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A SIMPLIFIED PROCEDURE FOR PROBABILITY-BASED $\phi_u=0$ STABILITY ANALYSIS

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ABSTRACT

The objective of this study is to provide a simple yet accurate procedure for the probabilistic stability analysis of an embankment constructed on a soft clay with finite thickness. The analysis is based on the conventional $\phi_u=0$ and circular arc slip surface assumptions.

The probabilistic soil profile description with respect to the undrained shear strength within a clay deposit is given first introducing a simple stochastic process. This is shown to be a general expression for the spatial variation of the strength of clays including the conventional deterministic soil profile as a special case. The probabilistic description of the soil profile is, however, transformed into the set of eight different deterministic soil profiles, each of which is a three-layered system with three different strengthes. Through the composition of these multi-layered systems, it is demonstrated that all the possibility in the distribution of a factor of safety can be covered by only eight times of execution of the deterministic, and then, conventional slope stability analysis for eight deterministic soil profiles. From this, the layered system is called in this study an equivalent multi-layered system.

The developed procedure is illustrated in the case studies including a multi-failure mode problem, from which the simplicity as well as the accuracy of the proposed method is verified for the use in practical design.

Key words : clay, design, earthfill, heterogeneity, layered system, safety factor, soft ground, stability analysis, statistical analysis (**IGC** : E 6/H 4)

INTRODUCTION

The conventional $\phi_u=0$ stability analysis is newly formulated assuming that the spatial variation of the undrained shear strength of clay follows a stochastic process.

The accuracy of the conventional procedure itself is considered to have been widely accepted when the testing manner is appropriate in which strength values are selected (e. g., Peck, 1967; Nakase, 1967; Wu, 1974; Matsuo and Asaoka, 1976; Hanzawa et al, 1980). In practical design, however, all design alternatives are required to satisfy the design criterion that the computed factor of safety be no less than a prescribed allowable value. This is mostly because of the uncertainty in soil profile modelling as well as the uncertainty in the method of stability analysis used. In this study, the former

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source of uncertainty is mainly discussed.

The mathematical expression for the uncertainty in soil profile modelling can be made perhaps only by introducing the concept of stochastic process. Therefore, if one follows a probabilistic soil profile description, the stability analysis is also considered to be made in a probabilistic way. Although many literatures have been accumulated in this research field (e. g., Biernatovsky, 1969; Wu and Kraft, 1967, 1970; Matsuo and Kuroda, 1974; Alonso, 1976; Matsuo and Asaoka, 1976; Morla Catelan and Cornell, 1976; Tang et al, 1976, Vanmarcke, 1977 b, etc), there still exist two difficulties.

The first difficulty comes from the probabilistic soil profile modelling itself because of the limited number of available soil data. A new simplified procedure for this problem is introduced in the successive section. The second one is the difficulty associated with the minimum factor of safety search in the random field of a shear strength distribution. The reason for this is because the location of the slip surface which minimizes the factor of safety becomes also random when the strength distribution is random. To avoid this difficulty, it has been usual in many literatures that the critical slip surface is fixed at the location determined from the mean value of shear strengthes. In the present study, however, this problem is solved completely in a different manner. As a result, it is shown that the probabilistic approach to the $\phi_u=0$ stability analysis is almost equivalent to some trials of the conventional deterministic analysis for some varieties of deterministic soil profiles. The simplicity of the presented procedure is illustrated in case studies.

PROBABILISTIC SOIL PROFILE DESCRIPTION

Even within homogeneous-like clay deposit, considerable variation is found from point to point in measuring undrained shear strength c_u. The idea of using stochastic process (or probability model) to describe the spatial variability of c_u is based on the assumption that a series of c_{u1}, c_{u2}, ……, c_{uN} of N successive observations is regarded as a sample realization from an infinite population of such series that could have been generated by the stochastic process. Being based on this idea, the value of undrained shear strength of any point is estimated with its probability and what one can do is only to establish the stochastic model which describes the spatial variability of c_u in a probabilistic manner (probabilistic soil profile description). The conventional deterministic soil profile is, thus, considered as a very special case since the deterministic model always gives the value of c_u with probability of one.

Recent developments in the use of probability theory in soil mechanics design have provided considerable amount of literatures on the description of the spatial variability of the undrained shear strength of clays (e. g., Hooper and Butler, 1966; Lumb, 1966; 1970; Matsuo and Asaoka, 1977; Kuroda et al, 1981). In these literatures, special efforts were made on the following three topics, that is, (i) the in-situ heterogeneity of clays, especially, depth-dependent nature of c_u, (ii) the effect of sample disturbance, test inperfections and human factors, and (iii) statistical uncertainty which comes from the limited availability of information (i. e., the limited number of soil samples and/or testing). This paper deals mainly with the first problem and the other two sources of uncertainty are only briefly dealt with in the discussion section.

To express the spatial (vertical) variability of c_u, one needs to determine at least the mean value function $\mu(z)$ and the covariance function $\mathrm{cov}(z, z')$ (a second moment random process). A general procedure for obtaining estimates of these functions was proposed by Vanmarcke (1977 a), who also discussed the application of the probabilistic soil profile to slope stability analysis. When large amount of data is available, Vanmarcke's method gives accurate estimates for the above functions. In the case of relatively thin clay layers, however, the number of soil samples along vertical direction is always limited, and then the less sophisticated method is desired. Asaoka and Grivas (1982) suggested the applicability of a first order autoregressive model to describe the vertical variability of c_u, which is introduced in this section.

Let $c_u(z)$ denote the value of the undrained shear strength at depth z. It is assumed that the undrained strength is measured at constant vertical interval Δz and $c_{ui}=c_u(\Delta z \cdot i)$ is generated by the following model:

$$c_{ui}=\alpha_0+\alpha_1 c_{ui-1}+\sigma\varepsilon_i \qquad (1)$$

where the errors ε_i, $i=1, 2, \cdots\cdots, N$ are independent random variable of each other and have zero mean and unity deviation while α_0, α_1 and σ are constant coefficients, sometimes referred to as the parameters. When the random process (1) has reached at the stationary state, the mean and the covariance functions for $c_u(z)$ are expressed as

$$\mu(z)=\frac{\alpha_0}{1-\alpha_1} \quad (=\text{constant}) \qquad (2)$$

$$\text{cov}(z, z')=\frac{\sigma^2}{1-\alpha_1{}^2}\exp\left(-\frac{|z-z'|}{l}\right) \qquad (3)$$

respectively, in which

$$l=-\frac{\Delta z}{\ln \alpha_1} \qquad (4)$$

is the decay parameter describing how rapidly the correlation decreases for increasing interval $|z-z'|$. The parameters α_0, α_1 and σ in Eq. (1) can be estimated by the least squares method using N observations $c_{u1}, c_{u2}, \cdots\cdots, c_{uN}$ taken along a vertical direction. To verify the stationarity of the data, the parameters in the following random process β_0, β_1 and σ should be estimated:

$$v_j=\beta_0+\beta_1 v_{j-1}+\sigma\varepsilon_j \qquad (5)$$

in which

$$v_j=c_{uN+1-j}, \quad J=1, 2, \cdots\cdots, N. \qquad (6)$$

That is, v_j is an inverse process of c_{ui}. If the estimates for β_0 and β_1 are close enough to those of α_0 and α_1, respectively, then c_{ui} is considered to have already reached at the stationary state.

When some boreholes are available in a given designing section, the mean values of Eqs. (2) and (3) of each borehole are to be used. The illustrative example for the proposed method is given in Table 1 which is the results from the data shown in Fig. 1.

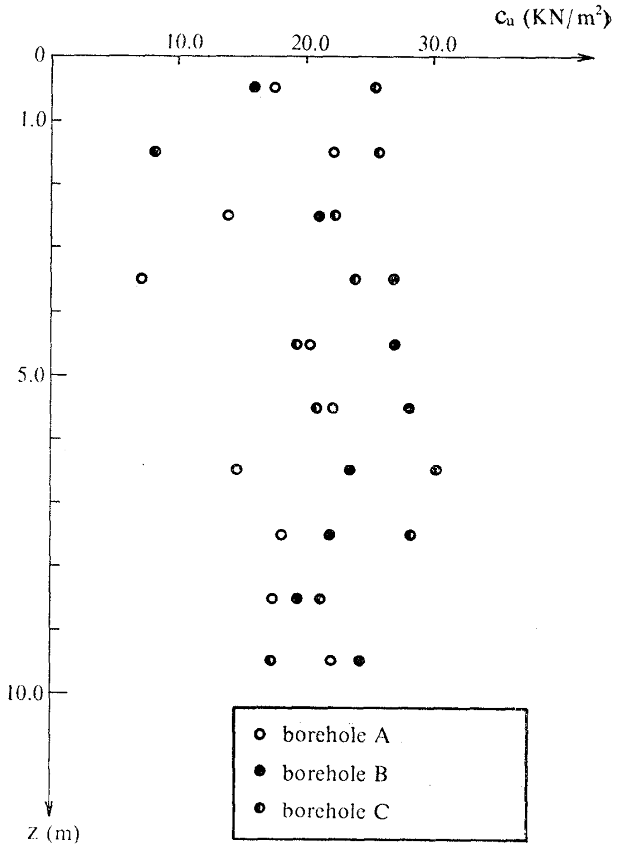

Fig. 1. Observations of the spatial variation of c_u

Table 1. Numerical values of parameters

Sample size	10×3
α_0	11.68 (kN/m²)
α_1	0.437
β_0	11.47 (kN/m²)
β_1	0.455
$(\alpha_0, \alpha_1)\simeq(\beta_0, \beta_1)$?	Yes
σ	5.00 (kN/m²)
$\mu(z)$	20.78 (kN/m²)
l in Eq. (3)	1.21 (m)

In some actual situations, especially in the case of a thick normally consolidated clay deposit, $\mu(z)$ should be determined as a linear function of z. In such case, the similar method by Asaoka and Grivas (1982) and Asaoka et al (1981) are recommended, the discussion of which is beyond the scope of this paper. Only the fact that $\mu(z)$ and $\text{cov}(z, z')$ can be easily determined even from the limited number of soil samples is emphasized in this section.

EQUIVALENT MULTI-LAYERED SYSTEM

The general formulation for the factor of safety in the $\phi_u=0$ stability analysis is given as follows:

$$F_s=\min_L\left[\frac{r\int_L c_u(z)dL}{M_o}\right] \qquad (7)$$

in which

F_s : factor of safety
L : circular arc slip surface
r : radius of L
M_o : overturning moment, and

the operator $\min_L[\ \ ]$ denotes the search of the slip surface L which minimize the quantity in brackets. Since $c_u(z)$ is a stochastic process with z, its integrated value

$$R_L=\int_L c_u(z)dL \qquad (8)$$

becomes also a random variable. Therefore, the distribution for determination of F_s given in Eq. (7) depends on the joint probability distribution of $R_L, R_{L'}, \cdots\cdots$, in which $L, L'\cdots\cdots$ denote the possible slip surfaces. The equivalent multi-layered system given in this section is designed so that the same joint distribution for $R_L, R_{L'}, \cdots\cdots$ may be obtained to that of the original stochastic system at least its first and the second probability moments for any arbitrary combination of possible slip surfaces.

In Fig. 2 is shown the original system while Fig. 3 shows the equivalent three-layered system, in which c_{ui}, $i=1, 2, 3$ are mutually independ random variables and their means and the standard deviations are expressed by μ_i and σ_i, $i=1, 2, 3$, respectively. In other words, in the equivalent multi-layered system, each layer with thickness Δ has its own strength c_{ui} which is perfectly correlated within the layer and perfectly uncorrelated to the strengthes of the other two layers. The parameters μ_i and σ_i, $i=1, 2, 3$ can be determined in the following manner. First, since the discussion is restricted here to a base failure mode, all the

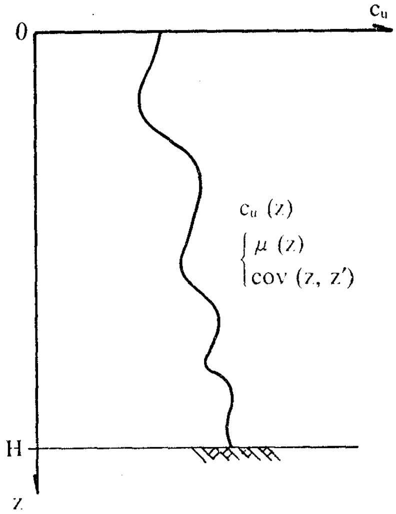

Fig. 2. Original system

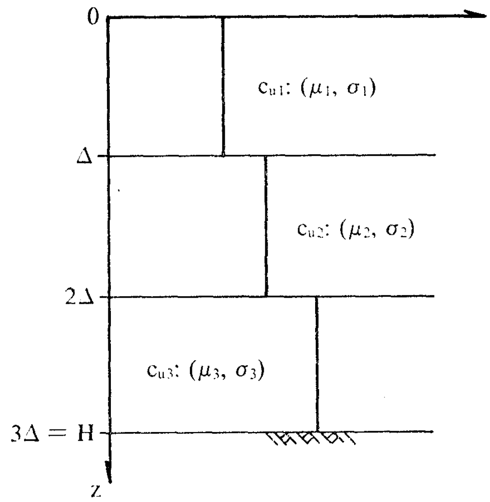

Fig. 3. Equivalent multi-layered system

possible slip surfaces are noted to pass from the top of a clay deposit to its bottom. From this, when the line element dL is expressed by $w(z)dz$, the weight function $w(z)$ is naturally defined on the range of $0\leqq z\leqq H$, in which $z=0$ and $z=H$ denote the top and the bottom of a clay deposit, respectively. Now, let $w(z)$ be assumed to be well approximated by $w(z)=a_0+a_1z+a_2z^2$, a polynomial function of the second order. Then, the equivalent multi-layered system in Fig. 3 can be found to give the same value of $E[R_L]$ (the first probability moment of the joint distribution for $R_L, R_{L'}, \cdots\cdots$) as that of the original system whatever values a_0, a_1 and a_2 may take (i. e., for any arbitrary slip surface L), if μ_i, $i=1,2,3$ satisfies

$$\left.\begin{aligned}&\mu_1\Delta+\mu_2\Delta+\mu_3\Delta=\int_0^H\mu(z)dz\\&\mu_1\frac{\Delta^2}{2}+\mu_2\frac{3\Delta^2}{2}+\mu_3\frac{5\Delta^2}{2}=\int_0^H\mu(z)zdz\\&\mu_1\frac{\Delta^3}{3}+\mu_2\frac{7\Delta^3}{3}+\mu_3\frac{19\Delta^3}{3}=\int_0^H\mu(z)z^2dz\end{aligned}\right\}\tag{9}$$

in which $\mu(z)$ in the right hand sides of Eqs. (9) is the mean value function of a stochastic process of $c_u(z)$. The μ_i, $i=1,2,3$ are easily determined by solving Eqs. (9) simultaneously.

Following the same logic, for the equivalent multi-layered system to give approximately the same value of $E[R_L\cdot R_{L'}]$ (the second probability moment of the joint distribution of $R_L, R_{L'}$) as that of the original system for any arbitrary L and L', the next simultaneous equations for σ_i, $i=1,2,3$ should be satisfied, that is,

$$\left.\begin{aligned}&\sigma_1^2\Delta^2+\sigma_2^2\Delta^2+\sigma_3^2\Delta^2\\&\quad=\int_0^H\int_0^H\mathrm{cov}(z,z')dzdz'\\&\sigma_1^2\frac{\Delta^3}{2}+\sigma_2^2\frac{3\Delta^3}{2}+\sigma_3^2\frac{5\Delta^3}{2}\\&\quad=\int_0^H\int_0^H\mathrm{cov}(z,z')zdzdz'\\&\sigma_1^2\left(\frac{\Delta^2}{2}\right)^2+\sigma_2^2\left(\frac{3\Delta^2}{2}\right)^2+\sigma_3^2\left(\frac{5\Delta^2}{2}\right)^2\\&\quad=\int_0^H\int_0^H\mathrm{cov}(z,z')zz'dzdz'\end{aligned}\right\}\tag{10}$$

where $\mathrm{cov}(z,z')$ is a covariance function of the original system, $c_u(z)$. Thus, the parameters σ_i, $i=1,2,3$ can also be determined from Eqs. (10).

If one needs more than the second order approximation for $w(z)$, the multi-layered system with more than three layers should be employed. It will be shown, however, that the three layered system is enough for usual engineering purposes through examining the illustrative examples in this study.

Using the data given in Fig. (1) and/or Table 1, Eqs. (9) and (10) determine the equivalent multi-layered system, the parameters of which is tabulated in Table 2.

Table 2. Parameters for Fig. 3

$\mu_1=\mu_2=\mu_3$	20.78 (kN/m²)
σ_1	4.51 (kN/m²)
σ_2	6.97 (kN/m²)
σ_3	4.46 (kN/m²)

This example is continuously discussed in the successive sections.

POINT ESTIMATES OF PROBABILITY MOMENTS FOR F_s

Now, it has been found that the factor of safety F_s given in Eq. (7) can be approximately regarded as a function of three discrete random variables, c_{u1}, c_{u2} and c_{u3}, while the F_s was originally considered as a functional of a continuous stochastic process, $c_u(z)$. This problem transformation makes the distribution search for F_s extremely easy.

For simplicity, F_s in Eq. (7) is rewritten here as

$$F_s=F_s(c_{u1},c_{u2},c_{u3}|G)\tag{11}$$

in which the symbol G is employed to express the loading condition on a clay deposit and some other conditions which must be taken into consideration for the slope stability analysis. In any case, F_s should be noted as a function of three random variables c_{u1}, c_{u2} and c_{u3}. The moments of the probabilistic function Eq. (11) up to the second order are readily estimated using the theory of point estimates developed by Rosenblueth (1975), as follows:

$$E[F_s]\simeq\frac{1}{8}(F_{s+++}+F_{s++-}+\cdots\cdots+F_{s---})\tag{12}$$

$$\mathrm{var}[F_s]\simeq\frac{1}{8}(F_{s+++}^2+F_{s++-}^2+\cdots\cdots+F_{s---}^2)-E[F_s^2]\tag{13}$$

where $E[\;]$ and $\mathrm{var}[\;]$ are the mean and the variance of the random variable in brackets and eight values $F_{s+++}, F_{s++-}, \cdots\cdots$, F_{s---} denote

$$\left.\begin{aligned}&F_{s+++}=F_s(\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3+\sigma_3|G)\\&F_{s++-}=F_s(\mu_1+\sigma_1,\mu_2+\sigma_2,\mu_3-\sigma_3|G)\\&\quad\vdots\\&F_{s---}=F_s(\mu_1-\sigma_1,\mu_2-\sigma_2,\mu_3-\sigma_3|G),\end{aligned}\right.\tag{14}$$

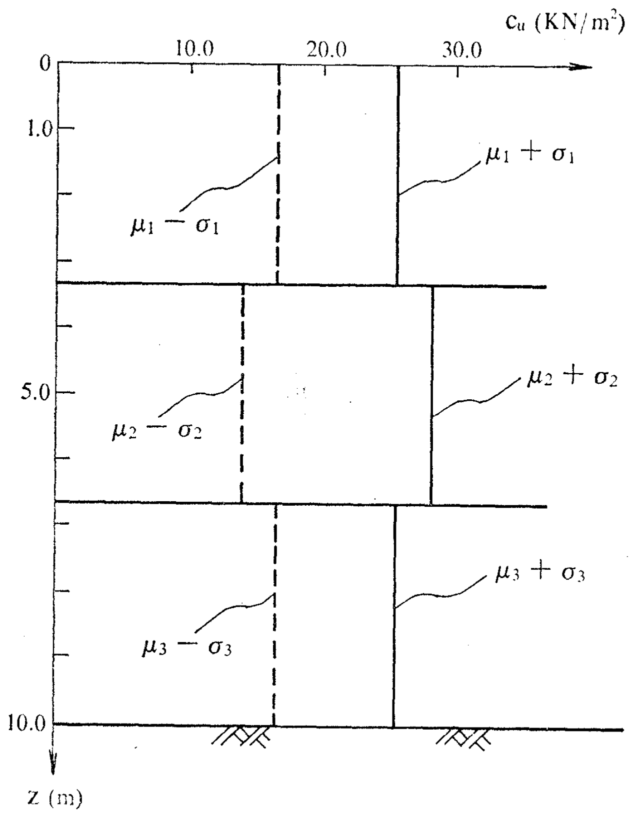

Fig. 4. Equivalent multi-layered system to Fig. 1

respectively. That is, $F_{s+++}, F_{s++-}, \cdots\cdots, F_{s---}$ are the factors of safety for the possible combinations of a plus or a minus sign in $\mu_i \pm \sigma_i$, $i=1, 2, 3$. It must be now recalled that, if c_{ui}, $i=1, 2, 3$ in Eq. (11) are deterministic values like $\mu_i \pm \sigma_i$, the F_s can be determined by the conventional $\phi_u=0$ stability analysis in a usual way. Therefore, Eq. (12) and/or Eq. (13) shows that a probabilistic slope stability analysis is no more than the combination of eight conventional, deterministic, stability analyses for eight deterministic soil profiles, $\mu_i \pm \sigma_i$, $i=1, 2, 3$. Shown in Fig. 4 is the eight soil profiles obtained from Table 2.

The comparison between Fig. 4 and the original observations given in Fig. 1 will give the intuitive interpretation for probabilistic approach developed in this study. That is, in conventional analysis, the mean value of Fig. 1 is only the soil profile to be analyzed and some other uncertain possibilities in the soil profile must be covered by the factor of safety, the value of which is subjectively selected by experienced engineers. In a probabilistic approach, on the other hand, eight deterministic soil profiles are to be analyzed, the results of which give the distribution of the possible factor of safety, at least its first and the second moments. Therefore, the decision of an optimum design alternative can be made examining whether the distribution of a factor of safety is acceptable or not, the index of which will presumably be the "probability of failure" (i. e., the probability that the factor of safety is less than unity).

ILLUSTRATIVE EXAMPLES AND DISCUSSION

The developed probabilistic approach is first discussed in the illustrative example of embankment stability on a soft clay deposit. The cross section of the embankment is shown in Fig. 5. The measured undrained strengthes of the clay deposit are plotted as has been given in Fig. 1 and then Table 1 provides the probabilistic description for the subsoil profile. Since a base failure mode is most probable in the case of Fig. 5, the equivalent multi-layered system obtained in Fig. 4 can be directly applied to this case study. Two kinds of banking height are analyzed; one is 5.5 m height while the other is 6.0 m height. In the stability analysis given here, the shear resistance of banking materials is neglected as is usual with the case of cohesionless low embankment construction (Nakase, 1967, 1970).

The results of the analysis are summerized in Tables 3 and 4, which correspond to 5.5 m height and 6.0 m height, respectively.

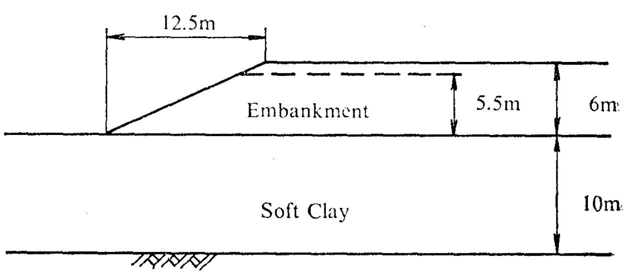

Fig. 5. Embankment cross section for illustrative example

Table 3. Probabilistic stability analysis for H=5.5 m

	1^{+++}	2^{++-}	3^{+-+}	4^{+--}	5^{-++}	6^{-+-}	7^{--+}	8^{---}
$c_{u1}:\mu_1\pm\sigma_1$ (kN/m²)	25.29	25.29	25.29	25.29	16.27	16.27	16.27	16.27
$c_{u2}:\mu_2\pm\sigma_2$ (kN/m²)	27.75	27.75	13.81	13.81	27.75	27.75	13.81	13.81
$c_{u3}:\mu_3\pm\sigma_3$ (kN/m²)	25.24	16.32	25.24	16.32	25.24	16.32	25.24	16.32
F_s	1.62	1.31	1.40	1.10	1.50	1.19	1.29	0.99
$E[F_s]$ (Eq. 12)	1.30							
$\sqrt{\mathrm{var}[F_s]}$ ($\sqrt{\text{Eq. 13}}$)	0.196							

Table 4. Probabilistic stability analysis for H=6.0 m

	1^{+++}	2^{++-}	3^{+-+}	4^{+--}	5^{-++}	6^{-+-}	7^{--+}	8^{---}
$c_{u1}:\mu_1\pm\sigma_1$ (kN/m²)	25.29	25.29	25.29	25.29	16.27	16.27	16.27	16.27
$c_{u2}:\mu_2\pm\sigma_2$ (kN/m²)	27.75	27.75	13.81	13.81	27.75	27.75	13.81	13.81
$c_{u3}:\mu_3\pm\sigma_3$ (kN/m²)	25.24	16.32	25.24	16.32	25.24	16.32	25.24	16.32
F_s	1.49	1.20	1.29	1.00	1.38	1.09	1.18	0.90
$E[F_s]$ (Eq. 12)	1.19							
$\sqrt{\mathrm{var}[F_s]}$ ($\sqrt{\text{Eq. 13}}$)	0.180							

In these tables, eight factors of safety are calculated from eight different soil profiles. Using these values, the mean and the standard deviation of a possible factor of safety are also obtained as listed in the tables. These results are discussed over the following four points:

(1) To check the consistency of the proposed method with the conventional analysis, Nakase's stability chart (Nakase, 1970) was applied to this problem using the mean value of the measured undrained strengthes given in Fig. 1. From this, it was verified that the mean values of the factors of safety obtained in Tables 3 and 4 exactly agreed with those by Nakase's stability chart.

(2) Observing the numerical relationship between F_s and $\mu_i\pm\sigma_i$, $i=1, 2, 3$, in Tables 3 and 4, it can be seen that the F_s given in Eq. (11) is almost the linear function with respect to c_{u1}, c_{u2}, and c_{u3} in their wide ranges. This equation is expressed as $F_s=0.01275\ c_{u1}+0.01506\ c_{u2}+0.03419\ c_{u3}+0.01150$ in the case of Table 3 and the equation becomes $F_s=0.01192\ c_{u1}+0.01417\ c_{u2}+0.03223\ c_{u3}-0.02057$ for Table 4. From this fact, as far as the mean value of F_s is concerned, the following simple equation is considered to give a satisfactory result, that is,

$$E[F_s]\simeq\frac{1}{2}(F_{s+++}+F_{s---}) \qquad (15)$$

The accuracy of the above equation is easily verified using the data given in Tables 3 and 4.

(3) The distribution function for F_s can be well approximated by the normal distribution, which is also certified by the linearity in the function of F_s with respect to c_{u1}, c_{u2} and c_{u3}, since the undrained strength is well known to follow a normal-like distribution. The probability of failure at G-condition can be, therefore, well evaluated by

$$P_f(G)\simeq\Phi\left[\frac{1-E[F_s]}{\sqrt{\mathrm{var}[F_s]}}\right] \qquad (16)$$

Where Φ denotes the cumulative distribution function of a standardized normal random variable. Applying Eq. (16) for the case study, $P_f(H=5.5\ \mathrm{m})=6.4\%$ and $P_f(H=6.0\ \mathrm{m})=14.4\%$ are obtained. Using these two values, the probability of failure at $H=6.0$ m given the condition that the embankment is safe up to $H=5.5$ m, for instance, can also be obtained as follows:

$$\mathrm{Prob.}[F_s\leqq 1 \text{ at } H=6.0\,\mathrm{m}\,|\,F_s>1 \text{ up to } H=5.5\,\mathrm{m}] =\frac{P_f(H=6.0\,\mathrm{m})-P_f(H=5.5\,\mathrm{m})}{1-P_f(H=5.5\,\mathrm{m})}=8.5\%$$

The general discussion for the above equation can be found in Matsuo and Asaoka, 1978.

(4) The distribution for F_s discussed in this study should be noted to have been derived only from the probabilistic soil profile

description. There are many other sources which bring uncertainty into the stability analysis. For example, a different method of soil testing will give a different result. Furthermore, $\phi_u=0$ and circular arc slip surface assumptions are known to have their own analytical errors. Even in the probabilistic description, the statistical error in the parameter estimates which comes from the limitation of the number of available soil samples is neglected. All these effects are desirable to be taken into consideration in a practical design. It is, however, still important to know the possible range of F_s due to the spatial variability of c_u because the contribution of this variability to the total uncertainty of F_s is known to be almost the same as the contribution of the other sources of uncertainty mentioned above (e. g., Wu, 1974, Matsuo and Asaoka, 1976, Tang et al, 1976, etc.).

APPLICATION TO A MULTI-FAILURE MODE PROBLEM

In this section, the developed procedure is applied to a double failure mode as shown in Fig. 6, in which the original subsoil system consists of two different clay layers. As illustrated in the figure, two base failures are analyzed, one is the base failure on the surface of the lower clay deposit while the other is the failure which reaches to the surface of the bed rock. The problem is to find the correlation between two failure modes as well as to determine the distribution of the factor of safety for each failure mode. This can be done introducing the equivalent multi-layered system as shown in Fig. 7. To the upper part in the original system are corresponding the first three layers with the strengthes c_{u1}, c_{u2} and c_{u3}, the parameters of which are determined by Eqs. (9) and (10), replacing $\mu(z)$, $\mathrm{cov}(z, z')$, H and Δ with $\mu_{\mathrm{I}}(z)$, $\mathrm{cov}_{\mathrm{I}}(z, z')$, H_{I} and Δ_{I}, respectively. These notations are given in Figs. 6 and 7. Using these parameters, since the values for $E[R_{L_{\mathrm{I}}}]$ and $E[R_{L_{\mathrm{I}}}\cdot R_{L_{\mathrm{I}}'}]$ in the equivalent multi-layered system become the same as those of the original system, respectively, for any given slip surfaces $L_{\mathrm{I}}, L_{\mathrm{I}}'$ passing from $z=0$ to $z=H_{\mathrm{I}}$, the distribution for $F_{s\mathrm{I}}$ (the factor of safety for the first failure mode) are naturally expected to be common between the original and the equivalent multi-layered system. As for the lower clay deposit in the original system, the parameters of the equivalent multi-layered system can be determined solving the following two simultaneous equations with respect to μ_i, and σ_i, $i=4, 5, 6$:

$$\begin{pmatrix} \int_{H_1}^{H_2} \mu(z)dz \\ \int_{H_1}^{H_2} z\mu(z)dz \\ \int_{H_1}^{H_2} z^2\mu(z)dz \end{pmatrix} = \begin{pmatrix} \Delta_{\mathrm{II}} & , & \Delta_{\mathrm{II}} & , & \Delta_{\mathrm{II}} \\ \Delta_{\mathrm{II}}\cdot H_1+\frac{1}{2}\Delta_{\mathrm{II}}^2 & , & \Delta_{\mathrm{II}}\cdot H_1+\frac{3}{2}\Delta_{\mathrm{II}}^2 & , & \Delta_{\mathrm{II}}\cdot H_1+\frac{5}{2}\Delta_{\mathrm{II}}^2 \\ \Delta_{\mathrm{II}}\cdot H_1^2+\Delta_{\mathrm{II}}^2\cdot H_1+\frac{1}{3}\Delta_{\mathrm{II}}^3, & & \Delta_{\mathrm{II}}\cdot H_1^2+3\Delta_{\mathrm{II}}^2\cdot H_1+\frac{7}{3}\Delta_{\mathrm{II}}^3, & & \Delta_{\mathrm{II}}\cdot H_1^2+5\Delta_{\mathrm{II}}^2\cdot H_1+\frac{19}{3}\Delta_{\mathrm{II}}^3 \end{pmatrix} \cdot \begin{pmatrix} \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix} \quad (17)$$

$$\begin{pmatrix} \int_{H_1}^{H_2}\int_{H_1}^{H_2} \mathrm{cov}(zz')dzdz' \\ \int_{H_1}^{H_2}\int_{H_1}^{H_2} \mathrm{cov}(zz')zdzdz' \\ \int_{H_1}^{H_2}\int_{H_1}^{H_2} \mathrm{cov}(zz')zz'dzdz' \end{pmatrix} = \begin{pmatrix} \Delta_{\mathrm{II}}^2 & , & \Delta_{\mathrm{II}}^2 & , & \Delta_{\mathrm{II}}^2 \\ \Delta_{\mathrm{II}}^2\cdot H_1+\frac{1}{2}\Delta_{\mathrm{II}}^3 & , & \Delta_{\mathrm{II}}^2\cdot H_1+\frac{3}{2}\Delta_{\mathrm{II}}^3 & , & \Delta_{\mathrm{II}}^2\cdot H_1+\frac{5}{2}\Delta_{\mathrm{II}}^3 \\ \frac{1}{4}\Delta_{\mathrm{II}}^2(2H_1+\Delta_{\mathrm{II}})^2, & & \frac{1}{4}\Delta_{\mathrm{II}}^2(2H_1+3\Delta_{\mathrm{II}})^2, & & \frac{1}{4}\Delta_{\mathrm{II}}^2(2H_1+5\Delta_{\mathrm{II}})^2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_4^2 \\ \sigma_5^2 \\ \sigma_6^2 \end{pmatrix} \quad (18)$$

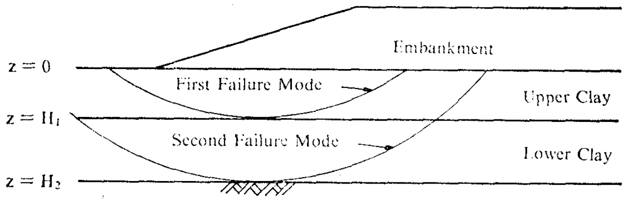

Fig. 6. Double failure mode problem

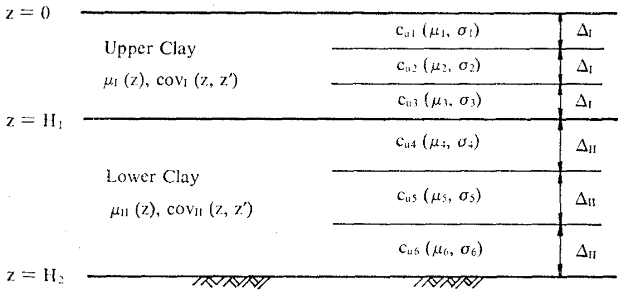

Fig. 7. Equivalent multi-layered system for Fig. 6

in which $\mu_{II}(z)$ and $\text{cov}_{II}(z, z')$ are the probabilistic soil profile description for $c_u(z)$ of the lower clay deposit. Eqs. (17) and (18) were derived so that the values for $E[R_{LII}]$ and $E[R_{LII} \cdot R_{LII'}]$ calculated from the equivalent multi-layered system might become the same as those of the original system, respectively, for any arbitrary slip surfaces L_{II} and L_{II}' as long as they pass through the surface of the bed rock below the lower clay deposit.

Following the same logic of the equivalent multi-layered system given in the former section, the factor of safety for each failure mode is now expressed as a probabilistic function of discrete random variables c_{ui}, $i=1, 2, \cdots\cdots, 6$, as follows:

$$F_{sI} = F_{sI}(c_{u1}, c_{u2}, c_{u3}) \qquad (19)$$

$$F_{sII} = F_{sII}(c_{u1}, c_{u2}, c_{u3}, c_{u4}, c_{u5}, c_{u6}) \qquad (20)$$

in which F_{sI} and F_{sII} denote the factors of safety for the shallow and the deep base failure modes, respectively. Then, as previously introduced, the point estimates theory becomes applicable and the first and second probability moments for the joint distribution of F_{sI} and F_{sII} can be estimated in the following manner:

$$\left.\begin{aligned}
E[F_{sI}] &= \frac{1}{8}\{F_{sI+++} + F_{sI++-} + \cdots\cdots + F_{sI---}\} \\
\text{var}[F_{sI}] &= \frac{1}{8}\{F_{sI+++}^2 + F_{sI++-}^2 + \cdots\cdots + F_{sI---}^2\} - E[F_{sI}]^2 \\
E[F_{sII}] &= \frac{1}{64}\sum F_{sII} \\
\text{var}[F_{sII}] &= \frac{1}{64}\sum F_{sII}^2 - E[F_{sII}]^2 \\
\text{cov}[F_{sI}, F_{sII}] &= \frac{1}{64}\{\sum F_{sI} \cdot F_{sII}\} \\
&\quad - E[F_{sI}] \cdot E[F_{sII}]
\end{aligned}\right\} \qquad (21)$$

in which $\sum F_{sII}$, $\sum F_{sII}^2$ and $\sum F_{sI} \cdot F_{sII}$ denote the sums of the sixty four different values of the functions, F_{sII}(=Eq. (20)), F_{sII}^2(=Eq. (20)^2) and $F_{sI} \cdot F_{sII}$(Eq. (19)·Eq. (20)), respectively, calculated from the substitution of $\mu_i \pm \sigma_i$ into c_{ui}, $i=1, 2, \cdots\cdots 6$. This is merely because the number of the possible combinations of plus and/or minus signs for $i=1, 2, \cdots\cdots, 6$ is $2^6=64$. As the results of this procedure, it can be said that the joint distribution for F_{sI} and F_{sII} is obtainable by analyzing the stability for sixty four varieties of a layered soil systems.

As previously discussed, both F_{sI} and F_{sII} are considered to follow approximately a normal distribution, the probabilities such as Prob $[F_{sI} < 1.0]$, Prob $[F_{sII} < 1.0]$, and Prob$[F_{sI}$ and/or $F_{sII} < 1.0]$ can be calculated from the two dimensional normal distribution for F_{sI} and F_{sII}, the parameters of which are given in Eqs. (21).

The procedure developed in this section was applied to the case study of the embankment stability at the Ebetsu test fill section, the portion of the Central Express Way in Hokkaido constructed by the Japan Highway Public Corporation. The cross section of the embankment and the subsoil layers are il-

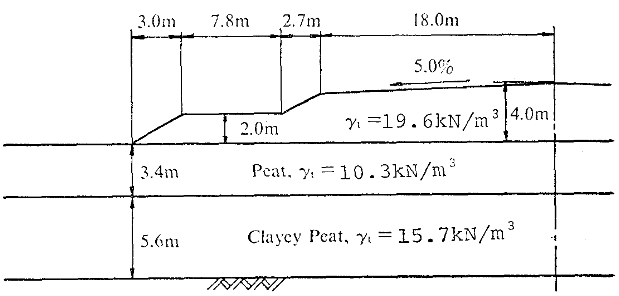

Fig. 8. Embankment cross section for case study

Table 5. Probabilistic soil profile description for case study

pea	$\mu_I(z)$	12.25 (kN/m²)
	$\mathrm{cov}_I(z, z')$	$24.01\times\exp\left(-\frac{\lvert z-z'\rvert}{2.0}\right)$ (kN/m²)²
clayey peat	$\mu_2(z)$	17.54 (kN/m²)
	$\mathrm{cov}_{II}(z, z')$	$43.11\times\exp\left(-\frac{\lvert z-z'\rvert}{2.0}\right)$ (kN/m²)²

Table 6. Numerical results of case study

	CASE 1*	CASE 2**
$E[F_{sI}]$	1.78	1.18
$\sqrt{\mathrm{var}[F_{sI}]}$	0.48	0.28
$E[F_{sII}]$	1.36	1.21
$\sqrt{\mathrm{var}[F_{sII}]}$	0.28	0.24
$\rho(F_{sI}, F_{sII})$***	0.25	0.26
Prob$[F_{sI}\leq 1]$	5.3(%)	25.1(%)
Prob$[F_{sII}\leq 1]$	10.0(%)	18.4(%)
Prob$[F_{sI}\leq 1$ and/or $F_{sII}\leq 1]$	14.2(%)	36.8(%)

* CASE 1 : Shear resistance of banking materials is neglected and vertical tension cracks are assumed within an embankment.

** CASE 2 : Shear resistance of banking materials is introduced along a potential slip surface with parameters of $c=0$ and $\phi=30°$ (0.523 radian).

*** $\rho(,)$: Correlation coefficient between the two values in parentheses.

lustrated in Fig. 8. As shown in this figure, the upper part of the soft layers is made of peat while the lower of soft clay with peat. In Table 5 are listed the parameters of the probabilistic soil profile of the undrained shear strengthes for this layered system. Two failure modes were naturally expected, one is the failure within the peat deposit while the other is the failure including both upper and lower clay deposit. Since each soil deposit was not so thick that these two types of failure were analyzed assuming the base failure mode as suggested from the conventional deterministic stability analysis. That is, the first one is failure with the slip surface which touches the surface of the lower clay layer (the first failure mode) while the second one is the slip failure on the top of the bed rock (the second failure mode). The application of the developed probabilistic approach was tried to this problem, the results of which are summarized in Table 6. As shown in the table, two cases were analyzed, one is the case where the shear resistance of the banking materials was taken into the analysis with the parameters of $c=0$ and $\phi=30°$, while in the other case the resistance was neglected.

In the actual condition of this embankment, it should be stated that the large lateral deformation was observed from the top to the bottom of the original two-layered system, although an actual slip failure was not experienced.

CONCLUSIONS

The conventional $\phi_u=0$ stability analysis was newly formulated under the condition of uncertainty in soil profile description. The developed procedure includes the simple scheme by which the spatial variation of undrained shear strengthes should be modelled for the purpose of $\phi_u=0$ analysis. On the basis of the analysis performed and results obtained in this study, the following conclusions can be drawn:

(1) Using the limited number of strength data of clays, a stochastic model is naturally introduced to express the soil profile. The simple procedure for this was presented.

(2) The probabilistic soil profile description was shown to be approximately equivalent to the set of eight different deterministic soil profiles, each of which is a three-layered system with three different strengthes (equivalent multi-layered system). From this, it was demonstrated that the possible distribution of the factor of safety in $\phi_u=0$ analysis could be found by only eight trials of the conventional deterministic stability analyses.

(3) The simplicity as well as the accuracy of the developed procedure was verified for the use in practice through examining the case studies including a double failure mode problem.

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