SOILS AND FOUNDATIONS Vol. 25, No. 4, 123-128, Dec. 1985
Japanese Society of Soil Mechanics and Foundation Engineering

TECHNICAL NOTE

RELATIONSHIP AMONG TRESCA, MISES, MOHR-COULOMB AND MATSUOKA-NAKAI FAILURE CRITERIA

HAJIME MATSUOKA* and TERUO NAKAI**

ABSTRACT

The new failure criterion of $J_1 \cdot J_2/J_3$=const. which has been proposed on the basis of the "spatially mobilized plane (SMP)" is introduced as a failure criterion for granular materials (J_1, J_2 and J_3 : the first, second and third effective stress invariants). This failure criterion is called the Matsuoka-Nakai ("SMP") criterion. The Tresca and Mises criteria are usually adopted as failure criteria for metals, and the Mohr-Coulomb and Matsuoka-Nakai criteria for granular materials such as soils. The Matsuoka-Nakai criterion has a concept of averaging the Mohr-Coulomb criterion, in such a way that the Mises criterion has a concept of averaging the Tresca criterion. Such interesting relationship among the Tresca, Mises, Mohr-Coulomb and Matsuoka-Nakai criteria is discussed by reviewing their physical meaning, their forms of equations and their shapes in three-dimensional stress space.

Key words : failure, granular material, sand, yield (**IGC** : D 6)

INTRODUCTION

There have been proposed such famous failure criteria as the Tresca, Mises and Mohr-Coulomb. The Tresca and Mises criteria are usually adopted as failure criteria for metals, and the Mohr-Coulomb criterion for granular materials such as soils. In recent years, J_1^3/J_3=const. by Lade and Duncan (1975) and $J_1 \cdot J_2/J_3$=const. by the authors (1974, 1976, 1977) have also been suggested as failure criteria for soils (J_1, J_2 and J_3 : the first, second and third effective stress invariants). In this paper, the new failure criterion of $J_1 \cdot J_2/J_3$=const. which has been proposed on the basis of the "spatially mobilized plane (SMP)" is examined, and the mutual relationship among the Tresca, Mises, Mohr-Coulomb and Matsuoka-Nakai ("SMP") criteria is discussed.

* Associate Professor, Department of Civil Engineering, Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya.
** Associate Professor, ditto.

Manuscript was received for review on April 1, 1985.
Written discussions on this note should be submitted before July 1, 1986, to the Japanese Society of Soil Mechanics and Foundation Engineering, Sugayama Bldg. 4F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.

PROPOSED FAILURE CRITERION FOR GRANULAR MATERIALS BASED ON "SPATIALLY MOBILIZED PLANE (SMP)"

The behaviour of soil particles is considered to be governed by the frictional law, i.e., the shear-normal stress ratio. The plane on which the shear-normal stress ratio is maximized has been called "mobilized plane", and considered to be the plane where the friction between soil particles is mobilized more than on any other plane (Murayama, 1966). The stress state on the "mobilized plane" corresponds to the point at which a straight line from the origin is tangent to the Mohr's stress circle.

Under three different principal stresses, three Mohr's stress circles are drawn and three "mobilized planes" whose stress states correspond to points P_1, P_2 and P_3 in Fig. 1 exist between the respective two principal stress directions as shown in Fig. 2(a). These three two-dimensional mobilized planes have been named "compoundedly mobilized planes (CMP)" as a whole (Matsuoka, 1974). Paying further attention to the plane ABC having these three mobilized planes AB, BC and AC as the three sides, it has been named "spatially mobilized plane (SMP)" as in Fig. 2(b) (Matsuoka and Nakai, 1974, 1977). The "spatially mobilized plane (SMP)" has a concept of averaging the three mobilized planes, and the stress state on the SMP corresponds to point P in Fig. 1. The values at points A, B and C in Fig. 2(b) are proportional to the square roots of the respective principal stresses according to the following equation.

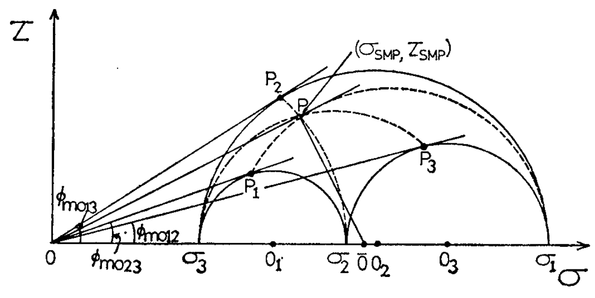

Fig. 1. Shear and normal stresses on mobilized planes and on the spatially mobilized plane (Satake, 1978)

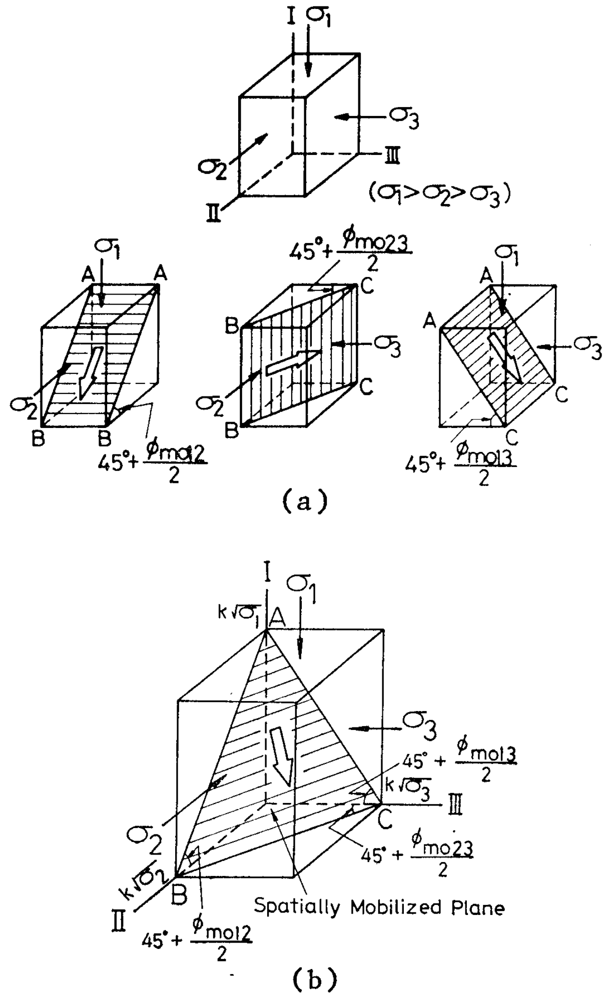

Fig. 2. Geometrical representation of mobilized planes and the spatially mobilized plane in three-dimensional space

$$\tan\left(45°+\frac{\phi_{moij}}{2}\right)=\sqrt{\frac{1+\sin\phi_{moij}}{1-\sin\phi_{moij}}}=\sqrt{\frac{\sigma_i}{\sigma_j}}$$
$$(i, j=1, 2, 3\ ;\ i<j) \qquad (1)$$

Therefore, the SMP coincides with the octahedral plane in an isotropic stress condition ($\phi_{moij}=0$) and is variable with possible changes in the stress ratios. In its physical significance, it is considered to be the plane on which soil particles are best mobilized on the average in the three-dimensional stresses. The direction cosines (a_1, a_2 and a_3) of the normal to the SMP are expressed as follows:

$$a_i=\sqrt{J_3/(\sigma_i J_2)} \qquad (i=1, 2, 3) \qquad (2)$$

where J_1, J_2 and J_3 are the first, second and third effective stress invariants which are given by the following equations.

$$\begin{aligned}J_1&=\sigma_1+\sigma_2+\sigma_3\\ J_2&=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\\ J_3&=\sigma_1\sigma_2\sigma_3\end{aligned}\tag{3}$$

Throughout this paper, the term "stress" is to be interpreted as "effective stress".

Taking Eq. (2) into consideration, the normal stress σ_{SMP}, the shear stress τ_{SMP} and the shear-normal stress ratio τ_{SMP}/σ_{SMP} on the SMP can be expressed as follows :

$$\sigma_{SMP}=\sigma_1\cdot a_1^2+\sigma_2\cdot a_2^2+\sigma_3\cdot a_3^2=\frac{3J_3}{J_2}\tag{4}$$

$$\begin{aligned}\tau_{SMP}&=\sqrt{\begin{matrix}(\sigma_1-\sigma_2)^2\cdot a_1^2\cdot a_2^2+(\sigma_2-\sigma_3)^2\cdot a_2^2\cdot a_3^2\\+(\sigma_3-\sigma_1)^2\cdot a_3^2\cdot a_1^2\end{matrix}}\\&=\frac{\sqrt{J_1J_2J_3-9J_3^2}}{J_2}\end{aligned}\tag{5}$$

$$\begin{aligned}\frac{\tau_{SMP}}{\sigma_{SMP}}&=\sqrt{\frac{J_1J_2-9J_3}{9J_3}}\\&=\frac{2}{3}\sqrt{\frac{(\sigma_1-\sigma_2)^2}{4\sigma_1\sigma_2}+\frac{(\sigma_2-\sigma_3)^2}{4\sigma_2\sigma_3}+\frac{(\sigma_3-\sigma_1)^2}{4\sigma_3\sigma_1}}\\&=\frac{2}{3}\sqrt{\tan^2\phi_{mo12}+\tan^2\phi_{mo23}+\tan^2\phi_{mo13}}\end{aligned}\tag{6}$$

where $\tan\phi_{moij}=(\sigma_i-\sigma_j)/2\sqrt{\sigma_i\sigma_j}$ $(i,\ j=1,\ 2,\ 3\ ;\ i<j)$ and ϕ_{moij} is seen in Figs. 1 and 2. It is understood from Eq. (6) that τ_{SMP}/σ_{SMP} is in the form of the square mean of $\tan\phi_{moij}$. This is one of the reasons why the SMP has a concept of averaging the three mobilized planes.

On the assumption that soils fail when this stress ratio τ_{SMP}/σ_{SMP} reaches a limiting value, one can derive quite a simple form of failure criterion for soils (Matsuoka and Nakai, 1974, 1977 ; Matsuoka, 1976).

$$J_1\cdot J_2/J_3=\text{const.}\tag{7}$$

or

$$\tan^2\phi_{mo12}+\tan^2\phi_{mo23}+\tan^2\phi_{mo13}=\text{const.}\tag{8}$$

It is seen from Eq. (7) that the proposed failure criterion is one of the simplest non-

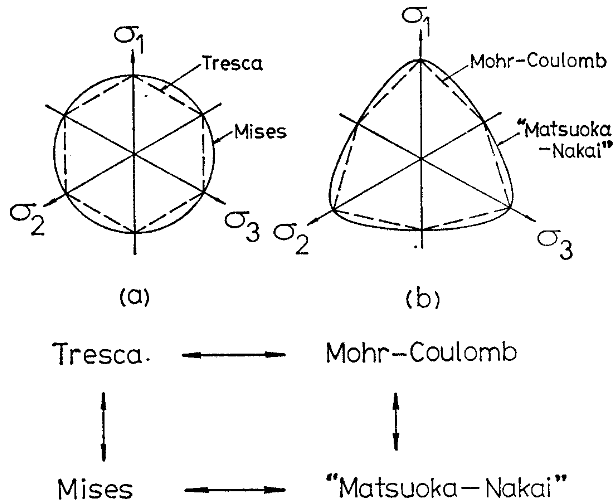

Fig. 3. Mutual relationship among the Tresca, Mises, Mohr-Coulomb and Matsuoka-Nakai ("SMP") failure criteria described on an octahedral plane

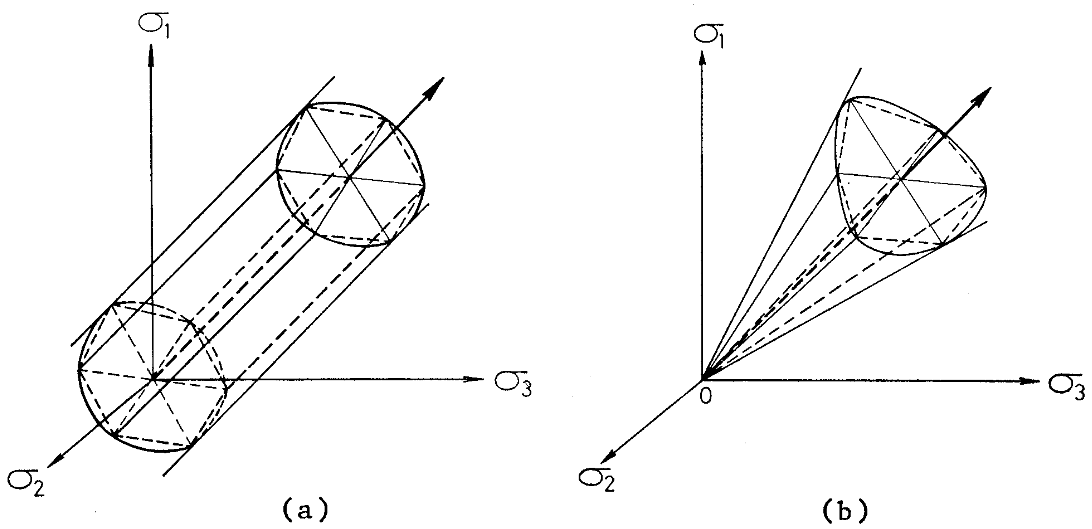

Fig. 4. Shapes of the Tresca and Mises criteria, and the Mohr-Coulomb and Matsuoka-Nakai("SMP") criteria in three-dimensional stresses

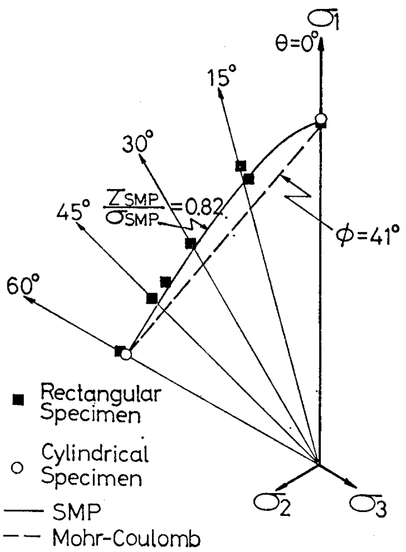

Fig. 5. Comparison of the Matsuoka-Nakai ("SMP") and Mohr-Coulomb criteria with stress states at failure on octahedral plane obtained by triaxial compression, triaxial extension and true triaxial tests on sand

dimensional quantities using all of the stress invariants J_1, J_2 and J_3. If this criterion is expressed on the octahedral plane, one obtaines a smooth convex curve circumscribed with the hexagon representing the Mohr-Coulomb criterion as shown in Fig. 3(b). The three-dimensional shapes of the proposed Matsuoka-Nakai criterion and the Mohr-Coulomb criterion in the principal stress space are drawn in Fig. 4(b) when the cohesion $c=0$.

Fig. 5 shows the comparison of the Matsuoka-Nakai ("SMP") and Mohr-Coulomb criteria with the stress states at failure on the octahedral plane obtained by the triaxial compression, triaxial extension and true triaxial tests on Toyoura sand (Nakai and Matsuoka, 1983). It is seen from Fig. 5 that the measured values agree well with the proposed Matsuoka-Nakai criterion drawn by the solid curve.

MUTUAL RELATIONSHIP AMONG FOUR KINDS OF FAILURE CRITERIA

Fig. 6 shows the shear and normal stresses on planes (called "45° planes") on which the shear stress is maximized under the respective two principal stresses (P_1, P_2 and P_3), as well as the shear stress τ_{oct} and the normal stress σ_{oct} on the octahedral plane (P). Fig. 1 shows the shear and normal stresses on the "mobilized planes" on which the shear-normal stress ratio is maximized under the respective two principal stresses

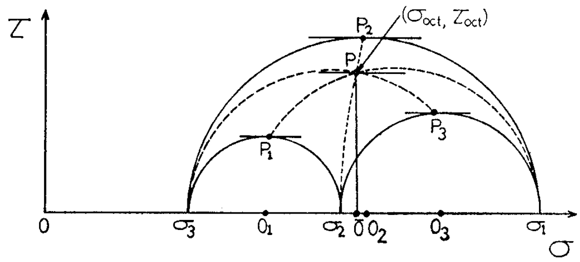

Fig. 6. Shear and normal stresses on 45° planes and on an octahedral plane (Satake, 1978)

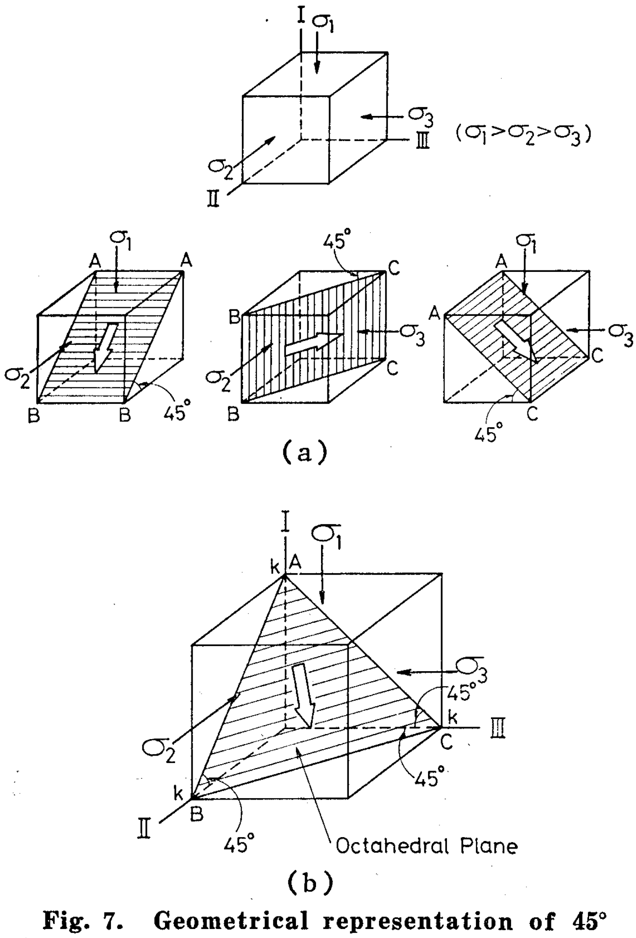

Fig. 7. Geometrical representation of 45° planes and an octahedral plane in three-dimensional space

(P_1, P_2 and P_3), and the shear stress τ_{SMP} and the normal stress σ_{SMP} on the "spatially mobilized plane (SMP)" (P) (Satake, 1978). The 45° planes and octahedral plane are shown in Fig. 7, and the "mobilized planes" and "spatially mobilized plane" are shown in Fig. 2. These figures show that the "spatially mobilized plane" is a plane whose three sides are "mobilized planes" even as the three sides of the octahedral plane are 45° planes.

In Fig. 6, $\tau_{max}=P_2O_2=$const. expresses the Tresca criterion and $\tau_{oct}=P\bar{O}=$const. the Mises criterion. In Fig. 1, $(\tau/\sigma)_{max}=\tan\angle P_2OO_2=$const. represents the Mohr-Coulomb criterion and $\tau_{SMP}/\sigma_{SMP}=\tan\angle PO\bar{O}=$const. the Matsuoka-Nakai ("SMP") criterion when the cohesion $c=0$. These four kinds of failure criteria are described as follows.

Tresca criterion :

$$\tau_{max}=P_2O_2=\frac{\sigma_1-\sigma_3}{2}=\text{const.} \qquad (9)$$

Mises criterion :

$$\tau_{oct}=P\bar{O}=\frac{2}{3}\sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\left(\frac{\sigma_2-\sigma_3}{2}\right)^2+\left(\frac{\sigma_3-\sigma_1}{2}\right)^2}=\text{const.} \qquad (10)$$

Mohr-Coulomb criterion :

$$(\tau/\sigma)_{max}=\tan\angle P_2OO_2=\frac{\sigma_1-\sigma_3}{2\sqrt{\sigma_1\sigma_3}}=\text{const.} \qquad (11)$$

Matsuoka-Nakai ("SMP") criterion :

$$\tau_{SMP}/\sigma_{SMP}=\tan\angle PO\bar{O}=\frac{2}{3}\sqrt{\left(\frac{\sigma_1-\sigma_2}{2\sqrt{\sigma_1\sigma_2}}\right)^2+\left(\frac{\sigma_2-\sigma_3}{2\sqrt{\sigma_2\sigma_3}}\right)^2+\left(\frac{\sigma_3-\sigma_1}{2\sqrt{\sigma_3\sigma_1}}\right)^2}=\text{const.} \qquad (12)$$

It is seen from Eqs. (9)-(12) that τ_{oct} is in the form of the square mean of τ_{max}, and τ_{SMP}/σ_{SMP} the square mean of $(\tau/\sigma)_{max}$. Satake (1982) has also suggested that the Matsuoka-Nakai criterion is in the statistically modified form of the Mohr-Coulomb criterion, as well as the Mises criterion is in that form of the Tresca criterion.

Fig. 3 shows the mutual relationship among these four kinds of failure criteria described on the octahedral plane. The shapes of the Tresca and Mises criteria, and the Mohr-Coulomb and Matsuoka-Nakai ("SMP") criteria in three-dimensional stress space are shown in Figs. 4(a) and (b). The Tresca and Mises criteria are considered to be the failure criteria for metals, and the Mohr-Coulomb and Matsuoka-nakai criteria the failure criteria for granular materials which are governed by the frictional law (i.e., shear-normal stress ratio).

CONCLUDING REMARKS

The interesting mutual relationship among the Tresca, Mises, Mohr-Coulomb and Matsuoka-Nakai ("SMP") failure criteria was presented by reviewing their mechanical meaning, their forms of equations and their shapes in three-dimensional stress space. It was found by this review that the Matsuoka-Nakai criterion had a concept of averaging the Mohr-Coulomb criterion, as well as the Mises criterion had a concept of averaging the Tresca criterion. Comparing these four kinds of failure criteria, it was shown that the proposed Matsuoka-Nakai criterion held a definite position as a failure criterion for granular materials such as soils.

ACKNOWLEDGEMENT

The authors would like to express their appreciation to Professor M. Satake of Tohoku University for his helpful advice and encouragement during the course of this study.

REFERENCES

1) Lade, P. V. and Duncan, J. M. (1975) : "Elasto-plastic stress-strain theory for cohesionless soil," Proc. ASCE, Vol. 101, No. GT 10, pp. 1037-1053.
2) Matsuoka, H. (1974) : "Stress-strain relationships of sand based on the mobilized plane," Soils and Foundations, Vol. 14, No. 2, pp. 47-61.
3) Matsuoka, H. (1976) : "On the significance of the spatial mobilized plane," Soils and Foundations, Vol. 16, No. 1, pp. 91-100.
4) Matsuoka, H. and Nakai, T. (1974) : "Stress-deformation and strength characteristics of soil under three different principal stresses," Proc. JSCE, No. 232, pp. 59-70.

5) Matsuoka, H. and Nakai, T. (1977) : "Stress-strain relationship of soil based on the 'SMP'," Proc. Specialty Session 9, Constitutive Equations of Soils, 9th Int. Conf. SMFE, pp. 153-162.
6) Murayama, S. (1966) : "A theoretical consideration on a behaviour of sand," Proc. IUTAM Symposium on Rheology and Soil Mechanics, Grenoble, pp. 146-159.
7) Nakai, T. and Matsuoka, H. (1983) : "Shear behaviors of sand and clay under three-dimensional stress condition," Soils and Foundations, Vol. 23, No. 2, pp. 26-42.
8) Satake, M. (1978) : Comment in Proc. U. S.-Japan Seminar on Continuum Mechanical and Statistical Approaches in the Mechanics of Granular Materials, Gakujutsu Bunken Fukyu-kai, Tokyo, p. 154.
9) Satake, M. (1982) : "On equivalent Mohr circle in granular materials," The Univ. New South Wales (UNICIV Rep. No. R-205).