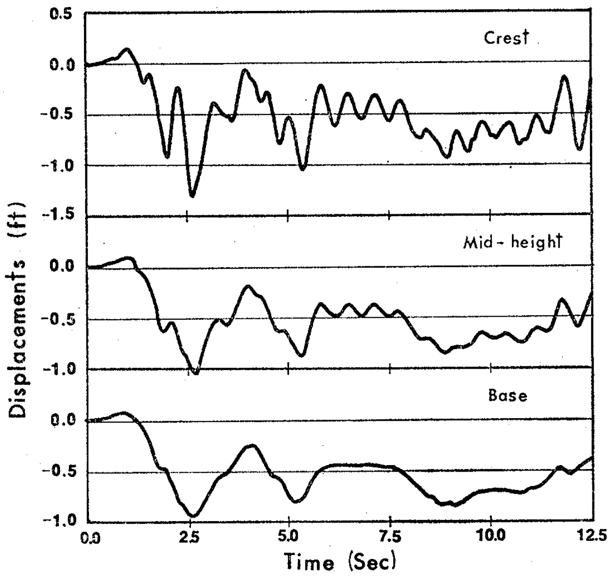

Fig. 7. Absolute displacements of a 322 ft. (98.1 m) high earth dam. El Centro, 1940, N-S earthquake. Strain-softening soil behavior.

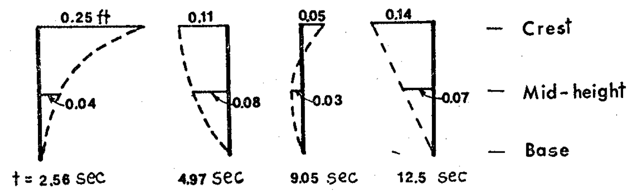

Fig. 8. Instant displacements of a 322 ft. (98.1 m) earth dam relative to its base. El Centro, 1940, N-S earthquake. Strain-softening soil behavior.

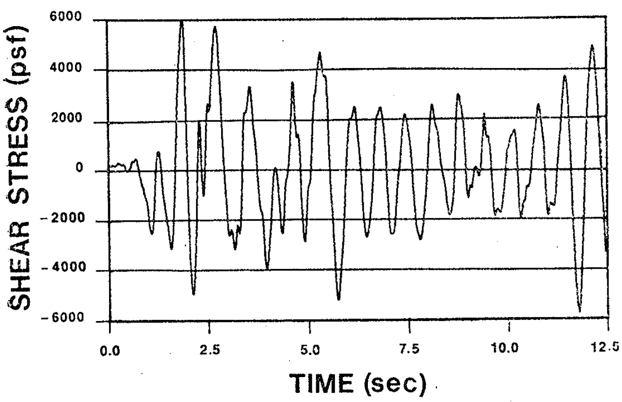

Fig. 9. Shear stress at the base of a 322 ft. (98.1 m) high earth dam. El Centro, 1940, N-S earthquake. Strain softening soil behavior.

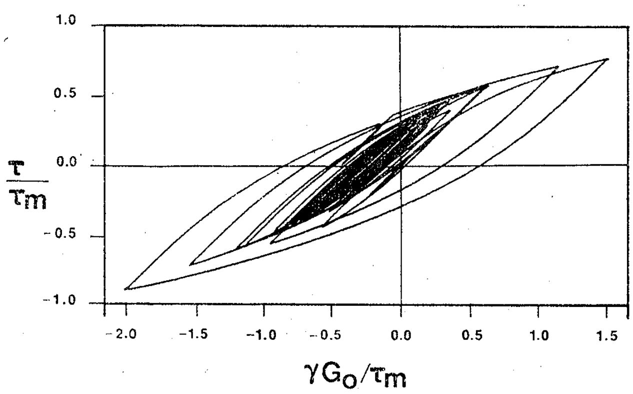

Fig. 10. Ramberg-Osgood normalized stress-strain diagram 154.5 ft. (47.1 m) below crest of a 322 ft. (98.1 m) high earth dam. El Centro, 1940, N-S earthquake.

stresses at the dam base developed at about the same time span, as shown in Fig. 9.

Computations for this example, including the Ramberg-Osgood nonlinear behavior, required approximately 110 seconds of execution time on the IBM 360/67 computer of the University of Michigan for 12.5 sec of the earthquake excitation.

SUMMARY AND CONCLUSIONS

The shear response of earth dams to earthquakes is investigated by studying the one-dimensional propagation of shear waves through wedge-shaped earth dams with truncated crests. A horizontal seismic excitation is considered to apply at the dam base. The earth dam material is assumed viscoelastic, or strain-softening. The material properties, such as shear modulus, viscosity, and density, may vary with depth or be constant throughout the dam cross section. The partial differential equations of state and motion are presented and two methods are used for their solution.

A closed form solution involving Hankel functions with complex arguments is developed in case the soil is viscoelastic and the dam base is subject to harmonic excitations. This method is extended to cases of random seismic vibrations by employing Fourier analysis in conjunction with a least squares criterion. To study resonance effects, response curves can be obtained for viscously damped tapered dam cross sections with truncated crests, by using the analytical solution developed.

The method of characteristics is used to solve the two linear hyperbolic partial differential equations of state and motion. The solution accounts for wave reflections from the truncated crest and the underlying rigid dam base, with excitation provided at the base.

Because of the one-dimensional concept, results obtained by both methods are satisfactory along the dam's axis of symmetry. The analytical method and the method of characteristics compare favorably in all case studies examined. If an earthquake record is analyzed in a sufficient number of harmonics, the results obtained by the analytical method are identical with those obtained by the method of characteristics. An additional confirmation emanates from a comparison of results from the method of characteristics to results from Seed and Martin's analytical solution. Despite the different damping mechanisms used, the two methods are in good agreement. A case of strain-softening nonlinear soil behavior demonstrates the applicability of the method of characteristics to dynamic problems where analytical methods are unable to provide solutions.

Application of the method of characteristics to the problem of shear wave propagation through earth dams with truncated crests illustrates some of the advantages of the method: relative simplicity; applicability to problems of purely transient nature; accuracy of results; and low computer cost. The method of characteristics provides versatility in handling different descriptions of dynamic response. Other "strain-softening" material behavior laws could be used successfully in the model. The method of characteristics also provides the means of calculating the response of earth dams resting on flexible foundation without additional complications.

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NOTATION

A = constant evaluated from boundary conditions

A_0 = horizontal cross-sectional area of earth dam (L^2)
B = constant evaluated from boundary conditions
C^+, C^- = refer to characteristics equations
E = 0.5772156649…, Euler's constant
e = 2.71828…, the base of natural (Napierian) logarithms
$F(z)$ = amplitude function (L)
G_0 = initial shear modulus at zero strain level (F/L^2)
G = shear modulus (F/L^2)
H = height of an earth dam (L)
$H_q^{(1)}$, $H_q^{(2)}$ = Hankel functions of order q
h = amount of dam crest truncation (L)
i = unit of complex number, equal to $\sqrt{-1}$
J_q = Bessel functions of the first kind, of order q
K = integer
K_0 = coefficient of earth pressure at rest
L_q, l_q = real parts of Bessel functions of the first and second kind of order q
M_q, m_q = imaginary parts of Bessel functions of the first and second kind, of order q
m, n = integers
q = order of Bessel functions (0 or 1)
R = auxiliary variable
R_0 = exponent in Ramberg-Osgood relation
r = modulus of complex number
S, T = auxiliary variables used in Bessel functions
T_n = natural period, in seconds
t = time, in seconds
u = horizontal soil particle displacement (L)
V = horizontal soil particle velocity (L/s)
$V_C^* = (\partial V/\partial z)_C$
v^* = complex shear wave volocity (L/s)
v_s = shear wave velocity (L/s)
W = amplitude of known harmonic displacement (L)
Y_q = Bessel functions of the second kind, of order q
Z = complex number
z = vertical distance (L)
α = slope of earth dam sides
γ = shearing strain
γ_1 = particular shearing strain at stress reversal
γ_s = unit weight of soil (F/L^3)
Δt = time interval, in seconds
Δz = distance interval (L)
θ = multipliers in characteristics method
μ = soil viscosity (Fs/L^2)
π = 3.141592…
ρ = mass density of soil (M/L^3)
σ_z = effective confining pressure (F/L^2)
τ = shearing stress (F/L^2)
τ_1 = particular shearing stress at stress reversal (F/L^2)
τ_m = maximum shear stress (F/L^2)
τ_y = yield shear stress (F/L^2)
ϕ = amplitude of complex numbers

φ = angle of internal friction, in radians
ω = angular frequency, in radians per second
ω_n = natural frequency, in radians per second

REFERENCES

1) Ambraseys, N. N. (1960): "On the shear response of a two-dimensional truncated wedge subjected to an arbitrary disturbance," Bull. Seism. Soc. of Am., Vol. 50, No. 1, pp. 45-56.
2) Ambraseys, N. N. (1960): "The seismic stability of earth dams," Proc. 2nd World Conf. Earthquake Eng., Tokyo, Japan, Vol. II.
3) Ambraseys, N. N. (1960): "On the seismic behavior of earth dams," Proc. 2nd World Conf. Earthquake Eng., Tokyo, Japan.
4) Ambraseys, N. N., and Sarma, S. K. (1967): "The response of earth dams to strong earthquakes," Geotechnique, Vol. 17, No. 3, pp. 181-213.
5) Hatanaka, M. (1955): "Fundamental considerations on the earthquake resistant properties of the earth dam," Bull. No. 11, Disaster Prevention Research Institute, Kyoto University.
6) IBM (1970): System/360 Scientific Subroutine Package, Version III, Programmer's Manual, Publication GH 20-0205-4, 5th Edition, pp. 276-278.
7) McLachlan, N. W. (1955): Bessel Functions for Engineers, Oxford Claredon Press, 2nd Edition, p. 240.
8) Mononobe, N., Takata, A. and Matsumura, M. (1936): "Seismic stability of the earth dam," *Proceedings*, 2nd Congress on Large Dams, Washington, D. C., Vol. IV, pp. 435-442.
9) National Bureau of Standards (1943): Mathematical Tables Project, Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments, Columbia University Press, New York, p. 403.
10) National Bureau of Standards (1950): Computation Laboratory Table of the Bessel Function $Y_0(z)$ and $Y_1(z)$ for Complex Arguments, Columbia University Press, New York, p. 427.
11) Okamoto, S., Hakuno, M., Kato, K., and Kawakami, F. (1965): "On the dynamical behavior of an earth dam during earthquake," *Proceedings*, 3rd World Conference on Earthquake Engineering, New Zealand.
12) Okamoto, S. (1973): Introduction to Earthquake Engineering, University of Tokyo Press, Tokyo, Japan, pp. 427-490.
13) Papadakis, C. N. (1973): "Soil Transients by Characteristics Method," thesis presented to the University of Michigan, Ann Arbor, 1973, in partial fulfillment of the requirements for the degree of Doctor of Philosophy. (Report UMEE-73 R 10, Department of Civil Engineering, p. 192)
14) Ralston, A. and Wilf, H. S. (1960): Mathematical Methods for Digital Computers, John Wiley and Sons, New York, Vol. I, pp. 258-262.
15) Richart, F. E., Jr., Hall, J. R., Jr. and Woods, R. D. (1970): Vibrations of Soils and Foundations, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, p. 44.
16) Seed, H. B. and Martin, G. R. (1966): "Seismic coefficient in earth dam design," Journal of the Soil Mechanics and Foundation Division, ASCE, Vol. 92, No. SM 3, Proc. Paper 4824, pp. 25-58.
17) Streeter, V. L., Wylie, E. B. and Richart, F. E., Jr. (1974): "Soil motion computations by characteristics method," Journal of the Geotechnical Engineering Division, ASCE, Vol. 100, No. GT 3, Proc. Paper 10410, pp. 247-263.

APPENDIX

Bessel Functions with Complex Arguments

In complex form Bessel functions of the first kind, of order q equal to zero or one, may be written as (+ for the first quadrant of the complex plane and − for the fourth):

$$J_q[r\exp(i\phi)] = l_q(r,\ \phi) \pm i m_q(r,\ \phi) \tag{42}$$

where

$$l_q(r,\ \phi) = \sum_{K=0}^{\infty} \frac{(-1)^K (r/2)^{2K+q}}{K!\,(K+q)!} \cos(2K+q)\phi \tag{43}$$

$$m_q(r,\phi)=\sum_{K=0}^{\infty}\frac{(-1)^K(r/2)^{2K+q}}{K!(K+q)!}\sin(2K+q)\phi \tag{44}$$

When 26 terms of the above series are considered, the truncation error induced by omitting the rest of the terms and by working in single precision was found to be of the order of 10^{-4}.

Bessel functions of the second kind, of order zero or one, may be written as (+ for the first quadrant of the complex plane and − for the fourth):

$$Y_q[r\exp(i\phi)]=L_q(r,\ \phi)\pm iM_q(r,\ \phi) \tag{45}$$

$$L_0(r,\ \phi)=\frac{2}{\pi}\left[l_0(r,\ \phi)\left(E+\ln\frac{r}{2}\right)-\phi m_0(r,\ \phi)+S(r,\ \phi)\right] \tag{46}$$

$$M_0(r,\ \phi)=\frac{2}{\pi}\left[m_0(r,\phi)\left(E+\ln\frac{r}{2}\right)+\phi l_0(r,\ \phi)+T(r,\ \phi)\right] \tag{47}$$

in which

$$S(r,\ \phi)=\sum_{K=1}^{\infty}\frac{(-1)^{K+1}(r/2)^{2K}}{K!K!}\left(1+\frac{1}{2}+\cdots+\frac{1}{K}\right)\cos 2K\phi \tag{48}$$

$$T(r,\ \phi)=\sum_{K=1}^{\infty}\frac{(-1)^{K+1}(r/2)^{2K}}{K!K!}\left(1+\frac{1}{2}+\cdots+\frac{1}{K}\right)\sin 2K\phi \tag{49}$$

Bessel functions of the second kind, of order one can be found from the "cross-relation" between Bessel functions of both kinds (7):

$$J_0(Z)\,Y_1(Z)-J_1(Z)\,Y_0(Z)=-\frac{2}{\pi Z} \tag{50}$$

where Z is complex. This leads to the following relations between real and imaginary parts of the Bessel functions:

$$l_0L_1-m_0M_1=l_1L_0-m_1M_0-2\cos\phi/\pi r \tag{51}$$

$$m_0L_1+l_0M_1=l_1M_0+m_1L_0+2\sin\phi/\pi r \tag{52}$$

From Eqs. (52) and (53), M_1 and L_1 are obtained in order to calculate $Y_1[r\exp(i\phi)]$.

The use of existing tables (9, 10) to evaluate Bessel functions of the first or second kind and of order zero or one, for any complex argument, was found to be excessively time-consuming because of the necessity of double interpolations in the values of r and ϕ. Therefore, a computer program was written in FORTRAN IV language (13) to calculate any Bessel function of the first or second kind and of order zero or one with complex argument. Single precision proved to give satisfactory results.

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