SOILS AND FOUNDATIONS Vol. 23, No. 4, Dec. 1983
Japanese Society of Soil Mechanics and Foundation Engineering

UPPER BOUND LIMIT ANALYSIS OF SOIL WITH NON-LINEAR FAILURE CRITERION

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ABSTRACT

The paper discusses the effects of non-linearity in the failure criterion of soil on the upper bound solution procedure. By considering the inherent non-linearity of the failure criterion, it is demonstrated that the upper bound procedure yields not only the minimum value of and the external load and the failure mechanism, but also the stress distribution along the slip surface. The stress distribution so obtained satisfies a Kötter type differential equation and guaranties the global equilibrium of the sliding mass. This result is valid for both linear and non-linear failure criteria. When the failure criterion is linear, it is demonstrated that for a simple failure mechanism the classical solution procedure provides only a partial solution which does not include the normal stress distribution. This is due to the different consequence of the normality requirement in the linear and non-linear cases.

Key words: bearing capacity, failure, plasticity, slip surface, stability analysis, (failure criteria), (limit analysis), (non-linearity) (**IGC**: E 0/E 3/E 6)

INTRODUCTION

The upper bound theorem of plasticity provides a convenient framework for the derivation of approximate solutions to various stability problems. The approach has been successfully applied to a number of soil mechanics problems (e. g. Finn, 1967) such as bearing capacity (e. g. Chen and Davidson, 1973), slope stability (e. g. Chen and Giger, 1971) and earth pressure calculations (e. g. Chen and Rosenfarb, 1973). These applications have always utilized a linear Mohr-Coulomb failure envelope. Although this is generally a reasonable approximation, there is evidence that in many cases the failure envelope of soils is not well represented by a straight line (e. g. Vesic and Clough, 1968; Lee and Seed, 1967). The significance of this departure from linearity may be appreciated by the fact that it has been used to explain scale effects in bearing capacity problems (e. g. De Beer, 1970; Vesic, 1975).

The validity of the upper bound theorem is not dependent on the shape of the failure envelope and therefore it is, in principle, applicable to materials with non linear failure envelopes. In this paper it is demonstrated that there is a fundamental difference in the procedures used for applying the theorem to materials with linear and non linear failure envelopes. This difference is due to the different roles played by the normality criterion in these two cases.

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Manuscript was received for review on February 1, 1983.
Written discussions on this paper should be submitted before October 1, 1984.

THE UPPER BOUND THEOREM

The upper bound theorem of plasticity (Drucker et al, 1952) has been presented in various ways; for the present purpose it will be stated in the following fashion:

"The load calculated by equating the rate of work of all external forces with the rate of energy dissipation in the plastic zone is not lower than the "true" limit load, provided that only kinematically admissible collapse mechanisms are considered."

The "true" limit load is that determined from a complete plasticity solution of the problem. A kinematically admissible collapse mechanism is a field of plastic displacement increments (velocities) which satisfied the following conditions:

(a) The velocity field is such that no gaps or penetrations occur.

(b) The velocity field satisfies the velocity boundary conditions.

(c) The rate of work produced by the external forces is non negative.

(d) The velocity field is consistent with the associative flow rule of plasticity (i. e. the normality requirement).

The solution procedure using the upper bound theorem generally consists of the following stages:

(a) A class of kinematically acceptable collapse mechanisms is postulated in terms of some geometrical variables.

(b) For any value of the geometrical variables, an estimate of the collapse load is obtained by equating the rate of external work with the rate of internal dissipation. The estimate of the collapse load is therefore a function of the geometrical variables defining the collapse mechanism.

(c) The lowest value of these estimates represents the least upper bound value for this class of collapse mechanisms.

(d) Other classes of collapse mechanisms may be considered, and the least upper bound obtained from all classes is taken as the best estimate of the true collapse load.

THE FRAMEWORK CONSIDERED

Discontinuous velocity fields are kinematically admissible, and have been sound useful for the derivation of upper bound solutions. The simplest type of such a field consists of the movement of a single rigid body on a surface of discontinuity. As the purpose of this note is to discuss the implication of non-linearity of the strength envelope and not to derive particular solutions, this simple mechanism will be used throughout the following presentation. The approach will be developed by referring to a particular problem-that of the bearing capacity of a strip footing on the upper surface of a slope (Fig. 1). The soil is assumed isotropic and homogeneous and no pore pressures exist.

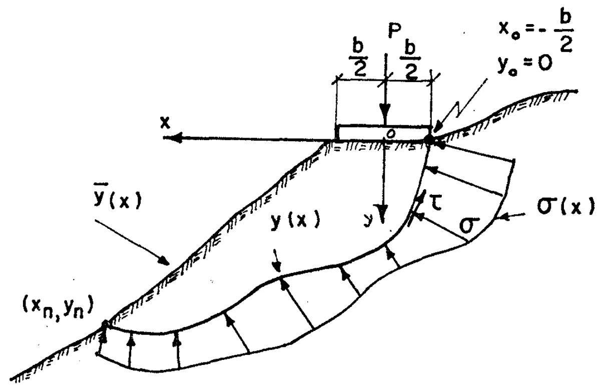

Fig. 1. The scheme considered-A loaded strip footing on a slope surface

The failure criterion of the soil is expressed by:

$$\tau=f(\sigma)=c+\psi_s(\sigma)\cdot\sigma \qquad (1)$$

where τ and σ are the shear and normal stresses respectively on the failure surface; c is the cohesion and $\psi_s(\sigma)=\tan[\phi_s(\sigma)]$ where ϕ_s is a stress dependent secant friction angle (see Fig. 2).

The expression $\tau=f(\sigma)$ is the functional relation defining the possibly non linear failure criterion.

In addition to the secant quantities ψ_s and ϕ_s, it is convenient to introduce the tangential friction angle ϕ_t and tangential friction coefficient $\psi_t=\tan\phi_t$. The relation between ψ_s and ψ_t is:

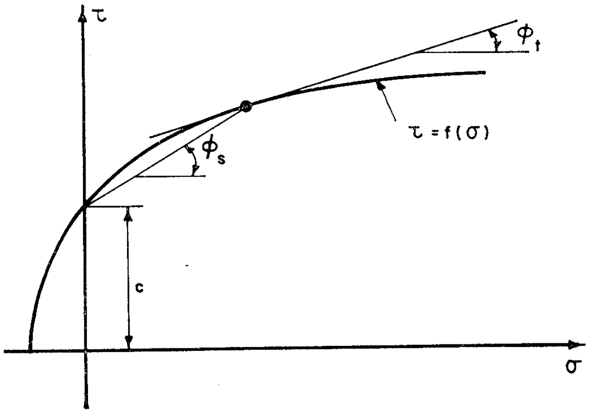

Fig. 2. A non-linear failure criterion

$$\phi_t \equiv \frac{df}{d\sigma} = \phi_s + \sigma\frac{d\phi_s}{d\sigma} \qquad (2)$$

THE UPPER BOUND APPROACH

Formulation of the Problem

The upper bound procedure requires that the rate of external work be equated to the rate of dissipation within the plastic zone. This requirement may be expressed as:

$$\dot{W}_{ex}(P, \beta_i) = D(\beta_i) \qquad (3)$$

where $\dot{W}_{ex}$=rate of external work, P=applied load, β_i=a set of geometrical parameters defining the failure mechanism, and D= rate of dissipation.

For the present purpose, it is convenient to write Eq. (3) as follows:

$$\dot{W} = \dot{W}(P, \beta_i) = D - \dot{W}_{ex} = 0 \qquad (4)$$

where $\dot{W}$ may be defined as the total virtual work of the system.

Considering a single rigid body, $\dot{W}$ is given by (e.g. Synge and Griffith, 1959),

$$\dot{W} = \dot{u}H + \dot{v}V + \dot{\Omega}M = 0 \qquad (5)$$

where M is the resultant moment about some reference point.

$\dot{u}$, $\dot{v}$ are the rates of horizontal and vertical virtual displacement of the reference point.

In the present case, the reference point is taken as point 0 in Fig. 1, through which the external load P is applied.

H, V are the resultant horizontal and vertical forces acting on the rigid body.

$\dot{\Omega}$ is the rate of virtual rotation of the rigid body.

Referring to Fig. 1, expressions for H, V and M may be developed (e.g. Garber and Baker, 1977), as follows:

$$H = \int_{x_0}^{x_n} [\sigma(\phi_s - y') + c]dx \qquad (6a)$$

$$V = \int_{x_0}^{x_n} [\sigma(1 + \phi_s y') + cy' - \gamma(y - \bar{y})]dx - P \qquad (6b)$$

$$M = \int_{x_0}^{x_n} \{\sigma[\phi_s(y - y'x) - (x + y'y)] + c(y - y'x) + \gamma(y - \bar{y})x\}\, dx \qquad (6c)$$

where $y = y(x)$ = the equation describing the discontinuity (i.e. the slip surface), $\sigma = \sigma(x)$ = the normal stress acting on the slip surface, x_0, x_n are the end points of the slip surface, $y' = dy/dx$, $\bar{y} = \bar{y}(x)$ = the equation describing the soil surface, and γ = total unit weight of the soil

Taking $\dot{v}$ as unity with $\dot{u}$ and $\dot{\Omega}$ then taking their compatible values and defining $V_1 = V + P$, Eq. (5) may be solved for P yielding

$$P = V_1 + H\dot{u} + M\dot{\Omega} \qquad (7)$$

Introducing the expressions for H, V_1, and M from Eq. (6) into Eq. (7), the following equation is obtained:

$$P = V_1 + H\dot{u} + M\dot{\Omega} = G[y(x), \sigma(x)] = \int_{x_0}^{x_n} g\,dx \qquad (8)$$

where G is the functional relation between the load P and the unknown functions $y(x)$ and $\sigma(x)$.

The function g is given by

$$g = \dot{\Omega}[(\phi_s g_1 + g_2)\sigma + (cg_1 - \gamma g_3)] \qquad (9)$$

where

$$g_1 = \left(\frac{\dot{u}}{\dot{\Omega}} + y\right) + y'\left(\frac{1}{\dot{\Omega}} - x\right) \qquad (10a)$$

$$g_2 = \left(\frac{1}{\dot{\Omega}} - x\right) - y'\left(\frac{\dot{u}}{\dot{\Omega}} + y\right) \qquad (10b)$$

$$g_3 = \left(\frac{1}{\dot{\Omega}} - x\right)(y - \bar{y}) \qquad (10c)$$

In developing Eqs. (8) to (10), it has been assumed that $\dot{\Omega}$ is not zero. Consequently this set of equations applies to a rotational mode of failure. An alternative set of equations for a translational failure mode can be derived by setting $\dot{\Omega}=0$ in Eq. (7).

Comparing Eqs. (5) and (8), it is clear that G represents that part of the total virtual work which is not related to the external load P. This part includes the work $\dot{W}_r$ done by the body forces (weight) and the dissipated work D. On the basis of Eq. (9), the following identifications may be made:

$$G=D-\dot{W}_r \tag{11a}$$

$$D=\dot{\Omega}\int_{x_0}^{x_n}[(\psi_s g_1+g_2)\sigma+cg_1]dx \tag{11b}$$

$$\dot{W}_r=\dot{\Omega}_r\int_{x_0}^{x_n} g_3 dx \tag{11c}$$

As a results of equating the rates of external work and internal dissipation, Eq. (8) has been obtained, expressing the load P as a functional of $y(x)$ and $\sigma(x)$. According to the upper bound procedure, the least upper bound value of P is therefore obtained by minimizing this functional relation with respect to the parameters β_i defining the failure mechanism. The minimization must be subjected to the constraint that the failure mechanism is kinematically admissible. This procedure may be summarized in the following equations:

$$P_m=\min_{\beta_i}[G] \tag{12a}$$

$$K(\beta_i)=0 \tag{12b}$$

where P_m is the minimum value of P, and $K(\beta_i)$ represents the kinematic constraints.

The Kinematic Constraints

In the present case, for a smooth footing which is free to rotate, there are no velocity boundary conditions.

The requirement of non negative rate of external work will be satisfied if both the point of application of the load and the centre of gravity of the sliding mass move downwards.

Attention is now turned to the normality requirement. According to Hill (1959), the requirement that the plastic velocity field satisfy the associative flow rule is equivalent to the requirement that the rate of plastic work (i.e. dissipation) is stationary with respect to the stress, for all stress systems satisfying the failure criterion. Consequently, the normality requirement is equivalent to the following expression:

$$\frac{\delta D}{\delta\sigma}=0 \tag{13}$$

where δ is the variational operator. Referring to Eq. (11), since $\dot{W}_r$ is independent of σ, Eq. (13) may be replaced by

$$\frac{\delta G}{\delta\sigma}=0 \tag{14}$$

Since G is a functional of $\sigma(x)$, this stationary requirement may be expressed by the following Euler equation (Elsgolc, 1972).

$$\frac{\delta G}{\delta\sigma}=\frac{d}{dx}\left[\frac{\partial g}{\partial\sigma'}\right]-\frac{\partial g}{\partial\sigma}=0 \tag{15}$$

From Eqs. (15), (9) and (2) it follows that

$$g_1\psi_t+g_2=0 \tag{16}$$

Substituting g_1 and g_2 from Eqs. (10):

$$\psi_t\left[\left(\frac{\dot{u}}{\dot{\Omega}}+y\right)+y'\left(\frac{1}{\dot{\Omega}}-x\right)\right]+\left[\left(\frac{1}{\dot{\Omega}}-x\right)-y'\left(\frac{\dot{u}}{\dot{\Omega}}+y\right)\right]=0 \tag{17}$$

Eq. (17) can be simplified considerably using the following coordinate transformation:

$$y=-\frac{\dot{u}}{\dot{\Omega}}+r\cos\theta \tag{18a}$$

$$x=\frac{1}{\dot{\Omega}}-r\sin\theta \tag{18b}$$

The coordinates r and θ represent a polar system centred at the point

$$x_c=-\frac{\dot{u}}{\dot{\Omega}},\quad y_c=\frac{1}{\dot{\Omega}}\quad \text{(see Fig. 3)}$$

Eq. (17) then yields:

$$\frac{dr}{d\theta}=-\psi_t(\sigma)r \tag{19}$$

Eq. (19) is a well known condition (e.g. Davis, 1968) for kinematic admissibility in the case of rigid body rotation. This equation was derived from the requirement of normality; the geometric significance of this requirement is that the plastic velocity vector, Δ, at the slip surface acts at an angle ϕ_t to the surface. Combining this with Eq. (19):

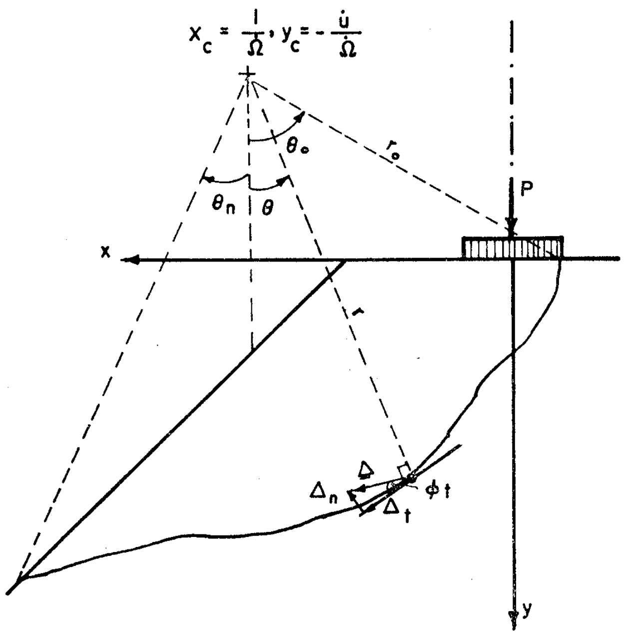

Fig. 3. The kinematic constraints of the system

$$\frac{\Delta_N}{\Delta_T}=\psi_t(\sigma)=-\frac{1}{r}\frac{dr}{d\theta} \qquad (20)$$

where Δ_N and Δ_T are components of Δ normal and tangential to the slip surface (Fig. 3).

It is instructive now to consider separately the following two limiting cases:

(a) Linear Failure Criterion—The assumption of linearity has the following consequences:

1. Eq. (20) represents a severe kinematic constraint, since $\psi_t=\psi_s=\psi=$const. and therefore only a single value of Δ_N/Δ_T is possible all along the slip surface.

2. With a constant ψ, Eq. (20) can be integrated, yielding:

$$r=Ae^{-\psi\theta} \qquad (21)$$

where A is an integration constant

3. In order to obtain an expression for the rate of dissipation, D, which includes the effect of the normality requirement, Eqs. (16) and (2) are substituted into Eq. (11 b), resulting with

$$D=\dot{\Omega}\int_{x_0}^{x_n}\left(c-\sigma^2\frac{d\psi_s}{d\sigma}\right)g_1dx \qquad (22)$$

For ψ_s constant, this expression simplifies to

$$D=\dot{\Omega}c\int_{x_0}^{x_n}g_1dx \qquad (23)$$

Using Eqs. (10 a), (18) and (19), Eq. (23) yields:

$$D=\dot{\Omega}c\int_{\theta_n}^{\theta_0}r^2(\theta)d\theta \qquad (24)$$

Eq. (24) is essentially the same as the expression presented by Chen (1975) for the case of a linear failure criterion.

(b) Non—Linear Failure Criterion—A limiting case of a non-linear failure criterion is considered (Fig. 2) where ϕ_t varied from infinity to zero as the normal stress increases; this has the following consequences:

1. Eq. (20) restricts the form of the slip surface only to the extent that $dr/d\theta$ must be negative, but the function $r(\theta)$ may be arbitrary otherwise. This freedom in the form of $r(\theta)$ is due to the fact that any vector Δ will be normal to the failure curve at some point; the point at which this normality occurs defines a unique value of σ(Fig. 4(a)). Hence, Eq. (20) constitutes, in effect, a relationship between the functions $r(\theta)$ and $\sigma(\theta)$. This is in contrast to the linear case where the vector Δ is normal at

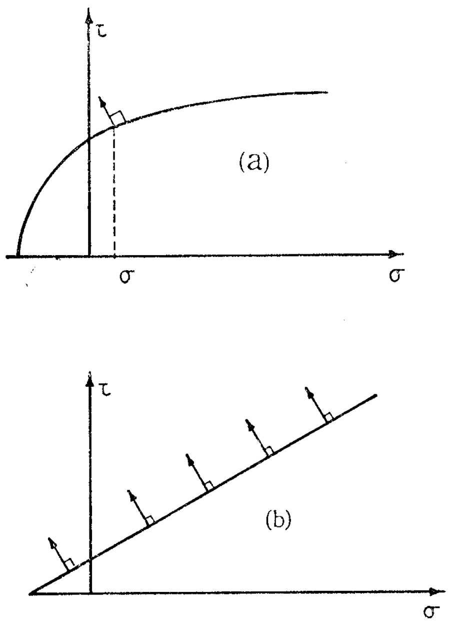

Fig. 4. Consequence of the normality requirement for (a) non-linear, (b) linear, failure criterion

every point along the linear failure curve and therefore cannot be associated with a unique value of σ (Fig. 4(b)).

2. Eq. (20) cannot be integrated without prior specification of the function $\sigma(\theta)$. Consequently it is not possible to obtain an equation such as Eq. (21) defining the shape of the failure surface.

3. The rate of dissipation is given by Eq. (22); in this case it depends on σ.

Solution Procedures

Two solution procedures are considered. Procedure A, which follows directly from Eq. (12) is based on minimization of the functional G with respect to the geometrical parameters β_i; in procedure B this minimization is done indirectly by solving a system of equations. The set of parameters β_i is defferent for linear and non linear failure criteria due to the different roles played by the normality constraint in these two cases. Consequently it is necessary to consider solution procedures separately for the linear and non linear cases.

(a) Procedure A, Linear Failure Criterion—Substituting Eq. (21) into Eq. (24), and using Eq. (11 a), it follows that G is a function of A, $\dot{u}$ and $\dot{\Omega}$. The set of parameters β_i defining the failure mechanism (see Eq. (3)) is therefore $(A, \dot{u}, \dot{\Omega})$. Consequently, Eq. (12) can be written as:

$$P_m = \min_{(A, \dot{u}, \dot{\Omega})} G(A, \dot{u}, \dot{\Omega}) \tag{25a}$$

$$y(x=-b/2) = y_0(A, \dot{u}, \dot{\Omega}) = 0 \tag{25b}$$

where Eq. (25 b) is a geometrical boundary condition requiring that the slip surface starts at the footing edge (see Fig. 1): Note that $\dot{u}$ and $\dot{\Omega}$ define the origin of the polar coordinate system (see Fig. 3) so that the minimization in Eq. (25 a) locates the critical center of the coordinate system, which is also the center of rotation. When $\dot{\Omega}\to 0$ the center of rotation is located at infinity; this situation corresponds to a translation mode of failure. Consequently translation can be viwed as a limiting case of rotation and need not be studied separately.

Eq. (25) represent the classical solution procedure (e. g. Chen, 1975), which yields the minimum load P_m and the slip surface $y(x)$, but not the normal stress distribution $\sigma(x)$.

(b) Procedure A, Non-Linear Failure Criterion—In this case the normality constraint constitutes a relation between the slip surface and normal stress functions. This relation makes it possible to consider G as a function of $y(x)$ only (see Eq. (8)). Consequently, the set of parameters β_i defining the failure mechanism, is the totality of values of the function $y(x)$. Therefore, Eq. (12 a) defines the variational problem of minimizing G with respect to the function $y(x)$, i. e.

$$\frac{\delta G}{\delta y} = 0 \tag{26}$$

Eq. (26) is a necessary but not sufficient condition for a minimum. However due to the upper bound theorem, the existence of minimum is guaranteed and there is no need to study the nature of the stationary point using the second variation $(\delta^2 G/\delta^2 y)$.

Eq. (26) is equivalent to the following Euler equation

$$\frac{\delta G}{\delta y} = \frac{d}{dx}\left[\frac{\partial g}{\partial y'}\right] - \frac{\partial g}{\partial y} = 0 \tag{27}$$

Substituting from Eqs. (9) and (10), and using the differential Eq. (19) and the coordinate transformation (18):

$$\frac{d\sigma}{d\theta} - 2\,\psi_s(\sigma)\sigma - 2c + \gamma r \sin\theta = 0 \tag{28}$$

The form of this equation is the same as that of one of the Kötter equations (e. g. Davis, 1968). It is however a generalization of the usual Kötter equation since it is valid for a non linear failure criterion.

Eqs. (19) and (28) constitute a pair of simultaneous differential equations for the determination of the forms of $r(\theta)$ and $\sigma(\theta)$; the solution depends on the particular function $\psi(\sigma)$ considered, but can always be written as follows;

$$r = r(\theta | A, B, \dot{u}, \dot{\Omega}) \tag{29a}$$

$$\sigma = \sigma(\theta | A, B, \dot{u}, \dot{\Omega}) \tag{29b}$$

where A, B are two integration constants and notation $(\theta|A, B, \dot{u}, \dot{\Omega})$ is used to emphasize the dependence of the solution on the four unknown parameters $A, B, \dot{u}, \dot{\Omega}$

Substituting Eq. (29) into the definition of G makes this quantity a function of the four parameters $A, B, \dot{u}, \dot{\Omega}$. Hence the solution procedure given by Eq. (12) becomes:

$$P_m = \min_{A, B, \dot{u}, \dot{\Omega}} G(A, B, \dot{u}, \dot{\Omega}) \qquad (30a)$$

$$y_0(A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (30b)$$

Eqs. (30) yield the minimum load P_m, the slip surface $y(x)$ and the normal stress distribution $\sigma(x)$.

(c) Procedure B—Both Eqs. (25) and (30) require minimization of G with respect to $\dot{u}$ and $\dot{\Omega}$. This requirement is equivalent to: $\partial G/\partial \dot{u}=0$; $(\partial G/\partial \dot{\Omega})=0$. From Eq. (8), It is seen that $(\partial G/\partial \dot{u})=H$ and $(\partial G/\partial \dot{\Omega})=M$. Consequently, the minimization with respect to $\dot{u}$ and $\dot{\Omega}$ is equivalent to the satisfaction of the equilibrium conditions $H=0$, $M=0$. Furthermore, from Eq. (5) also $V=0$, so that all three equilibrium equations are satisfied. This result is due to the equivalence between the virtual work statement $\dot{W}=0$, and the equilibrium requirement.

Non-Linear Failure Criterion—Since minimization of G with respect to $\dot{u}$ and $\dot{\Omega}$ is equivalent to $H=0$, $M=0$, Eqs. (30) can be rewritten as:

$$P_m = \min_{A, B} G(A, B, \dot{u}, \dot{\Omega}) \qquad (31)$$

Subject to:

$$y_0 = (A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (32a)$$

$$H = H(A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (32b)$$

$$M = M(A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (32c)$$

Eqs. (32) may be solved to express three of the parameters $A, B, \dot{u}, \dot{\Omega}$ in terms of the fourth; hence the minimization of G in Eq. (31) has to be done in terms of one parameter only. Furthermore, even this minimization can be avoided by the use of the transversality condition which fixes the critical location of the exit point x_n, y_n (Fig. 1) and is expressed by (Elsgolc, 1972):

$$\left[g + (\bar{y}' - y')\frac{\partial g}{\partial y'}\right]_{x=x_n} = 0 \qquad (33)$$

where $\bar{y}' = \bar{y}'(x)$ is the slope of the soil surface.

Substituting Eqs. (29) into Eq. (33), the transversality equation can be expressed as $T(A, B, \dot{u}, \dot{\Omega}) = 0$.

Solution procedure B therefore reduces to determination of the four constants A_0, B_0, $\dot{u}_0$, $\dot{\Omega}_0$ as a solution of the set of simultaneous equations:

$$y_0(A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (34a)$$

$$H(A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (34b)$$

$$M(A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (34c)$$

$$T(A, B, \dot{u}, \dot{\Omega}) = 0 \qquad (34d)$$

Following determination of the constants A_0, B_0, $\dot{u}_0$, $\dot{\Omega}_0$ the minimum value P_m is found from the relation:

$$P_m = G(A_0, B_0, \dot{u}_0, \dot{\Omega}_0) \qquad (34e)$$

Linear Failure Criterion—Application of Eqs. (34) requires that the function $\sigma(x)$ be specified; consequently it appears that this approach cannot be used for a linear failure criterion. It is possible, however, to view a linear criterion as a limiting case of a general non-linear one. Even the slightest non linearity will yield a unique point of normality between the displacement vector Δ and the failure criterion, resulting in an association between slip surface and normal stress distribution. From this point of view, Eq. (28) with a constant ψ_s defines the limiting form of the normal stress distribution as the non-linearity tends to zero.

Substituting the value of r from Eq. (21) into Eq. (28), and integrating, the following results is obtained (Baker, 1981);

$$\sigma = B\exp(2\psi\theta) + \frac{A\gamma}{1+9\psi^2}(\cos\theta + 3\psi\sin\theta)\exp(-\psi\theta) - c\frac{1-\exp(2\psi\theta)}{\psi} \qquad (35)$$

It is seen, then that the assumption of linearity decouples the pair of simultaneous differential Eqs. (19) and (29) making it possible to solve them in turn. With the form of the function σ given by Eq. (35) it is now possible to follow the same approach as was used for a non linear criterion, resulting with a system of equations the same as Eqs. (34) except that now y_0 and G are independent of B. This procedure yields P_m, $y(x)$ and $\sigma(x)$ even for the linear failure criterion.

CONCLUSIONS

The classical solution procedure for the linear failure criterion (Eq. (25)) provides only a partial solution to the problem; the introduction of the assumption of linearity at too early a stage of the analysis results in a loss of information regarding the normal stress distribution. The assumption of a linear failure criterion is, at best, a convenient approximation to a non-linear reality, and hence the stress function defined by Eq. (35) has relevance for most real soils. The suggested form of this distribution is supported by the fact that Eq. (35) is the solution of the Kötter equation (Eq. (28)) which has a physical basis.

The conventional solution procedure of upper bound problems requires minimization with respect to $\dot{u}$ and $\dot{\Omega}$. It has been shown here that this minimization is equivalent to satisfaction of global equilibrium for the sliding mass. This observation makes it possible to solve the problem by solving a system of four simultaneous equations. To the authors' knowledge, such a solution procedure has not been used before. It may be verified that this type of procedure is equivalent to the variational limit equilibrium approach formulated by Baker and Garber (1977). That approach has been used to solve problems of bearing capacity (Garber and Baker, 1977) and slope stability (Baker, 1981) for soils with a linear failure criterion.

The use of the upper bound approach is greatly simplified by assuming a linear failure criterion. This simplification is due to the decoupling of the stress distribution and the slip surface functions, making it possible to define the shape of the slip surface in advance. Even with this assumption, the possibility exists for evaluating the normal stress distribution along the slip surface, providing valuable, additional information not obtained by the classical procedures.

REFERENCES

1) Baker, R. and Garber, M. (1977): "Variational approach to slope stability," Proc., 9th ICSMFE, Vol. 2, Tokyo, pp. 9-12.
2) Baker, R. (1981): "Tensile strength, tension cracks, and stability of slopes," Soils and Foundations, Vol. 21, No. 2, pp. 1-17.
3) Chen, W. F. and Giger, M. W. (1971): "Limit analysis of stability of slopes," Journal of the Soil Mechanics and Foundations Division, Proc. of the ASCE, Vol. 97, SM 6, pp. 19-26.
4) Chen, W. F. and Rosenfarb, J. L. (1973): "Limit analysis solutions of earth pressure problems," Soils and Foundations, Vol. 13, No. 4, pp. 45-60.
5) Chen, W. F. and Davidson, H. L. (1973): "Bearing capacity determination by limit analysis," Journal of the Soil Mechanics Foundations Division, Proc. of the ASCE, Vol. 99, No. SM 6, pp. 433-449.
6) Chen, W. F. (1975): Limit Analysis and Soil Plasticity, Elsevier, Amsterdam.
7) Davis, E. H. (1968): "Theories of plasticity and the failure of soil masses," pp. 341-380, in "Soil Mechanics-Selected Topics," Edited by Lee, I. K., Butterworths, London.
8) De Beer, E. E. (1970): "Experimental determination of the shape factors and the bearing capacity factors of sand," Geotechnique, Vol. 20, No. 4, pp. 387-411.
9) Drucker, D. C., Greenberg, H. J. and Prager, W. (1952): "Extended limit design theorems for continuous media," Quarterly of Applied Mathematics, Vol. 9, pp. 381-389.
10) Elsgolc, L. E. (1962): Calculus of Variations, Pergamon Press, London
11) Finn, W. H. L. (1967): "Applications of limit plasticity in soil mechanics," Journal of the Soil Mechanics and Foundations Division, Proceedings of the ASCE, Vol. 93, No. SM 5, pp. 101-119.
12) Garber, M. and Baker, R. (1977): "Bearing capacity by variational method," Journal of the Geotechnical Engineering Division, Proceeding of the ASCE, Vol. 103, No. GT 11, pp. 1209-1225.
13) Hill, R. (1950): The Mathematical Theory of Plasticity, Clarendon Press, Oxford.

14) Lee, K. L. and Seed, H. B. (1967) : "Drained strength characteristics of sands," Journal of the Soil Mechanics and Foundations Division, Proceeding of the ASCE, Vol. 93, No. SM 6, pp. 117-141.

15) Synge, J. L. and Griffith, B. A. (1959) : Principles of Mechanics, McGraw-Hill Book Comp., New-York.

16) Vesic, A. S. and Clough, G. W. (1968) : "Behaviour of granular materials under high stress," Journal of the Soil Mechanics and Foundations Div., Proceeding of the ASCE, Vol. 94, No. SM 3, pp. 661-688.