SOILS AND FOUNDATIONS Vol. 31, No. 3, 138-143, Sept. 1991
Japanese Society of Soil Mechanics and Foundation Engineering

THE VARIATION OF SHEAR MODULUS OF A CLAY WITH PRESSURE AND OVERCONSOLIDATION RATIO

G. T. HOULSBY[i)] and C. P. WROTH[i)]

ABSTRACT

A method is described for expressing the elastic shear modulus of a clay as a power function of the applied pressure and the preconsolidation pressure. The method has the advantage that it incorporates the concept of normalisation of clay properties with respect to pressure, whilst allowing a realistic variation of the shear modulus with overconsolidation ratio to be described. A further advantage is that, since the shear strength is often expressed in a similar manner, the rigidity index G/s_u, which plays an important role in many geotechnical engineering analyses, can be expressed as a power function of the overconsolidation ratio. The new method is compared with some existing data showing the variation of the stiffness of clays.

Key words: clay, deformation, elasticity, shear strength, strain, stress, stress-strain curve (**IGC**: D6)

INTRODUCTION

Although clays rarely, if ever, exhibit fully recoverable behaviour, they do show a substantially elastic response for a certain range of stress paths. If the clays are approximately isotropic then it is convenient to express the elastic properties in terms of the bulk and shear moduli. Many engineering calculations require values of these moduli, especially the shear modulus, which can be difficult to measure directly. It is therefore useful to establish correlations between the shear modulus and other quantities, for instance the mean effective stress, overconsolidation ratio and undrained shear strength. It is the purpose of this note to examine how the shear modulus of a clay varies with pressure and overconsolidation ratio, and hence also deduce a relationship with the undrained shear strength.

ANALYSIS

The use of the term "elastic" is ambiguous for highly nonlinear materials such as soil. In some contexts, for instance dynamic loading tests, it is used to refer to the behaviour at very small strains (say less than 0.01%) for which very small hysteretic damping occurs (e.g. Hardin and Black, 1968). In the interpretation of some monotonic loading tests it is used only for a very small range of strain, typically 0.001%, for which fully recoverable behaviour is observed (e.g. Jardine et al., 1984). If the definition of elasticity is restricted to these cases then rather complex nonlinear

i) Oxford University, Department of Engineering Science, Parks Road, Oxford, OX1 3PJ, UK.
Manuscript was received for review on August 2, 1990.
Written discussions on this paper should be submitted before April 1, 1992, to the Japanese Society of Soil Mechanics and Foundation Engineering, Sugayama Bldg. 4 F, Kanda Awaji-cho 2-23, Chiyoda-ku, Tokyo 101, Japan. Upon request the closing date may be extended one month.

models must be used to describe the behaviour at slightly larger strains. A third possibility is to divide the behaviour of soils into a range for which the strains are broadly small and recoverable, although in practice some irrecoverable behaviour is observed, and a range for which the strains are larger and irrecoverable. This type of approach is exemplified by the Cam-Clay type of plasticity model in which a single work-hardening yield surface is employed and the behaviour within it is described as "elastic". Such a model offers the benefits of simplicity, although there will be shortcomings in the description of the elastic behaviour. This last approach is taken in this Paper.

The above considerations mean that the precise definition of the shear modulus is difficult. Fig. 1 shows three possible definitions which are in current use. The initial gradient of the τ—γ curve is given by G_0, G_{50} is the secant modulus to 50% of the failure stress τ_{max} (similar definitions to other proportions of τ_{max} are also used) and G_{ur} is the unload–reload modulus (which depends on the size of the unloading loop). A further modulus, G_{max} may be obtained from tests involving very small amplitude cycling, e.g. dynamic tests (G_{max} may be different from G_0 in certain circumstances). Although this last value is considered as representing most closely the actual elastic behaviour of the clay, each of the other values is relevant to particular engineering applications. The following discussion applies to each of the above definitions although the absolute values of the moduli defined in alternative ways will differ widely.

The behaviour of many clays conforms broadly to the pattern described by the critical state type of model (Schofield and Wroth, 1968), in

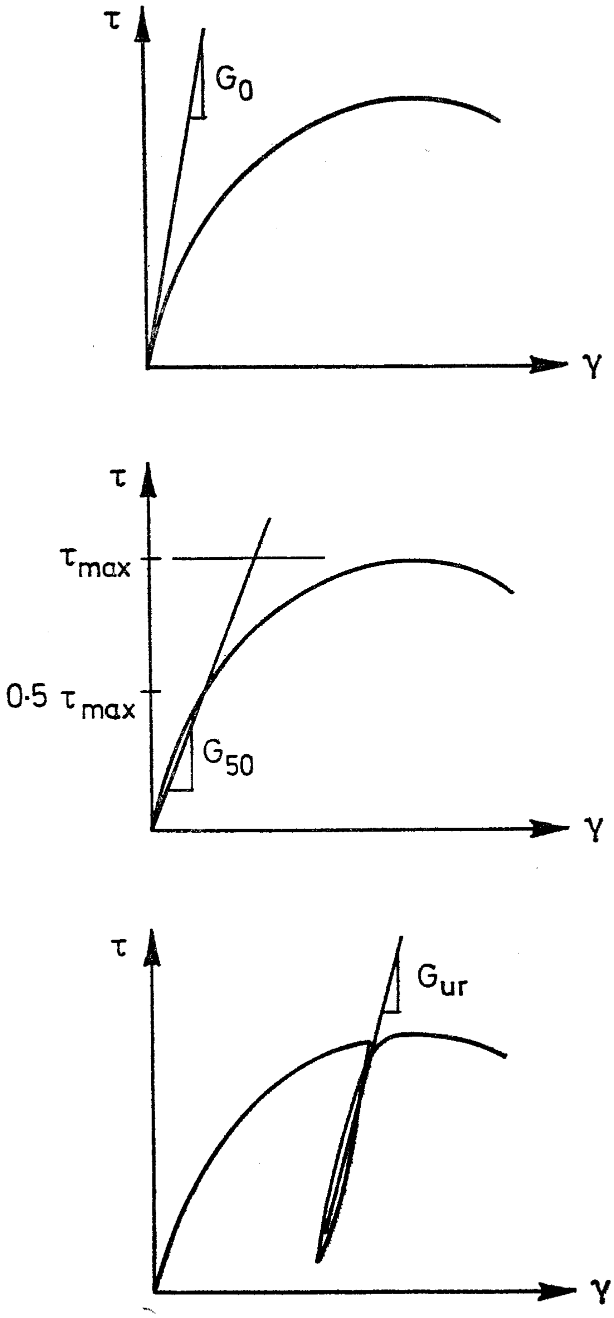

Fig. 1. Definitions of shear moduli

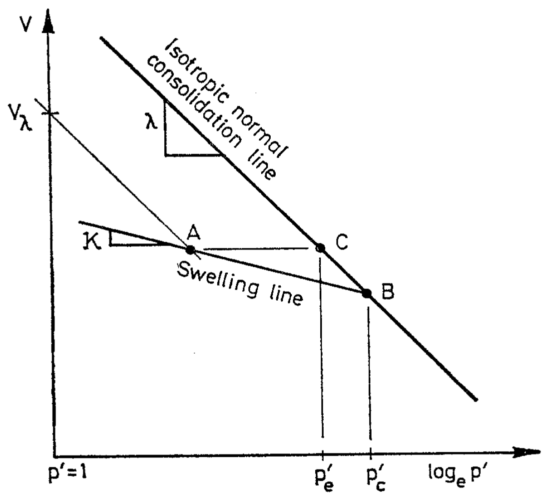

Fig. 2. Consolidation plots for a Critical State model

which case the elastic swelling lines are straight and of gradient $-\kappa$ in V–$\log_e p'$ space (Fig. 2), where V is the specific volume and p' the mean effective stress (consolidation lines are of gradient $-\lambda$ in the same plot). The increment in volumetric strain v (compressive positive) may then be determined as:

$$dv=-\frac{dV}{V}=\frac{\kappa dp'}{Vp'}=\frac{dp'}{K} \qquad (1)$$

where K is the bulk modulus. This implies a variable bulk modulus having the value $K=\frac{Vp'}{\kappa}$. A minor alteration which has various advantages (Butterfield, 1979; Wroth and Houlsby, 1980) is to assume straight consolidation and swelling lines in $\log_e V-\log_e p'$ space. If $-\kappa^*$ is the gradient of the swelling lines in this plot (and $-\lambda^*$ the gradient of the consolidation lines) then it follows that the increment in volumetric strain is given by:

$$dv=-\frac{dV}{V}=\frac{\kappa^*\, dp'}{p'}=\frac{dp'}{K} \qquad (2)$$

This results in a bulk modulus which is truly proportional to the mean stress since $K=\frac{p'}{\kappa^*}$. Although this form of relation is preferable to the more conventional approach, it is not examined further here.

The approach described above represents the normalisation of a soil property (in this case the bulk modulus) with respect to stress level (represented by the mean effective stress). This type of normalisation of soil properties with respect to pressure is central to both Critical State soil mechanics and also to the SHANSEP procedure (Ladd and Foott, 1974), and has proved a valuable aid in simplifying and understanding soil behaviour. It is reasonable to expect that a normalisation of the shear modulus with respect to the pressure may also prove useful, and therefore the variations of the value of the dimensionless quantity G/p' will be investigated.

It is important to note that any elastic model in which the shear modulus depends on the pressure, but the bulk modulus does not correspondingly depend on the shear stress, will result in non-conservative elastic behaviour, i.e. it will not be thermodynamically acceptable. This point is discussed by Zytynski et al. (1978) and also by Houlsby (1985). Such problems are not discussed further here, where the purpose is to investigate the trends in variation of the quantity G/p'. However, if the results are to be incorporated into any rigorously formulated mathematical model then some further minor modifications to the theory will be necessary.

Whilst maintaining the normalisation of soil properties with respect to pressure, the ratio G/p' need not be constant, but could for instance be a function of overconsolidation ratio. This type of result is in fact predicted by Critical State theory for the undrained shear strength, for which it can be shown (see for example Wroth, 1984):

$$\left[\frac{s_u}{p_o'}\right]=\left[\frac{s_u}{p_o'}\right]_{nc}R^{\Lambda} \qquad (3)$$

where s_u is the undrained shear strength, p_o' the initial mean effective stress, $(s_u/p_o')_{nc}$ is the strength/pressure ratio for a normally consolidated clay, R is the initial overconsolidation ratio defined as (p_c'/p_o') and Λ is a factor equal to $\frac{(\lambda-\kappa)}{\lambda}$ (which is also equal to $\frac{(\lambda^*-\kappa^*)}{\lambda^*}$). The quantity p_c' is the preconsolidation pressure defined as the intersection of the swelling line with the isotropic normal consolidation line (point B on Fig. 2 for a soil currently at stress point A). The formula above provides a convenient way of estimating both absolute values of undrained strength, and also trends of behaviour at different pressures and overconsolidation ratios. It is well supported by experimental evidence (Ladd et al., 1977).

Wroth et al. (1979) reported that a reasonable fit to the variation of G for clays could be obtained by using the expression:

$$\left[\frac{G}{p'}\right]=\left[\frac{G}{p'}\right]_{nc}(1+C\log_e(R)) \qquad (4)$$

where $(G/p')_{nc}$ is a constant for a given clay and C is a second constant. This formula is based on data of Webb (1967) showing an approximately linear variation of (G/p') with the parameter V_λ, which is defined as $(V+\lambda \log_e p')$ (see Fig. 2) and is linearly related to

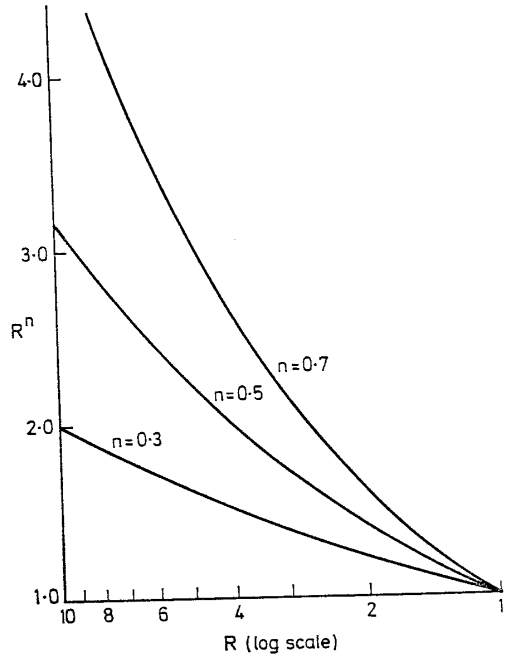

Fig. 3. R^n versus log (R)

$\log_e(R)$. (Wroth (1971) originally analysed these data using $e_\lambda = V_\lambda - 1$).

In the light of the variation of s_u with pressure an alternative expression

$$\left[\frac{G}{p'}\right]=\left[\frac{G}{p'}\right]_{nc} R^n \qquad (5)$$

is proposed. Fig. 3 shows a plot of R^n against $\log_e(R)$, and demonstrates that this approximates closely to a straight line for values of n up to about 0.7 and overconsolidation ratios less than about 10. The new formula is therefore expected to be consistent with the same data which suggested the original variation with $\log_e(R)$.

The new expression has certain advantages, for instance it implies:

$$G=\left[\frac{G}{p'}\right]_{nc} p' R^n=\left[\frac{G}{p'}\right]_{nc} p'^{(1-n)} p_c'^{n} \qquad (6)$$

i.e. G depends on p' to the power $(1-n)$ and p_c' to the power n. Since G may reasonably be expected to increase with p' for constant p_c' and to increase with p_c' for constant p' this immediately suggests on physical grounds that n should lie between 0 and 1. (No such simple argument could be used to limit the value of the parameter C).

For sands, Hertzian contact theory can be used to predict a variation of both bulk and shear moduli as power functions of pressure. The moduli are often expressed as proportional to p'^m with $m \approx 0.5$. If any similar mechanism is applicable to clay behaviour, then by comparison with Eq. (6), the value of $(1-n)$ for clays may also be expected to be about 0.5.

The second advantage of the new formula is that by combination with the expression for the undrained shear strength one may derive a simple relationship between G, s_u and R. Assuming that isotropic elasticity is applicable during the initial stages of an undrained shear test, then it follows that $p'=p_o'$, so that:

$$\left[\frac{G}{s_u}\right]=\left[\frac{G}{s_u}\right]_{nc} R^{(n-\Lambda)} \qquad (7)$$

which relates G directly to s_u. Since values of n may typically be about 0.5 and Λ about 0.75, the exponent $(n-\Lambda)$ may be expected to be approximately -0.25. This may be compared with the data from Ladd and Edgers (1972) (Fig. 4), assuming that some low G_{50} values for normally consolidated specimens are essentially due to additional strains caused by plastic deformation. The Λ value for Boston Blue Clay is about 0.8, and the shear modulus variation suggests a value of $(n-\Lambda)$ of -0.65, i.e. $n=0.15$ (perhaps a rather low value) and $\left[\frac{G_{50}}{s_u}\right]_{nc}=280.0$.

The non-dimensional parameter (G/s_u) defined by Vesic (1972) as the rigidity index I_r is particularly useful since it appears in many analyses, for instance in the cavity expansion

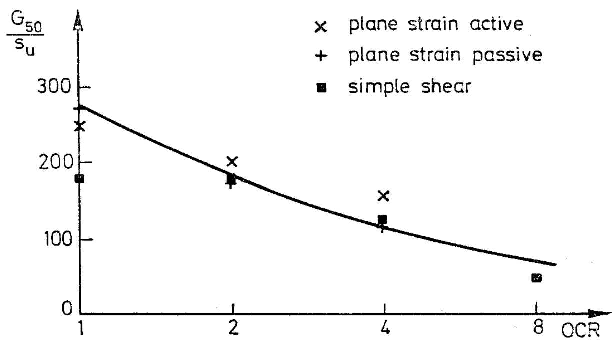

Fig. 4. $\left[\frac{G_{50}}{s_u}\right]$ values for Boston Blue Clay, after Ladd and Edgers (1972)

analysis of the pressuremeter test, pile installation or cone penetration test. Note that for a simple elsastic–perfectly plastic soil model (G/s_u) is related to the inverse of the engineering shear strain to failure, so that the strain to failure is predicted as varying as $R^{(\Lambda-n)}$, i.e. the soil behaves in a more *ductile* manner with increasing overconsolidation ratio (although the variation is rather small). In practice this trend is not observed since, especially at low overconsolidation ratios, considerable plastic strain occurs long before peak stress is reached.

Loudon (1967) presents shear test data in the form of shear strain contours for triaxial tests at different overconsolidation ratios. This information is indirect evidence of the shear modulus variation (although it confirms the non–linearity of the stress–strain response). The data are presented (see Fig. 5) in terms of the non–dimensional parameters (q/p_e') against (p'/p_e'), where p_e' is the equivalent pressure, defined as the value of the pressure on the isotropic normal consolidation line at the same specific volume (point C in Fig. 2 for a soil currently at stress point A). It may readily be shown that:

$$R=\left[\frac{p_c'}{p'}\right]=\left[\frac{p_e'}{p'}\right]^{1/\Lambda} \tag{8}$$

The shape of the shear strain contour may be deduced as follows:

$$q=3G\varepsilon=3\left[\frac{G}{p'}\right]_{nc}p'R^n\varepsilon \tag{9}$$

Substituting from equation (8):

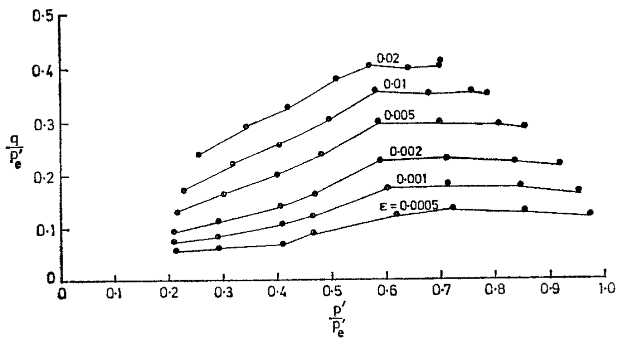

Fig. 5. Shear strain contours for kaolin, after Loudon (1967)

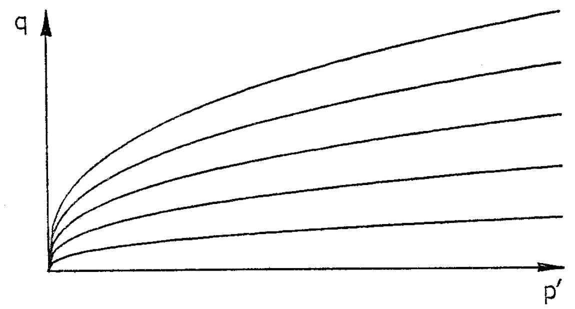

Fig. 6. Predicted shape of shear strain contours for $(1-n/\Lambda)=0.375$

$$\left[\frac{q}{p_e'}\right]=3\left[\frac{G}{p'}\right]_{nc}\left[\frac{p'}{p_e'}\right]^{(1-n/\Lambda)}\varepsilon \tag{10}$$

So for ε=constant, $\left[\frac{q}{p_e'}\right]$ varies with $\left[\frac{p'}{p_e'}\right]^{(1-n/\Lambda)}$. Taking typical values of $\Lambda=0.8$ and $n=0.5$ the exponent $(1-n/\Lambda)$ is 0.375, and the predicted shear strain contours are as shown in Fig. 6. It can be seen that the trends of behaviour observed experimentally are broadly matched by the theory.

SUMMARY AND CONCLUSIONS

Critical state models imply a variation of the elastic bulk modulus in the form $K/p'=\text{constant}=1/\kappa^*$. They also imply a variation of the undrained shear strength in the form $(s_u/p_o')=(s_u/p_o')_{nc}R^{\Lambda}$. In this note it is further suggested that the variation of shear modulus may be represented by an expression $(G/p')=(G/p')_{nc}R^n$ and this is supported by experimental data, resulting therefore in the relationship $(G/s_u)=(G/s_u)_{nc}R^{(n-\Lambda)}$ which may be convenient for estimating values of shear modulus from the relatively easily measured undrained strength.

NOTATION

C =Constant
e =Voids ratio $(=V-1)$
G =Shear modulus
I_r =Rigidity index G/s_u
K =Bulk modulus

m =Exponent
n =Exponent
p' =Mean effective stress
p_c' =Preconsolidation pressure
p_e' =Equivalent pressure
p_o' =Initial effective stress in undrained test
q =Deviator stress
R =Overconsolidation ratio
s_u =Undrained shear strength
v =Volumetric strain
V =Specific volume
V_λ =Defined as $V+\lambda \log_e p'$
γ =Engineering shear strain
κ =Gradient of swelling line in $V-\log_e p'$ space
κ^* =Gradient of swelling line in $\log_e V-\log_e p'$ space
λ =Gradient of consolidation line in $V-\log_e p'$ space
λ^* =Gradient of consolidation line in $\log_e V-\log_e p'$ space
Λ $=(\lambda-\kappa)/\lambda$
τ =Shear stress

REFERENCES

1) Butterfield, R. (1979): "A natural compression law for soils (an Advance on $e-\log_e p'$)," Géotechnique, Vol. 26, No. 4, pp. 469-480.
2) Hardin, B.O. and Black, W.L. (1968): "Vibration modulus of normally consolidated clay," Proc. ASCE, Vol. 94, No. SM2, pp. 353-369.
3) Houlsby, G.T. (1985): "The use of a variable shear modulus in elastic-plastic models for clays," Computers and Geotechnics, Vol. 1, No. 1, pp. 3-13.
4) Jardine, R.J., Symes, M.J. and Burland, J.B. (1984): "The measurement of soil stiffness in the triaxial apparatus," Géotechnique, Vol. 34. No. 3, pp. 323-340.
5) Ladd, C. and Edgers, L. (1972): "Consolidated-undrained direct simple shear tests on saturated clays," MIT Research Report, R72-82.
6) Ladd, C. and Foott, T. (1974): "New design procedure for the stability of soft clays," Proc. ASCE, Jour. Geotech. Eng. Div., Vol. 100, pp. 763-786.
7) Loudon, P.A. (1967): Some Deformation Characteristics of Kaolin, Ph. D. Thesis, University of Cambridge.
8) Schofield, A.N. and Wroth, C.P. (1968): Critical State Soil Mechanics, McGraw Hill, London.
9) Vesic, A.S. (1972): "Expansion of cavities in an infinite soil mass," Proc. ASCE, Jour Soil Mech. Found. Eng. Div., Vol. 98, No. SM3, pp. 265-290.
10) Webb, D.L. (1967): The Mechanical Properties of Undrained Samples of London Clay and Pierre Shale, Ph. D. Thesis, University of London.
11) Wroth, C.P. (1971): "Some aspects of the elastic behaviour of overconsolidated clay," Proc. Roscoe Memorial Symposium, Foulis, pp. 347-361.
12) Wroth, C.P. (1984): "The interpretation of in situ soil tests," 24th Rankine Lecture, Géotechnique, Vol. 34, No. 4, pp. 447-492.
13) Wroth, C.P. and Houlsby, G.T. (1980): "A critical state model for predicting the behaviour of clays," Proc. Workshop on Limit Equilibrium, Plasticity and Generalised Stress-Strain in Geotechnical Engineering, Montreal, pp. 592-627.
14) Wroth, C.P., Randolph, M.F., Houlsby, G.T. and Fahey, M. (1979): A Review of the Engineering Properties of Soils, with Particular Reference to the Shear Modulus, CUED/D-Soils TR75, Cambridge University Engineering Department.
15) Zytynski, M., Randolph, M.F., Nova, R. and Wroth, C.P. (1978): "On modelling the unloading-reloading behaviour of soils," Int. Jour. Num. and Anal. Methods in Geomech., Vol. 2, pp. 87-93.