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LIMIT ANALYSIS OF GEOTECHNICAL PROBLEMS BY APPLYING LOWER-BOUND THEOREM

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ABSTRACT

This paper develops a numerical procedure to provide an appropriate lower-bound solution for the wide range of problems of stability analysis. To represent the equation of equilibrium, the stress field is discretized in the similar manner as in FEM. To isolate a particular stress distribution, the problem to find the lower-bound solution is formulated as an optimization problem. When optimizing the bearing capacity, for instance, the problem is to find the stress distribution which maximizes the footing pressure within the limitations of satisfying the equations of equilibrium and of no-yield condition. The formulated optimization problem is solved numerically by a nonlinear programming technique. This procedure furnishes a reasonable solution for the problems not only of the bearing capacity analysis but also of the slope stability analysis. The results of several case studies by using the procedure are also reported.

Key words: bearing capacity, computer application, finite element method, foundation, plane strain, slope stability, stability analysis, stress distribution (**IGC**: E 3/E 6)

INTRODUCTION

In spite of the remarkable development of FEM (Finite Element Method), limit analysis is as yet the principal method of strength analysis in geotechnical engineering. Chen (1975) divided the methods of stability analysis into three groups, i. e., slip line method, limit equilibrium method, and limit analysis method. According to this classification, the limit analysis herein is the method based on the lower- and upper-bound theorems in plasticity. The most difficult aspect in applying the lower- and upper-bound theorems, is the skillful construction of discontinuous fields of stress and velocity which are related to the collapse mechanism. Moreover in the upper-bound theorem, the velocity fields due to the collapse mechanism must be compatible or kinematically admissible. On the other hand in the lower-bound theorem, the stress field must be statically admissible. These complicated conditions can be easily satisfied by the use of discretization technique in FEM. For instance, the finite element displacement ap-

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proach can fulfill both the conditions of compatibility and of equilibrium by employing the displacement function and the virtual displacement theorem respectively. The discretization technique in FEM makes it possible to apply the limit analysis method to wide range of problems which have arbitrary geometry and boundary conditions. In the field of geotechnical engineering, Frémond and Salençon (1973), Turgeman and Pastor (1982) and Tamura, Kobayashi and Sumi (1984) proposed such procedures which combined the upper-bound theorem and the technique in the finite element displacement approach. In comparison with the upper-bound approach, it has been known to be generally difficult to find the appropriate lower-bound solution. This is because of the difficulty in constructing a statically admissible stress distribution which satisfies the equation of equilibrium and which nowhere violates the yield criterion. By employing an analogous technique in FEM, Lysmer (1970) first developed a numerical procedure to find a lower-bound solution for the plane problems involving arbitrary geometry and stress boundary conditions. Following Lysmer's method, Pastor and Turgeman (1982) added extending conditions so that the method can deal with axial symmetry problems. In order to isolate a unique stress field, Lysmer formulated the problem to find the lower-bound solution as an optimization problem, which was solved numerically by a linear programming technique. Though Lysmer's method is highly rational and essential, his method seems to contain two points of shortcoming. One is too intricate discretization of stress field. The other is too compulsory linearization of Mohr-Coulomb yield criterion which has high nonlinearity. Owing to these defects, the solution by Lysmer's method is considerably influenced by subdivision system of the soil mass into elements. And this method requires a lot of computational effort when desiring to obtain highly precise solution. To compensate for such defficiency, this paper investigates a different numerical procedure to find the lower-bound solution by employing a nonlinear programming technique. It is noted that the plane strain condition is assumed throughout this paper.

PROBLEM FORMULATION

The conditions required to establish a lower-bound solution are essentially as follows (see Chen, 1975). 1) The stress distribution must everywhere satisfy the euqation of equilibrium. 2) The stress field at the boundary must satisfy the stress boundary conditions. 3) The stress field must nowhere violate the yield condition. These conditions are represented as follows.

Equation of Equilibrium : To represent the equation of equilibrium, the soil mass subject to analysis is subdivided into many elements in the same manner as in FEM. Within each element a set of stresses is assumed to be constant, because the stress is the independent variable to be determined by the lower-bound approach. The present procedure adopts the quadrilateral element which is composed of four constant strain triangular elements as illustrated in Fig. 1. The [B] matrix to calculate strains from nodal displacements in this quadrilateral element, is constructed by the superposition of the [B] matrix in each triangular element. And both the strain and stress are assumed to be constant throughout the quadrilateral element. By employing the principle of virtual displacement, the following equation defines a set of equivalent nodal forces which is statically equilibrium with the stress condition of element.

$$\{F_{xi}{}^m, F_{yi}{}^m, \cdots, F_{xl}{}^m, F_{yl}{}^m\}^T = \int_{V_m} [B]^T \{\sigma_x{}^m, \sigma_y{}^m, \tau_{xy}{}^m\}^T dv \qquad (1)$$

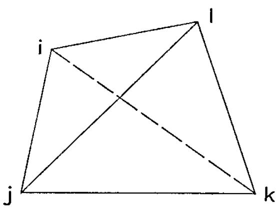

Fig. 1. Quadrilateral element

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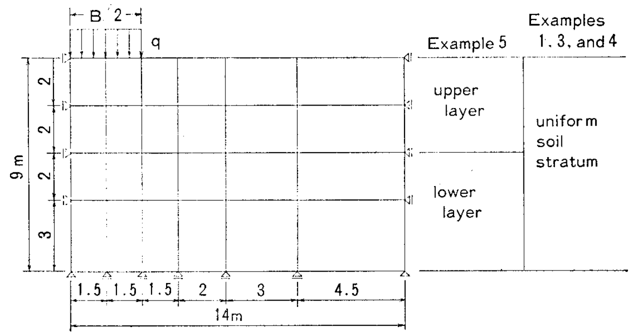

Fig. 2. Models in Examples 1, 2, 3 and 5

where $F_{xn}{}^m$, $F_{yn}{}^m$: respectively horizontal and vertical equivalent nodal forces, $\sigma_x{}^m$, $\sigma_y{}^m$, $\tau_{xy}{}^m$: horizontal and vertical normal stresses and shear stress in the element, V_m : volume of the element, m : element number, and n : node number. The equation of equilibrium is represented in the same manner as in FEM. That is, at each nodal point the sum of the equivalent nodal forces and external nodal loads must be equal to zero. When considering the bearing capacity problem as illustrated in Fig. 2, the following equation of equilibrium must be satisfied at the all points of node.

$$P_1{}^n=\sum_m F_{xn}{}^m+G_{xn}=0 \quad n=1,2,\cdots,N_p \qquad (2)$$

$$P_2{}^n=\sum_m F_{yn}{}^m+G_{yn}+H_n\cdot q=0 \quad n=1,2,\cdots,N_p \qquad (3)$$

where G_{xn}, G_{yn} : respectively horizontal and vertical nodal loads, H_n : coefficient to replace the uniform footing pressure q with the nodal load, and N_p : total number of nodal points. The term G_{yn} in Eq. (3) contains the weight of soil mass.

Boundary Conditions : The discretization technique described above is essentially founded on the finite element displacement approach. Then the conditions of equilibrium are not necessarily satisfied at the interfaces between two elements. For the similar reason, the present procedure cannot represent the arbitrary stress conditions at the boundary surface. For instance, consider the interface between the soil and the footing base, in the bearing capacity problem as shown in Fig. 2. For the perfectly rough footing base, the horizontal displacement are completely restricted at the corresponding points of node. And the equation of equilibrium as Eq. (2) is ignored at the nodal point (see Lysmer, 1970). The adequacy of this representation will be proved by the computational results. On the other hand, for the perfectly smooth base, the shear stress at the interface between the soil and the footing must be equal to zero. This boundary condition cannot be specified in this lower-bound analysis, as the analysis assumes a set of stresses to be constant within each element. Since Lysmer (1970) and Pastor and Turgeman (1982) use the linearly variable stress fields in each element, it is possible to specify the general stress conditions at the surface of boundary. When giving no special boundary condition for the footing base, the present procedure regards the interface as being intermediate between perfectly rough and perfectly smooth. Such a case is called 'pseudo-smooth' interface in this paper. However, the perfectly smooth interface between the soil and the structure may not exist in most of the actual engineering practices. Based on the upper-bound approach, Chen (1975) proves that a rather modest value of base friction is sufficient to create an essentially perfect rough condition. These facts make it sense in engineering to apply the present procedure by assuming the perfectly rough interface between the soil and the structure. In addition, the use of triangular element did not provide a stable solution in this lower-bound analysis. This result may be attributed to the matter that this analysis assumes a set of stresses to be constant in each element.

No-Yield Condition : When employing the Mohr-Coulomb yield criterion, the following inequality must be satisfied in each element.

$$P_3{}^m=\{(\sigma_x{}^m+\sigma_y{}^m)\sin\phi+2c\cos\phi\}^2-\{(\sigma_x{}^m-\sigma_y{}^m)^2+(2\tau_{xy}{}^m)^2\}\geqq 0 \quad m=1,2,\cdots,N_e \qquad (4)$$

where c : cohesion, ϕ : friction angle, and N_e : total number of elements. Eq. (4) represents that Mohr's stress circle is always located below the yield surface as illustrated in Fig. 3.

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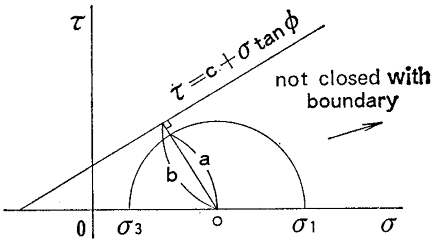

Fig. 3. Yield criterion

Furthermore assuming that soil mass has no tensile strength,

$$\sigma_x^m \geqq 0,\ \ \sigma_y^m \geqq 0 \quad m=1, 2, \cdots, N_e \qquad (5)$$

This constraint is effective to stabilize the iteration behaviour in the subsequent numerical analysis.

Problem Formulation (Problem I) : Based on Lysmer's method, the problem to find an appropriate lower-bound solution is formulated as an optimization problem which isolates the particular stress field. The constraints to be satisfied are Eqs. (2), (3), (4) and (5). When optimizing the bearing capacity as shown in Fig. 2, the objective function is to maximize the footing pressure q.

$$\text{minimize} : J=-q \qquad (6)$$

The unknown variables to be determined are the footing pressure q and stress components in each element. The maximized footing pressure is designated as the bearing capacity q^*. It is easy to extend the present procedure to layered soil deposits.

Problem II : As shown later in Example 3, when the bearing capacity becomes too large, the numerical procedure founded on Problem I cannot provide a reasonable solution. This may be because of the limitation in the employed nonlinear programming technique. This deficiency is easily covered by the following modification of Problem I.

$$\text{minimize} : J=-\bar{q}=-q/c \qquad (7)$$

subject to

$$\sum_m \bar{F}_{xn}^m+\bar{G}_{xn}=0 \qquad (8)$$

$$\sum_m \bar{F}_{yn}^m+\bar{G}_{yn}+H_n \cdot \bar{q}=0 \qquad (9)$$

$$\{(\bar{\sigma}_x^m+\bar{\sigma}_y^m)\sin\phi+2\cos\phi\}^2 - \{(\bar{\sigma}_x^m-\bar{\sigma}_y^m)^2+(2\bar{\tau}_{xy}^m)^2\} \geqq 0 \qquad (10)$$

$$\bar{\sigma}_x^m \geqq 0,\ \ \bar{\sigma}_y^m \geqq 0 \qquad (11)$$

where $\bar{x}$ denotes the quantity normalized by cohesion c such as $\bar{q}=q/c$. And the bearing capacity factor N_c is defined as $N_c=q^*/c$.

NUMERICAL ANALYSIS

SUMT : To solve the constrained optimization problem formulated above, the present procedure employs SUMT (Sequential Unconstrained Minimization Technique) interior point method proposed by Fiacco and McCormick (1968). This method achieves the minimization of objective function in the interior of the feasible region by avoiding the boundary which represents constraints. On the other hand, in SUMT exterior point method the movement of solution is from the outside or infeasible region toward the inside of the feasible region. The property of the interior point method is more compatible with the lower-bound approach in which the stress field must be strictly on the inside of the yield surface. Moreover in this particular problem, the solution by exterior point method is considerably influenced by the choice of penalty coefficient value.

The procedure in SUMT interior point method is outlined as follows. 1) Consider the optimization problem which is to find $\boldsymbol{x}^*$ solving

$$\text{minimize} : J=f(\boldsymbol{x}) \qquad (12)$$

subject to

$$\boldsymbol{h}(\boldsymbol{x})=\boldsymbol{0},\ \boldsymbol{g}(\boldsymbol{x}) \geqq \boldsymbol{0} \qquad (13)$$

2) Define the modified objective function $P(\boldsymbol{x}, \lambda_k)$ as

$$P(\boldsymbol{x}, \lambda_k)=f(\boldsymbol{x})+\lambda_k \sum_i g_i(\boldsymbol{x})^{-1} + \lambda_k^{-1/2} \sum_i h_i(\boldsymbol{x})^2 \qquad (14)$$

where λ_k is a positive number and is refered to penalty coefficient. 3) As a starting point, determine the initial value $\boldsymbol{x}_0$ in the interior of the feasible region. 4) Starting from $\boldsymbol{x}_0$, find an unconstrained minimum of $P(\boldsymbol{x}, \lambda_1)$ for some λ_1. Denote it by $\boldsymbol{x}(\lambda_1)$. 5) Starting from $\boldsymbol{x}(\lambda_1)$, find an unconstrained minimum of $P(\boldsymbol{x}, \lambda_2)$ where $\lambda_2 < \lambda_1$. 6) Proceed in this fashion, minimizing $P(\boldsymbol{x}, \lambda_k)$ for a strictly monotonously decreas-

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ing sequence $\{\lambda_k\}$. As $\lambda_k \rightarrow 0$, the sequence of unconstrained minima will approach a local constrained minimum $\boldsymbol{x}^*$. The modified objective function for Problem I is defined as

$$P(\boldsymbol{x}, \lambda_k) = -q + \alpha \left\{ \lambda_k^{-1/2} \sum_{n=1}^{N_p} (P_1^n)^2 + \lambda_k^{-1/2} \sum_{n=1}^{N_p} (P_2^n)^2 + \lambda_k \sum_{m=1}^{N_e} (P_3^m)^{-1} \right\} = -q + \alpha \{\lambda_k^{-1/2} Q_1 + \lambda_k^{-1/2} Q_2 + \lambda_k Q_3\} \quad (15)$$

where α : positive constant which adjusts the order of magnitude both of q and of the penalty terms. In SUMT, it is important to control the order of magnitude both of the original objective function and of the penalty terms in the modified objective function, so that the values of λ_k and α can be commonly used for the wide ranging values of strength parameters and of element stresses. In the formulation of Problem I, the order of magnitude of the penalty terms varies considerably with the strength parameter values. The formulation of Problem II avoids this difficulty by normalizing the penalty terms by cohesion c. The constraint by Eq. (5) can be simply dealt with at the iteration step as subsequently explained (see Pagurek and Woodside, 1968).

Iteration Procedure : In the search for the unconstrained minimum of the modified objective function $P(\boldsymbol{x}, \lambda_k)$ for a certain λ_k, the present procedure employs the conjugate gradient technique proposed by Fletcher and Reeves (1964). The Davidon's method (see Fletcher and Powell, 1963) requires too much computation time to obtain the search direction at each iteration step, due to a great number of decision variables to be determined. The iteration procedure in the conjugate gradient technique by Fletcher and Reeves is summarized as follows. 1) Set the initial values of decision variable $\boldsymbol{x}$. 2) Calculate the gradient $\boldsymbol{r}_i = \partial P(\boldsymbol{x}, \lambda_k)/\partial \boldsymbol{x}$, where i implies the iteration number. 3) $\boldsymbol{s}_i = -\boldsymbol{r}_i + (\boldsymbol{r}_i^T \boldsymbol{r}_i)/(\boldsymbol{r}_{i-1}^T \boldsymbol{r}_{i-1}) \cdot \boldsymbol{s}_{i-1}$. When the iteration number i coincides with the total number of decision variables, $\boldsymbol{s}_i = -\boldsymbol{r}_i$. 4) $\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \beta \boldsymbol{s}_i$, where β has to be determined so that β minimizes the modified objective function locally. To decide the value of β, the present procedure uses the one-dimensional search method by Sayama (1969). When stress component violates Eq. (5), take the boundary value. 5) Repeat the steps 1) to 4) until the following condition is satisfied.

$$|P(\boldsymbol{x}, \lambda_k)_i - P(\boldsymbol{x}, \lambda_k)_{i-1}/P(\boldsymbol{x}, \lambda_k)_i| \leqq 10^{-4} \quad (16)$$

Uniqueness of Solution : Under the several conditions, it is proved that SUMT interior point method gives a global minimum solution (see Kowalik and Osborne, 1968). In these conditions, the following ones are thought to be important from the practical point of view. 1) Both the original objective function $f(\boldsymbol{x})$ and the constraint equations are continuously twice differentiable. 2) $f(\boldsymbol{x})$ has a lower limit. 3) The region involving the decision variables is closed with boundary. 4) Both $f(\boldsymbol{x})$ and $g(\boldsymbol{x})^{-1}$ are convex functions. When these conditions are not satisfied, the solution by SUMT should be considered as a local minimum. In the present bearing capacity problem (Problem I and II), the original objective function $f(\boldsymbol{x})$ is only once differentialbe, and that $f(\boldsymbol{x})$ has neither lower limit nor convexity. As shown in Fig. 3, the region containing the element stress (decision variable) has not the boundary which restricts the expansion of Mohr's circle. This tendency becomes more remarkable with the increase in friction angle ϕ. In conclusion, the solution by the present procedure should be regarded as a local minimum. (The devision such as replacing the righthand side of Eq. (6) with $1/q$ did not provide a stable solution.) When applying the present procedure, one must take care both of the penalty coefficient value and of the initial value of decision variables. Many trials prove that the following selection provides a reasonable solution in general cases.

$$\left.\begin{array}{l} \lambda_1 = 1.0,\ \lambda_2 = 0.01,\ \lambda_3 = 0.0001 \\ \alpha\,(\text{see Eq. 15}) = 0.25\,N_e/B \\ \text{initial value}: q = 3c,\ \sigma_x^m = \sigma_y^m = c + \gamma_0 h, \\ \quad \tau_{xy}^m = 0 \end{array}\right\} \quad (17)$$

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where B : width of the strip footing (see Fig. 2), γ_0 : unit weight of the soil, and h : depth from the ground surface to the element.

Causes of Error : The solution by the present procedure involves the following errors. 1) In the particular problem in which no initial strains and no initial stresses exist, the finite element displacement approach is known to produce a smaller solution both of displacement and of stress than the correct solution (see Zienkiewicz, 1971). It is noted that the external loads are specified in the primary finite element analysis. When optimizing the bearing capacity or external load by use of the independent element stresses, the obtained bearing capacity may be greater than the precise solution under the condition. This tendency will become more pronounced as the rough discretization of soil mass as seen in Example 2. 2) SUMT furnishes at most an approximate solution roughly satisfying the equality constraints such as Eq. (3) which relates the footing pressure with the element stresses. 3) As pointed out previously, with the increase of friction angle ϕ, the effect that the region containing the element stresses is not closed with boundary, becomes more considerable. 4) The iteration procedure by a combination of SUMT and conjugate gradient technique may not be able to reach the strictly optimum solution. And that, the final result by the iteration procedure is a local optimum solution. 5) At the boundary where displacement is fixed, the stress field becomes close to the yield state. This phenomenon is remarkable in the weightless soil mass and at the corner element being lower and remote from the footing, as shown later in Figs. 5 through 11. This effect may prevent the footing pressure from approaching to more precise solution. The factors 1) through 3) cause the overestimation of bearing capacity, whereas the factors 4) and 5) yield the underestimated result. However the errors by the factors 1) and 2) may not be larger than the solution violates the lower-bound. So far as employing Eq. (17), the present procedure provides a stable lower-bound solution within a certain limit of friction angle.

CASE STUDIES IN BEARING CAPACITY PROBLEM

Throughout the examples of bearing capacity problem, the force acted on the footing is assumed to be normally and uniformly loaded. It is further assumed that the interface between the soil and the footing is either pseudo-smooth or perfectly rough. In most cases, however, the footing base is assumed to be pseudo-smooth except in Example 4. Because concerning the imponderable soil the result by the present procedure is little affected by such interface conditions. Based on the upper-bound approach, Chen (1975) draw the same conclusion concerning the imponderable soil. As shown in Fig. 2, the base of soil stratum is supposed to be rigid and pseudo-smooth, since this assumption provides more precise and stable solution in most cases.

Example 1 : The first example (see Fig. 2) considers a strip footing on a purely cohesive weightless soil. At first the cohesion c is supposed to be 1 tf/m² (9.8 kPa). Table 1 shows the performance of SUMT interior point method by use of the formulation of Problem I. Fig. 4 illustrates the iteration behaviour of the conjugate gradient technique at each SUMT stage. As seen in Fig. 4, the search for the optimum solution requires a number of iterations, due to the non-convexity in the original objective function as stated previously. The computation time required for the total iteration steps in this case, is about 8 minutes when using a combination of a personal computer NEC PC 9801, TALOS 68 K (CPU : MC 68000, 8 MHz), CP/M-68 K and SVS FORTRAN

Table 1. Performance in SUMT (Example 1)

stage	λ_k	$P(x, \lambda_k)$	q	Q_1	Q_2	Q_3
1	1.0	4.63	2.42	0.5470	0.1651	6.82
2	0.01	−4.30	4.66	0.0008	0.0039	31.47
3	0.0001	−4.59	4.67	0.0002	0.0006	44.51

λ_k : penalty coefficient, q : footing pressure (tf/m²), and Q_1, Q_2 and Q_3 : see Eq. (15). (1 tf/m²=9.8 kPa)

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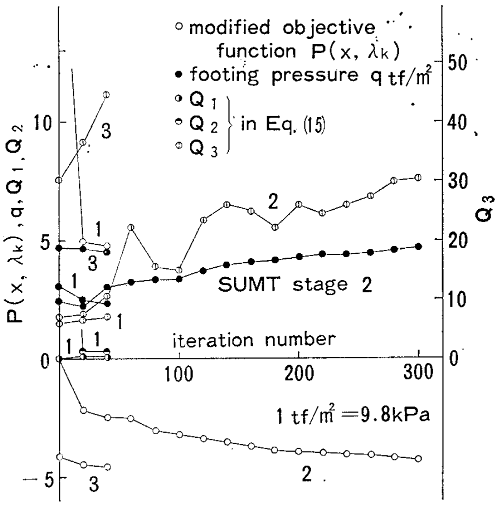
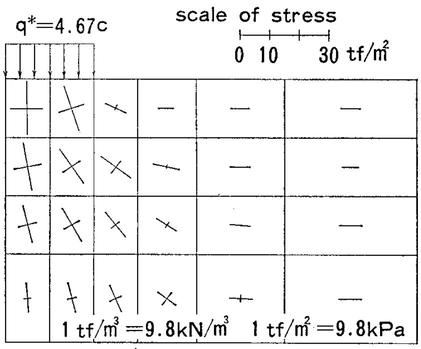


Fig. 4. Iteration behaviour in Example 1



(a) Principal stresses

1.04	1.04	1.20	1.24	1.20	1.17
1.01	1.01	1.02	1.01	1.13	1.24
1.03	1.02	1.05	1.18	1.10	1.06
1.02	1.04	1.23	1.87	1.11	1.01

(b) Safety factor *F*

Fig. 5. Results in Example 1 (c=1 tf/m², ϕ=0°, γ_0=0 tf/m³)

compiler. As seen in Table 1, the optimized bearing capacity q^* takes the underestimated value comparing with the lower-bound solution $q^*=5.0\,c$ obtained by Shield (1955). Fig. 5(a) shows the principal stress distribution by the present procedure. Fig. 5(b) shows the distribution of safety factor F which is defined as

$$F=b/a \qquad (18)$$

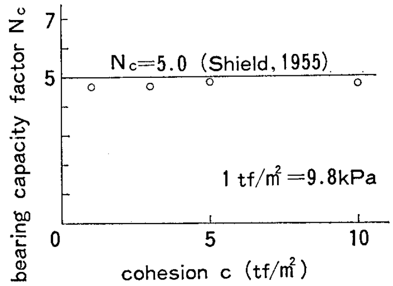
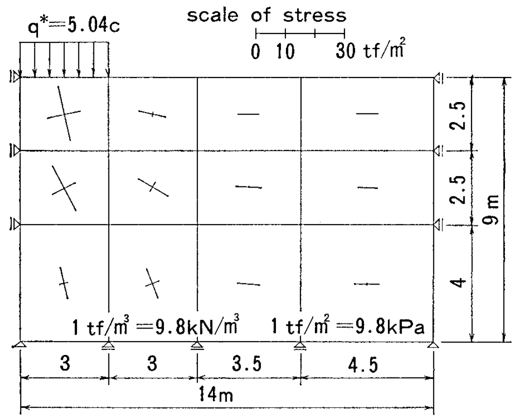


Fig. 6. Bearing capacity factor N_c in Example 1



(a) Principal stresses

1.02	1.04	1.09	1.04
1.00	1.00	1.01	1.18
1.00	1.17	1.04	1.01

(b) Safety factor *F*

Fig. 7. Results in Example 2 (c=1 tf/m², ϕ=0°, γ_0=0 tf/m³)

where both a and b are prescribed in Fig. 3. As seen in Fig. 5(b), the safety facter F becomes close to unity in the corner element being lower and far from the footing. This result is attributed to the constraint by boundary condition as stated previously. Fig. 6 shows the bearing capacity factor N_c by the present procedure (Problem I) under the various values of cohesion c.

Example 2 : To investigate the effect of element subdivision system, the second example considers the model illustrated in Fig. 7(a).

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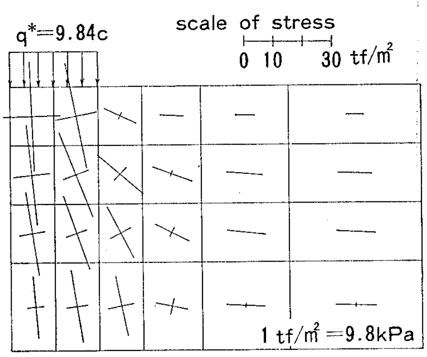
The calculated bearing capacity, principal stress distribution, and the distribution of safety factor F are shown in Figs. 7(a) and 7(b). The optimized bearing capacity is more precise one than the solution given in Example 1. This may be because of the overvaluation of footing pressure caused by the rough discretization of stress field as stated previously in the paragraph *Causes of Error*. It is important to note that the solution by the present procedure is not extremely influenced by the element subdivision system.

Example 3 : The third example (see Fig. 2) investigates a strip footing on a weightless soil which has both cohesion c and friction angle ϕ. Figs. 8(a) and 8(b) show the results under the certain values of soil parameters. Fig. 9 compares the bearing capacity factor N_c by the present procedure (Problem I) with the lower-bound solution by Shield (1955) and with the upper-bound solution by Chen (1975). As seen in Fig. 9, when the friction angle ϕ increases beyond 25°, the solution of Problem I becomes unstable.



(a) Principal stresses

1.20	1.09	1.10	1.25	1.31	1.35
1.02	1.00	1.10	1.00	1.00	1.03
1.00	1.00	1.19	1.18	1.00	1.00
1.00	1.00	1.25	1.56	1.00	1.00

1 tf/m³ =9.8kN/m³

(b) Safety factor F

Fig. 8. Results in Example 3 (c=1 tf/m², ϕ=20°, γ_0=0 tf/m³)

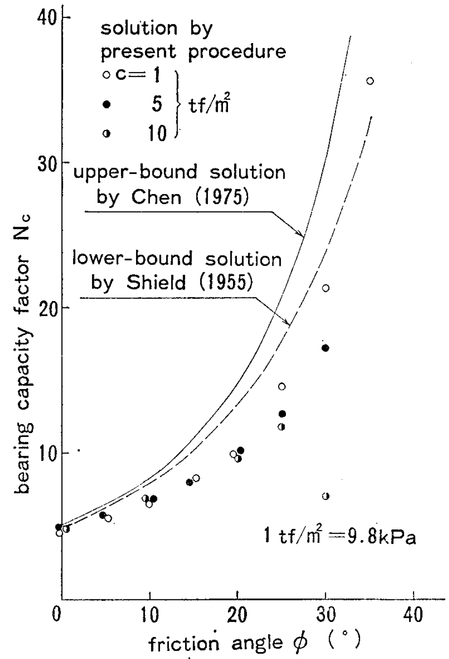


Fig. 9. Comparison between some solutions in Example 3 (weightless soil)

Generally the procedure by Problem I dose not provide a reasonable solution, when the bearing capacity q^* to be determined exceeds about 100 tf/m² (980 kPa). In such a case, one should employ the procedure by Problem II which corresponds to the case of cohesion c=1 tf/m² in Fig. 9. Even if employing the procedure by Problem II, Fig. 9 reveals that a reasonable solution cannot be obtained when friction angle ϕ exceeds 30°. This is because of the property of this lower-bound problem that the region containing the element stress is not closed with boundary as pointed out previously.

Example 4 : The fourth example (see Fig. 2) studies a strip footing on a ponderable soil. Figs. 10(a) through 11(b) show the results under the certain values of soil parameters. Fig. 12 compares the bearing capacity factor N_c by the present procedure with the two types of upper-bound solution by Chen (1975) according to the interface condition between the soil and the footing base. For the case of pseudo-smooth base, the lower-bound solution by the present procedure exceeds the upper-bound solution for the perfectly smooth base. This result means that the pseudo-smooth base is rather close to the

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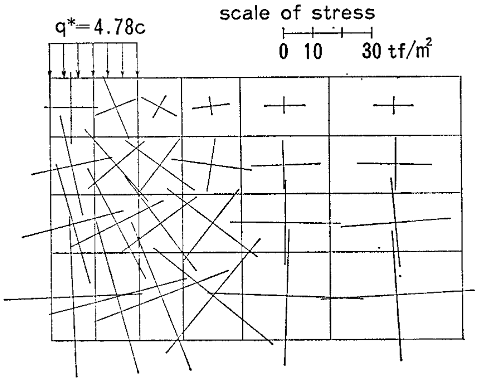


(a) **Principal stresses**

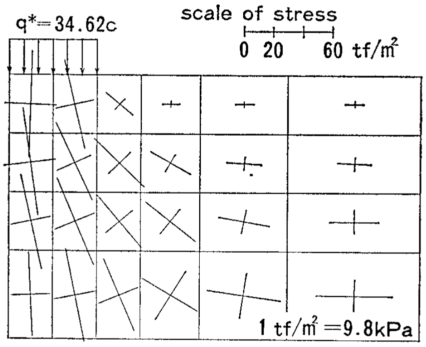
q* (loaded strip)

1.05	1.05	1.25	1.13	1.03	1.08
1.02	1.02	1.03	1.03	1.05	1.12
1.03	1.02	1.05	1.16	1.03	1.05
1.01	1.04	1.16	1.93	1.12	1.02

1 tf/m³ =9.8kN/m³ 1 tf/m² =9.8kPa

(b) **Safety factor *F***

Fig. 10. Results in Example 4 (c=1 tf/m², ϕ=0°, γ_0=1.5 tf/m³)



(a) **Principal stresses**

1.00	1.00	1.00	1.00	1.03	1.06
1.01	1.00	1.00	1.00	1.02	1.11
1.02	1.03	1.10	1.10	1.06	1.09
1.02	1.04	1.37	1.83	1.25	1.02

1 tf/m³ =9.8kN/m³

(b) **Safety factor *F***

Fig. 11. Results in Example 4 (c=1 tf/m², ϕ=20°, γ_0=1.5 tf/m³)

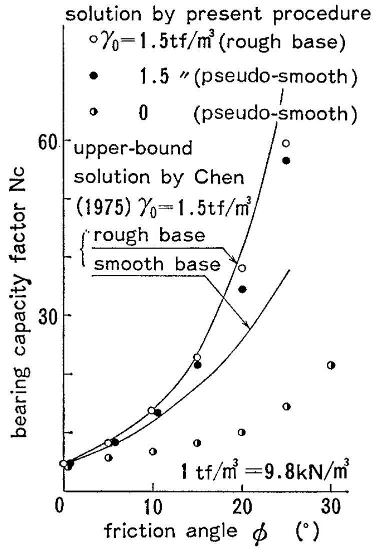


Fig. 12. Comparison between some solutions in Example 4

perfectly rough base than the perfectly smooth base in this bearing capacity problem.

Example 5: The fifth example (see Fig. 2) considers a strip footing on a multi-layered soil deposit. Figs. 13(a) and 13(b) show the

q*=4.43tf/m²

1.00	1.04	1.26	1.39	1.39	1.38	c=1tf/m², ϕ=0°, γ_0=0tf/m³
1.03	1.05	1.00	1.02	1.58	2.67	
1.96	2.06	2.24	2.45	2.64	2.70	c=5tf/m², ϕ=0°, γ_0=0tf/m³
2.11	2.32	2.80	4.65	2.81	2.45	

1 tf/m³ =9.8kN/m³ 1 tf/m² =9.8kPa

(a) **Weak upper-layer**

q*=8.48tf/m²

1.42	1.35	1.17	1.67	2.91	7.81	c=5tf/m², ϕ=0°, γ_0=0tf/m³
1.33	1.24	1.27	1.23	1.04	1.00	
1.02	1.02	1.04	1.33	1.10	1.01	c=1tf/m², ϕ=0°, γ_0=0tf/m³
1.02	1.02	1.02	1.03	1.02	1.00	

1 tf/m³ =9.8kN/m³ 1 tf/m² =9.8kPa

(b) **Weak lower-layer**

Fig. 13. Results in Example 5 (safety factor *F*)

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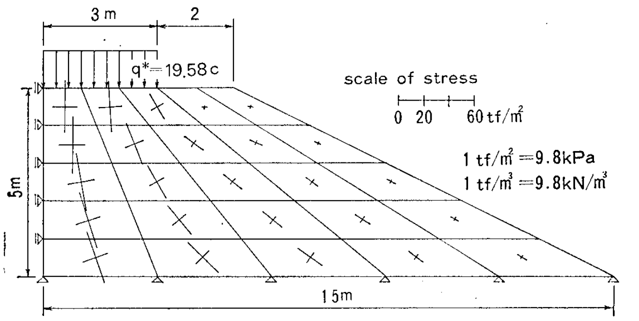
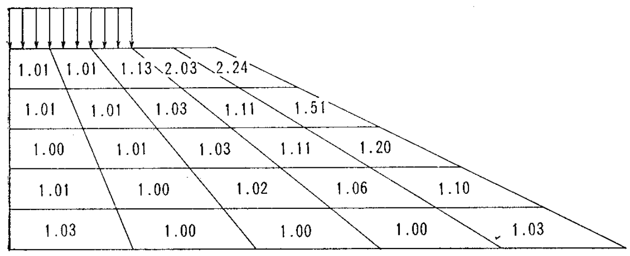


(a) Principal stresses



(b) Safety factor F

Fig. 14. Results in Example 6 ($c=1$ tf/m^2, $\phi=20°$, $\gamma_0=1.5$ tf/m^3)

results for the particular values of soil parameters. It may be an accident in numerical computation that the bearing capacity in Fig. 13(a) is less than that in Example 1.

Example 6 : The sixth example (see Fig. 14 a) investigates a strip footing on a slope. Being different from Figs. 2 through 13, the base of soil stratum is supposed to be perfectly rough. In this particular case the present procedure provides too small bearing capacity when assuming the pseudo-smooth base of soil stratum. This is because the boundary constraints give more important effect in this example. That is, the stress field at the toe of slope tends to become close to the yield state. This effect prevents the footing pressure from approaching the precise solution. The results are shown in Figs. 14(a) and 14(b). Kusakabe, Kimura and Yamaguchi (1981) gave a full study to this kind of bearing capacity problem.

APPLICATIONS TO SLOPE STABILITY ANALYSIS

Problem Formulation : When applying the present lower-bound approach to the slope stability analysis, the quantity to be maximized is the unit weight of soil which corresponds to the footing pressure in the bearing capacity analysis. The problem here (Problem III) is to find γ^* solving

$$\text{minimize} : J=-\gamma \qquad (19)$$

subject to Eqs. (2) through (5). The safety factor in this slope stability analysis, F_w is defined as

$$F_w=\gamma^*/\gamma_0 \qquad (20)$$

where γ_0 : primary unit weight of the soil, and γ^* : unit weight of soil provided by the present procedure, which corresponds to the maximum unit weight of soil to be sustained by the slope within the limitations of the lower-bound theorem. This Problem III can be solved in the same manner as in bearing capacity analysis. However the solution of Problem III is highly sensitive to the selection of penalty coefficient value in SUMT interior point method. Hence Problem III is converted to Problem IV as

$$\text{minimize} : J=-\bar{\gamma}=-\gamma/c \qquad (21)$$

subject to the modified form of Eqs. (2) through (5) which are normalized both by γ and by cohesion c. Eq. (17) is also valid in this Problem IV except replacing α with

$$\alpha=2.0\,N_e/S \qquad (22)$$

where S : sum of the areas of all elements.

Example 7 : The seventh example (see Fig. 15 a) considers a simple slope of a homogeneous soil with zero pore pressure. The base of soil stratum is supposed to be perfectly rough, for the same reason as stated in Example 6. Figs. 15(a) and 15(b) show the results by the present procedure together with the result by the simplified Bishop's method (see Chowdhury, 1978 ; Naruoka et al, 1977). The safety factor in the simplified Bishop's method, F_s is defined as the ratio between the shear strength and the mobilized shear stress. Fig. 16 compares these two safety factors concerning this example. The direct comparison of these two safety factor values has little reasonable meaning because of the different definitions of safety factor. However Fig. 17 is thought to suggest the characteristics of these two analysis methods.

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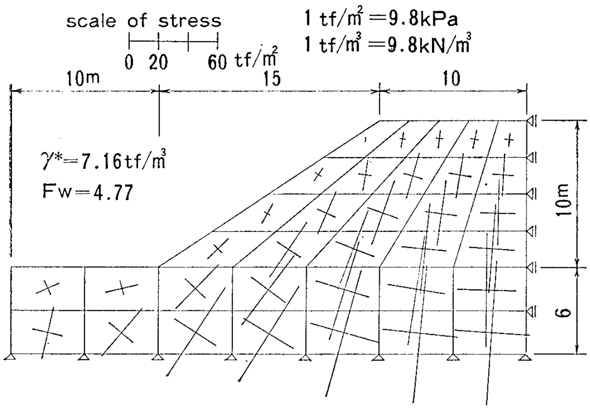
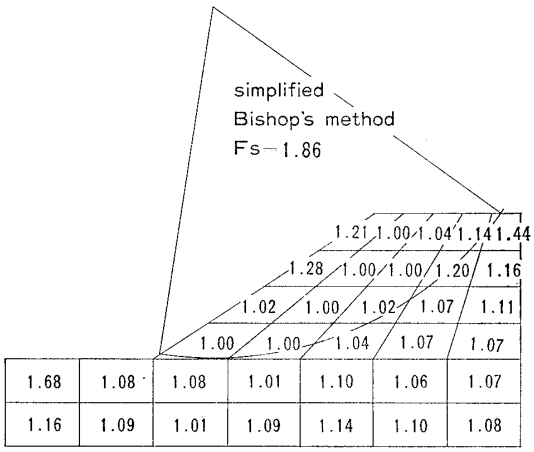


(a) Principal stresses



(b) Safety factor *F*

Fig. 15. Results in Example 7 (c=2 tf/m², ϕ=20°, γ_0=1.5 tf/m³)

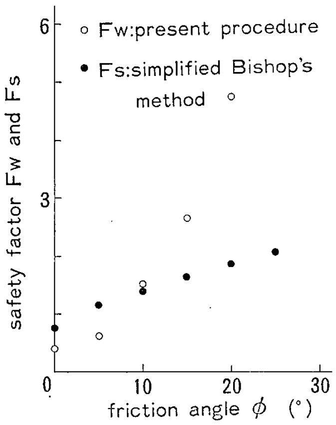


Fig. 16. Comparison between two solutions in Example 7

CONCLUSIONS

This paper developed a numerical procedure to provide an appropriate lower-bound solution for the wide range of stability problems. In order to avoid the defects in Lysmer's method such as too complex discretization of stress field and too compulsory linearization of Mohr-Coulomb yield criterion, the present procedure discretizes the stress field in the similar manner as in the finite element displacement approach, and employs a non-linear programming technique. To isolate a particular stress distribution, the problem to find the lower-bound solution is formulated as an optimization problem. When optimizing the bearing capacity, the problem is to determine the stress distribution which maximizes the footing pressure within the limitations of satisfying the equilibrium equation and the no-yield condition (Mohr-Coulomb yield criterion). Through the several case studies in bearing capacity analysis, it has been proved that the present procedure can successfully provide an appropriate and stable lower-bound solution for general soils which have cohesion, friction angle and its own weight, so far as the friction angle is not so large. This procedure furnishes a reasonable lower-bound solution for the problem not only of the bearing capacity analysis but also of the slope stability analysis. However this procedure cannot represent the arbitrary stress conditions at the boundary surface, because a set of stresses is assumed to be constant within each element. It is also difficult to apply the procedure to the problem of interaction between the soil and the structure, such as the problem of earth pressure. This is because the procedure considers the stress as the independent variable and assumes the soil mass as to be rigid-perfectly plastic material. Such insufficiency must be investigated in the future study.

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