RELATIONS BETWEEN PROCESS OF CUTTING AND UNIQUENESS OF SOLUTIONS

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SYNOPSIS

When an accurate stress analysis is to be made for excavation problems, it is desirable to make a step-by-step calculation by simulating the actual steps of excavation procedure. In this connection, there arise some fundamental questions as to whether the solutions are all identical when the order of cutting is changed, and whether the solution for multi-stage cutting equals that for a single-step cutting. To check the uniqueness of the solution for these different methods of cutting, the uniqueness proof was investigated by following the conventional method of approach which has been used extensively in the theory of elasticity and plasticity. The well-known theorem of virtual work was applied to the proof together with the stress-strain relationship of the material. It is shown that the uniqueness depends strongly upon the material properties, and that the sufficient condition for the uniqueness to be established is that the material is linear, time-independent and elastic throughout the cutting process.

1. INTRODUCTION

When a large-scale excavation is made in the field, it can not usually be done overnight. It is made on a step-by-step basis by following a certain process which is determined in each case by considering the economy and the safety of the construction. Among a number of conceivable methods which can lead to a given final design section of the cut, the only one process that can be considered the best is chosen. Corresponding to each conceivable construction process, the stress and the displacement in the surrounding ground change from one step to another. One construction procedure will produce a pattern of stress path and a different stress path will be formed when some other procedure is taken up. Therefore, there are an infinite number of stress paths corresponding to all the conceivable constructuion procedures. When the behavior of ground soils is such that the stress and displacement in the final design cross section are determined uniquely irrespective of the history of the stress change, it may be permissible to calculate the stresses and displacements at one time for the final configuration without tracing the intermediate paths. When the soil is assumed to be elastic and the change in the boundary condition is specified in the form of change in load or surface deformation without any change in the boundary configuration, the superposition principle in the linear elasticity proves that the solution

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is determined uniquely irrespective of the stress path through which the final state is reached. The problem involving determination of the stress and surface deflexion of an elastic soil ground loaded on some area of the surface is one example of this type of problem. However, when the boundary condition is specified in terms of the change in the boundary shape as is usually the case with cutting practice, path-independence of the solutions seems to be left open to question, even though the material is elastic. If path-dependence is actually the case even with an elastic soil, the final state of stress and deformation is dependent upon the stress-path through which the final shape is reached and the multiplicity of the solutions results corresponding to the conceivable construction procedure. This means that the final behavior of ground is to be assessed by simulating every step of construction procedure as closely as possible in the program of computation, because otherwise no definite solution is obtained. Therefore, a fundamental question arises whether the solution for an elastic ground is ever path-dependent or not, when the process of excavation is changed. When the soil can be considered elastic, the path-independency seems to be true from the study of Goodman and Brown.[13] However, they did not furnish the definite proof of uniqueness of the solution. In this paper, the question of uniqueness will be considered from a general point of view, using the virtual work theorem and the uniqueness for some particular material properties such as elasticity, pore elasticity and plasticity will be examined.

2. REVIEW OF THE PREVIOUS WORKS

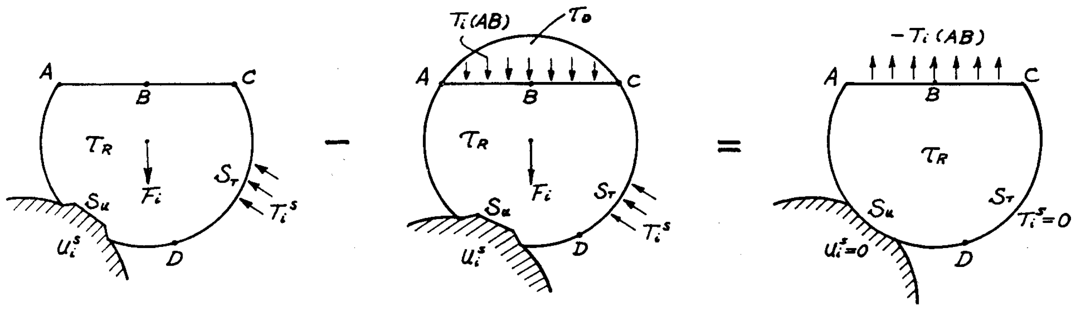
Investigation on the behavior of a ground due to excavation was initiated in connection with evaluation of stresses and displacements around a hole of a tunnel. Earlier attempt for this problem was made back in 1920's by H. Schmit[1] and N. Yamaguchi[2], who succeeded in determining stress and displacement fields due to gravity around a circular hole located far down from the ground surface. Following these contributions, a number of investigators such as Z. Anzo,[3] R. D. Mindlin[4] and Yi-Yuan Yu[5] extended the theory further to take into account the effect of ground surface or to include various configurations of tunnel openings other than a circular hole. Among them N. Yamaguchi was the first to introduce the concept of superposing two kinds of stress fields to make the boundary surface of a tunnel free from stress: the first one is due to existing stresses and the second one is an additional field which is equal in magnitude but opposite in sign to the existing stresses on the wall of tunnel hole. Since the latter stress fields are superposed to make the existing fields cancel on the boundary, it can be considered as an incremental field which is caused merely by removal of the mass, i.e., excavation. This method has an advantage over other methods in that the added stress field can be treated separately from the existing one. To be more specific, it can be applied even for such cases where the deformation characteristics which a material exhibits during creation of the existing stress is different from those which it exhibits during subsequent distrubance due to excavation. In general, the existing stress fields have been formed after a long history of tectonic movement of rocks or after gradual consolidation of soft soils, whereas excavation is generally made in a few months which are by far shorter than the period during which the existing stresses were formed. The stress-strain relationship which a material exhibits during these two major stress changes is, therefore, different in general.

Application of this method to the behavior of ground or slopes due to open cutting was delayed until quite recently, because of the analytical difficulty in handling the complex shapes of excavated boundaries. Now that the development of high-speed electronic digital computers has made it feasilbe to carry out a great deal of computation on the basis of the finite element method or lumped-parameter method, a number of problems ever considered difficult can now be solved without difficulty. An analysis along this line was presented by W. Finn[6] to predict elastic behaviors of slopes due to cutting. E. Dibiagio[7] presented useful results of computation for rectangular open-cuts in elastic grounds by using the method proposed by Yamaguchi. Different cases were considered, changing the geometry of cuts, the elastic modulus of soils, and the values of initial stresses. C. B. Brown and I. P. King[8] presented results of their works concerning the shear stress distribution within cut-down slopes, pointing out the importance of simulating excavation procedures in computing behaviors of slopes due to cutting. By means of a similar method T. Kawamoto[9] analyzed the behaviors of slopes consisting of anisotropic rocks. The changes in stress and displacement as the rock loses its strength gradually with time were studied. J. M. Duncan and P. Dunlop[10] centered their attention to the effect of initial horizontal stresses on the subsequent deformation of cuts made in stiff-fissured clays. A considerable amount of computation was performed using the finite element method to interpret various failure case histories in terms of the computed values. All the works mentioned above were done on the basis of the assumption that the soil or rock can be considered as a linear elastic material. More recently the problem of loosening of the wall and ceiling in a rock tunnel as excavation proceeds was investigated by M. Hayashi,[11] by successively tracing the change in stress which is caused by the development of cracks within rocks. Successive development of a plastic zone within soft clays which can occur as the ground is excavated without bracing was analyzed by T. Kokusho,[12] on the assumption that the soil can be approximated by an elastic-perfect plastic material.

All these works except the last two have been done on the basis of the tacit assumption that the solution for a final design section is uniquely determined, no matter what method of cutting may be employed. Since the uniqueness of the solution as explained in Introduction has not been studied in detail, it seems necessary to look back into this fundamental question again to give a sound basis for the cutting problem as a whole.

3. SYSTEM OF BASIC EQUATIONS AND BOUNDARY CONDITIONS FOR THE PROBLEM OF CUTTING

When a body under consideration is subjected to disturbance due to change in the boundary shape, the assignment of boundary conditions is obviously different from what is done when there is no boundary alteration. Therefore, it is necessary first to know how the boundary condition should be specified for such a problem. Before proceeding to the question of process of cutting, a fundamental aspect of the boundary value problem will be considered in this section. In order to make the illustration simpler, two-dimensional plane strain problem will be taken up hereafter throughout this paper.



a) after cutting b) before cutting c) change due to cutting

Fig. 1. Change in force system caused by cutting

(1) *Equations for the initial state of stresses.*

Assume that the entire body $\tau_R + \tau_0$ is initially in equilibrium, as shown in Fig. 1b, with the gravity forces and the external surface tractions T_x^s, T_y^s on the part S_T of the boundary, and the surface displacement u_x^s and u_y^s on the part S_v. The volume element τ_0 in Fig. 1b is going to be removed as a result of cutting. Since the behavior of the remaining part τ_R only is of major concrn to us, the equations which hold for this region will be exclusively considered in the following.

When confining our attention to the region τ_R, the forces $T_x(AC)$, $T_y(AC)$ which are acting on the body τ_R across the interface ABC must be considered as boundary tractions. Hence, the equilibrium, compatibility and boundary conditions for the region bounded by ABCD are written as follows:

$$\left.\begin{aligned}\frac{\partial \sigma_x^0}{\partial x} + \frac{\partial \tau_{xy}^0}{\partial y} + F_x = 0\\ \frac{\partial \tau_{xy}^0}{\partial x} + \frac{\partial \sigma_y^0}{\partial y} + F_y = 0\end{aligned}\right\} \tag{1}$$

$$e_x^0 = \frac{\partial u_x^0}{\partial x}, \quad e_y^0 = \frac{\partial u_y^0}{\partial y}, \quad \gamma_{xy}^0 = \frac{1}{2}\left(\frac{\partial u_x^0}{\partial y} + \frac{\partial u_y^0}{\partial x}\right) \tag{2}$$

$$\left.\begin{aligned}&T_x^s,\ T_y^s && \text{on} \quad S_T\\ &T_x(AC),\ T_y(AC) && \text{on} \quad S_{AC}\end{aligned}\right\} \tag{3}$$

$$u_x^s,\ u_y^s \qquad \text{on} \quad S_u \tag{4}$$

where σ_x^0, σ^0_x and τ^0_{xy} are components of stresses and u_x^0 and u_y^0 are those of displacements in the domain τ_R. From the principle of virtual work, the set of Equations (1), (2), (3) and (4) can be incorperated into one single equation as follows:

$$\int_{\tau_R}(\sigma_x^0 e_x^0 + \sigma_y^0 e_y^0 + 2\tau_{xy}^0\gamma_{xy}^0)d\tau = \int_{\tau_R}(F_x u_x^0 + F_y u_y^0)d\tau + \int_{S_T}(T_x^s u_x^0 + T_y^s u_y^0)dS$$
$$+ \int_{S_{AC}}[T_x(AC)u_x^0 + T_y(AC)u_y^0]dS + \int_{S_u}[T_x u_x^s + T_y u_y^s]dS \tag{5}$$

In Eq. (5) the forces T_x and T_y acting on the boundary S_u are related with the stresses σ_x, σ_y, and τ_{xy} as follows:

$$\left.\begin{aligned} T_x &= n_x\sigma_x + n_y\tau_{xy} \\ T_y &= n_x\tau_{xy} + n_x\sigma_y \end{aligned}\right\} \tag{6}$$

where n_x, n_y are the direction cosines of a surface element, taken positive when directed outward. The left-hand side of Eq. (5) represents the stored energy in the interior of the body τ_R, while the right-hand side expresses the total energy supplied externally by the body force and the surface forces. It is important to note that Eq. (5) is exactly equivalent to a set of equations given by (1), (2), (3) and (4). Therefore, instead of considering seperate equations individually, we can use one compact Eq. (5) to examine the physical nature of the basic boundary value problem. It should also be kept in mind that the set of Eqs. (1), (2), (3) and (4) does not contain any inforation as to the stress-strain relationship. Therefore, the equivalent Eq. (5) can be applied for any sort of continum body even if no information is available regarding the material properties. For this reason, the kinematic quantities e_x^0, e_y^0 ... need not necessarily be associated with the strains which are produced by the corresponding statical quantities σ_x^0, σ_y^0 .... In other words, completely fictitious kinematic quantities which have nothing to do with the corresponding statical quantities can be used in Eq. (5), as long as the kinematic quantities satisfy the conditions expressed by Eqs. (2) and (4). It is for this reason that Eq. (5) is usually called the theorem of virtual work.

(2) *Equations for the post-cut state*

Now, we will consider the set of equations which must be satisfied after the block τ_0 shown in Fig. 1b is detached from the block τ_R. The incremental changes in stress and displacement produced by the change in the boundary shape will be denoted by $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\tau_{xy}$ and Δu_x, Δu_y, ... Δe_x, Δe_y, ... , respectively. In this state the initial stresses σ_x^0, σ_y^0 ... plus the increments $\Delta\sigma_x$, $\Delta\sigma_y$, ... constitute the total stress system which must satisfy the equilibrium equation. Likewise the total strain and displacement system $e_x^0 + \Delta e_x$, $e_y^0 + \Delta e_y$, ... , $u_x^0 + \Delta u_x$, $u_y^0 + \Delta u_y$ ... must satisfy the compatibility equation, and we have

$$\left.\begin{aligned} \frac{\partial(\sigma_x^0 + \Delta\sigma_x)}{\partial x} + \frac{\partial(\tau_{xy}^0 + \Delta\tau_{xy})}{\partial y} + F_x = 0 \\ \frac{\partial(\tau_{xy}^0 + \Delta\tau_{xy})}{\partial x} + \frac{\partial(\sigma_y^0 + \Delta\sigma_y)}{\partial y} + F_y = 0 \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} e_x^0 + \Delta e_x = \frac{\partial(u_x^0 + \Delta u_x)}{\partial x}, \qquad e_y^0 + \Delta e_y = \frac{\partial(u_y^0 + \Delta u_y)}{\partial y} \\ \gamma_{xy}^0 + \Delta\gamma_{xy} = \frac{1}{2}\left(\frac{\partial(u_x^0 + \Delta u_x)}{\partial y} + \frac{\partial(u_y^0 + \Delta u_y)}{\partial x}\right) \end{aligned}\right\} \tag{8}$$

As far as the boundary conditions are concerned, the stresses on the boundary ABC are now equal to zero. It is to be noticed that on the boundary S_T, the external traction force does not change and, on S_u, the given displacement remains as it was before the cut was made. Therefore, the boundary conditions now become

$$T_x^s,\ T_y^s \quad \text{on} \quad S_T \tag{9}$$

$$u_x^s,\ u_y^s \quad \text{on} \quad S_u \tag{10}$$

It should be kept in mind here that the boundary conditon (9) can not be satisfied in case nowhere are the boundary conditions specified in terms of surface displacements. We will exclude such an exceptional case from the following discussions. If the basic equations (7) and (8) and the boundary conditions (9) and (10) are expressed in terms of the virtual work theorem, it follows that

$$\begin{aligned}\int_{\tau_R}&[(e_x^0 + \Delta e_x)(\sigma_x^0 + \Delta\sigma_x) + (e_y^0 + \Delta e_y)(\sigma_y^0 + \Delta\sigma_y) + 2(\gamma_{xy}^0 + \Delta\gamma_{xy})(\tau_{xy}^0 + \Delta\tau_{xy})]d\tau \\ &= \int_{\tau_R}[F_x(u_x^0 + \Delta u_x) + F_y(u_y^0 + \Delta u_y)]d\tau \\ &\quad + \int_{S_T}[T_x^s(u_x^0 + \Delta u_x) + T_y^s(u_y^0 + \Delta u_y)]dS \\ &\quad + \int_{S_u}[u_x^s(T_x + \Delta T_x) + u_y^s(T_y + \Delta T_y)]dS\end{aligned} \tag{11}$$

Now, subtracting Eq. (5) from Eq. (11) we get

$$\begin{aligned}\int_{\tau_R}&[\Delta\sigma_x\Delta e_x + \Delta\sigma_y\Delta e_y + 2\Delta\tau_{xy}\Delta\gamma_{xy}]d\tau + \int_{\tau_R}[e_x^0\Delta\sigma_x + e_y^0\Delta\sigma_y + 2\gamma_{xy}^0\Delta\tau_{xy}]d\tau \\ &\quad + \int_{\tau_R}[\sigma_x^0\Delta e_x + \sigma_y^0\Delta e_y + 2\tau_{xy}^0\Delta\gamma_{xy}]d\tau \\ &= \int_{\tau_R}[F_x\Delta u_x + F_y\Delta u_y]d\tau + \int_{S_T}[T_x^s\Delta u_x + T_y^s\Delta u_y]dS \\ &\quad + \int_{S_u}[u_x^s\Delta T_x + u_y^s\Delta T_y]dS - \int_{S_{AC}}[T_x(AC)u_x^0 + T_y(AC)u_y^0]dS\end{aligned} \tag{12}$$

The second term of the volume integral on the left side of Eq. (12) is transformed into a surface integral by using the divergence theorem of Gauss as follows:

$$\begin{aligned}\int_{\tau_R}&[\Delta\sigma_x e_x^0 + \Delta\sigma_y e_y^0 + \Delta\tau_{xy}\gamma_{xy}^0]d\tau = \int_{\tau_R}\left[\Delta\sigma_x\frac{\partial u_x^0}{\partial x} + \Delta\sigma_y\frac{\partial u_y^0}{\partial y} + \Delta\tau_{xy}\left(\frac{\partial u_x^0}{\partial y} + \frac{\partial u_y^0}{\partial x}\right)\right]d\tau \\ &= -\int_{\tau_R}\left[\left(\frac{\partial\Delta\sigma_x}{\partial x} + \frac{\partial\Delta\tau_{xy}}{\partial y}\right)u_x^0 + \left(\frac{\partial\Delta\tau_{xy}}{\partial x} + \frac{\partial\Delta\sigma_y}{\partial y}\right)u_y^0\right]d\tau + \int_S[\Delta T_x u_x^0 + \Delta T_y u_y^0]dS\end{aligned} \tag{13}$$

where
$$\Delta T_x = \Delta\sigma_x n_x + \Delta\tau_{xy}n_y, \quad \Delta T_y = \Delta\tau_{xy}n_x + \Delta\sigma_y n_y \tag{14}$$

The last term in the right side of Eq. (13) must disappear on the boundary S_T, because the surface traction is assumed unchanged during cutting. On the boundary S_u, u_x^0 and u_y^0 must be u_x^s and u_y^s, respectively, from Eq. (4). On S_{AC}, $\Delta T_x = -T_x(AC)$ and $\Delta T_y = -T_y(AC)$. Therefore, the last term becomes

$$\int_S[\Delta T_x u_x^0 + \Delta T_y \Delta u_y^0]dS = \int_{S_u}[u_x^s\Delta T_x + u_y^s\Delta T_y]dS - \int_{S_{AC}}[T_x(AC)u_x^0 + T_y(AC)u_y^0]dS \tag{15}$$

In a similar way, the volume integral in the third term on the left side of Eq. (12) is transformed as

$$\int_{\tau_R}[\sigma_x{}^0\Delta e_x+\sigma_y{}^0\Delta e_y+2\tau^0_{xy}\Delta\gamma_{xy}]d\tau=\int_{\tau_R}\left[\sigma_x{}^0\frac{\partial\Delta u_x}{\partial x}+\sigma_y{}^0\frac{\partial\Delta u_y}{\partial y}+\tau^0_{xy}\left(\frac{\partial\Delta u_x}{\partial y}+\frac{\partial\Delta u_y}{\partial x}\right)\right]d\tau$$

$$=-\int_{\tau_R}\left[\left(\frac{\partial\sigma_x{}^0}{\partial x}+\frac{\partial\tau^0_{xy}}{\partial y}\right)\Delta u_x+\left(\frac{\partial\tau^0_{xy}}{\partial x}+\frac{\partial\sigma_y{}^0}{\partial y}\right)\Delta u_y\right]d\tau+\int_S[T_x{}^0\Delta u_x+T_y{}^0\Delta u_y]dS \qquad (16)$$

The last term on the right side of Eq. (16) disappears on S_u, because the displacement is assumed confined there during cutting. On S_T, $T_x^0=T_x^s$, $T_y^0=T_y^s$ and on S_{AC}, $T_x^0=T_x(AC)$, $T_y^0=T_y(AC)$ from the boundary conditions (3). Therefore, the last term becomes

$$\int_S[T_x{}^0\Delta u_x+T_y{}^0\Delta u_y]dS$$

$$=\int_{S_T}[T_x{}^s\Delta u_x+T_y{}^s\Delta u_y]dS+\int_{S_{AC}}[T_x(AC)\Delta u_x+T_y(AC)\Delta u_y]dS \qquad (17)$$

In deriving the above relation, the compatibility relations given by

$$\Delta e_x=\frac{\partial\Delta u_x}{\partial x},\quad \Delta e_y=\frac{\partial\Delta u_y}{\partial y},\quad \Delta\gamma_{xy}=\frac{1}{2}\left(\frac{\partial\Delta u_y}{\partial x}+\frac{\partial\Delta u_x}{\partial y}\right) \qquad (18)$$

were used, because the Relation (18) is a direct consequence from Eqs. (2) and (8). Inserting Eqs. (13) and (16) into the Relation (12) together with Eqs. (1), (7), (15), and (17), one obtain,

$$\int_{\tau_R}[\Delta\sigma_x\Delta e_x+\Delta\sigma_y\Delta e_y+2\Delta\tau_{xy}\Delta\gamma_{xy}]d\tau=-\int_{S_{AC}}T_x(AC)\Delta u_x+T_y(AC)\Delta u_y]dS \qquad (19)$$

The above equation is the expression of the virtual work theorem for the incremental change in stress and strain, in the domain τ_R, which is caused by the removal of a part of the pre-stressed body. In looking at the equation, it should be emphasized that the induced stress and strain increments have nothing to do with the boundary tractions T_x^s, T_y^s on S_T and the boundary displacements u_x^s, u_y^s on S_u which are still being applied during cutting. In other words, the incremental stresses and strains should be determined with the modified boundary conditions in which the surface tractions being equal in magnitude and opposite in sign to the initial forces $T_x(AC)$, $T_y(AC)$ are imposed along the boundary S_{AC}, and with the surface traction and the displacement vanishing on S_T and S_u, respectively. Another point to be noted is that the only quantity in Eq. (19) which is connected with the previous history of loading is the surface tractions $T_x(AC)$ and $T_y(AC)$. In other words, the effect, on the subsequent incremental change, of the previous history of loading is taken into account only through the initial stresses along S_{AC}. It does not matter how the initial stresses have been reached through the complicated history of loading in the past. A single or a sequence of elastic, plastic, viscoelastic, or poroelastic process may have contributed to the formation of the current state of stresses. The only requirement that the initial stress system must fulfil is to satisfy the equations of equilibrium, compatibility and appropriate boundary conditions as indicated by Eqs. (1), (2), (3) and (4). In dealing with cutting problems in a natural soil deposit or rock foundation, it is almost impossible to know the existing stresses by tracing the past history of loading. Fortunately, we need not try to know the history all through. All we have to do is just to know the existing stresses. This can be done by direct measurement of earth pressure coefficient at rest or some other values, in-situ or in the laboratory.

4. PROBLEM SETTING

Consider a body as shown in Fig. 2. The entire mass bounded by AKFJIHG is initially in equilibrium with gravity forces F_x, F_y, the external forces T_x^s, T_y^s on the boundary S_T, and the surface displacements u_x^s, u_y^s on S_u. Suppose that a cut is to be made step-by-step until the final configuration ABCDEFK is reached. The initial state is disturbed by successive removal of pieces of masses which are denoted by 1,2, ... n in Fig. 2. The number of ways in which these masses are removed step-by-step is theoretically $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$. In considering the effect of excavation processes, there arise two questions: the one is whether the exchange of the order of cutting is essential and the other question is if multi-stage cutting can be replaced by a single one-step cutting. When attempting to examine these questions, there is no need to consider all of the possible ways of cutting. It is sufficient to see the effect only for two arbitrarily chosen masses. The reason for this is as follows. As far as the first question is concerned, it is apparent that the result obtained for two arbitrarily selected masses can be applicable to all other combinations of cutting order and it is not necessary to consider all of them. The second question can be reduced, as follows, to the problem in which only two masess are involved. Consider the n-th and $(n-1)$th blocks in Fig. 2. If it is proved that the simultaneous one-step removal of both masses yields the same answer as two-step separate removal of these masses, there is no need to separate them. The two masses can be jointly considered as one mass. The number of masses to be separately considered is then reduced by one, totaling now $(n-1)$. Therefore, it follows that the solution obtained by n-step removal of the soil masses is identical with what is obtained by $(n-1)$ step cutting. By repeating this reasoning, we can eventually reduce the number of cutting to one.

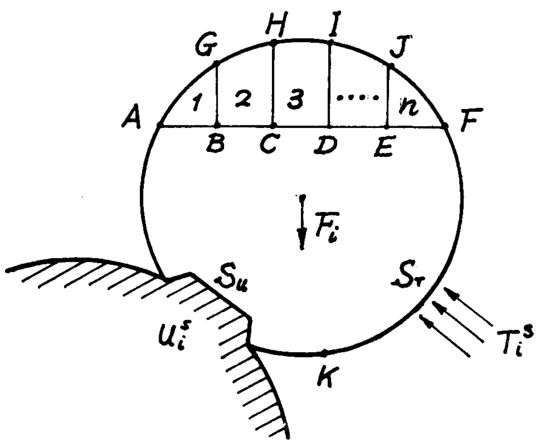


Fig. 2. A simplified model of excavation sequence

Thus, it can be said that the final solution for a cut obtained by successive removal of n-masses of soils is identical with that which is obtained by removing the entire blocks altogether at one time, if the solution for simultaneous removal of two arbitrarily selected masses is shown to coincide with that for separate removal of these two masses. It is, therefore, obvious that the proof of uniqueness for n-sequence of cutting can be reduced to that for two-step cutting.

5. VIRTUAL WORK EQUATIONS FOR ONE AND TWO-STEP CUTTING

Let us consider two masses shown in Fig. 3, in which parts τ_1 and τ_2 of the volume τ are going to be cut off eventually. There are three ways the excavation is made. The first method is to take off the volume τ_1 first and then to cut the volume τ_2 next. The second method is the reverse of the first. The third method is to remove the volume τ_1 and τ_2 altogether at a time. Now, if it is shown that the solution for the first removal of τ_1 followed by removal of τ_2 is identical with the simultaneous cutting of τ_1 and τ_2, it naturally follows that the reverse case in which τ_2 is cut first and then τ_1 yields the same solution as for the case of τ_1 to τ_2 cutting, because the masses τ_1 and τ_2 are selected quite arbitrarily without any restriction on its size or the relative position within the entire mass. Consequently, the first question raised above concerning interchangeability of cutting order can be automatically answered, if the proof is obtained as to the second question concerning replaceability of two-step cutting by one-step cutting. Then, to answer the whole question, it is sufficient to examine only two methods; two-step removal of τ_1 and τ_2, and simultaneous one-step removal of τ_1 and τ_2.

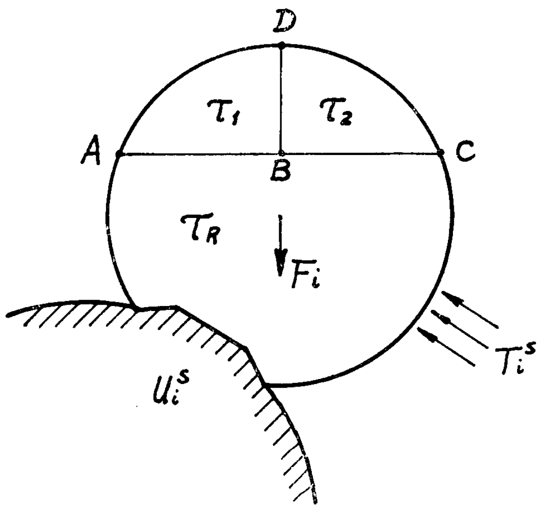


Fig. 3. A model of two-step cutting

(1) *Two-step cutting*

Consider the change in stress and strain which occurs when the volume τ_1 is removed from the body τ as shown in Fig. 3. As proved before, the change in stress and displacement in the volume τ_R which is caused by the removal of τ_1 is equivalent to that which is created by applying, on the boundary surfaces AB and BD, the forces which are of the same magnitude, but of opposite sign, to those initially existing there. At this time, the forces $T_x(BC)$ and $T_y(BC)$ which have acted along the interface BC change and their increments will be denoted by $\Delta T_x(BC)$ and $\Delta T_y(BC)$. Hence, denoting the changes in stress by $\Delta\sigma_x'$, $\Delta\sigma_y'$, $\Delta\tau_{xy}'$, the changes in strain by $\Delta e_x'$, $\Delta e_y'$, $\Delta\gamma_{xy}'$, and the changes in displacement by $\Delta u_x'$, $\Delta u_y'$, the virtual work equation which holds for these changes is from Eq. (19),

$$\int_{\tau_R}[\Delta\sigma_x'\Delta e_x' + \Delta\sigma_y'\Delta e_y' + 2\Delta\tau_{xy}'\Delta\gamma_{xy}']d\tau$$
$$= -\int_{S_{AB}}[T_x(AB)\Delta u_x' + T_y(AB)\Delta u_y']dS + \int_{S_{BC}}[\Delta T_x(BC)\Delta u_x' + \Delta T_y(BC)\Delta u_y']dS \tag{20}$$

Since the volume τ_R being considered is bounded by the surfaces AB and BC but not directly by BD, the effect of the removal of the initial stresses along the boundary BD does not appear explicitly in Eq. (20). Instead, the effect is included implicitly in the changes $\Delta T_x(BC)$ and $\Delta T_y(BC)$ which are necessary to keep the volume elements τ_2 and τ_R in contact with each other along the interface BC.

Next, remove further the volume τ_2. The changes in stress and strain will be denoted by $\Delta\sigma_x''$, $\Delta\sigma_y''$, $\Delta\tau_{xy}''$ and $\Delta e_x''$, $\Delta e_y''$, $\Delta\gamma_{xy}''$. The increments in displacement will be designated by $\Delta u_x''$ and $\Delta u_y''$. In this case, the removal of the initial stresses is achieved along the boundary BC. From the basic theorem demonstrated in Eq. (19) the vitrual equation reads

$$\int_{\tau_R}[\Delta\sigma_x''\Delta e_x'' + \Delta\sigma_y''\Delta e_y'' + 2\Delta\tau_{xy}''\Delta\gamma_{xy}'']d\tau$$
$$= -\int_{S_{BC}}[\{T_x(BC) + \Delta T_x(BC)\}\Delta u_x'' + \{T_y(BC) + \Delta T_y(BC)\}\Delta u_y'']dS \tag{21}$$

Eq. (20) relates the stresses $\Delta\sigma_x'$, $\Delta\sigma_y'$, $\Delta\tau_{xy}'$ with the boundary forces $T_x(AB)$, $T_y(AB)$, $\Delta T_x(BC)$, $\Delta T_x(BC)$, through equilibrium. The strains $\Delta e_x'$, $\Delta e_y'$, $\Delta\gamma_{xy}'$ in Eq. (20) are related with $\Delta u_x'$, $\Delta u_y'$ through the compatibility condition. Since any equilibrium stress system can be connected, through the virtual equation, with arbitrary strains satisfying compatibility, the stresses $\Delta\sigma_x'$, $\Delta\sigma_y'$, $\Delta\tau_{xy}'$ are combined with the strains $\Delta e_x''$, $\Delta e_y''$, $\Delta\gamma_{xy}''$ as follows:

$$\int_{\tau_R}[\Delta\sigma_x'\Delta e_x'' + \Delta\sigma_y'\Delta e_y'' + 2\Delta\tau_{xy}'\Delta\gamma_{xy}'']d\tau$$
$$= -\int_{S_{AB}}[T_x(AB)\Delta u_x'' + T_y(AB)\Delta u_y'']dS + \int_{S_{BC}}[\Delta T_x(BC)\Delta u_x'' + \Delta T_y(BC)\Delta u_y'']dS \tag{22}$$

Similarly, the stresses $\Delta\sigma_x''$, $\Delta\sigma_y''$, $\Delta\tau_{xy}''$ are combined with the strains $\Delta e_x'$, $\Delta e_y'$, $\Delta\gamma_{xy}'$ through the following virtual work equation:

$$\int_{\tau_R}[\Delta\sigma_x''\Delta e_x' + \Delta\sigma_y''\Delta e_y' + 2\Delta\tau_{xy}''\Delta\gamma_{xy}']d\tau$$
$$= -\int_{S_{BC}}[\{T_x(BC) + \Delta T_x(BC)\}\Delta u_x' + \{T_y(BC) + \Delta T_y(BC)\}\Delta u_y']dS \qquad (23)$$

Adding Equations (20), (21), (22), and (23), we get

$$\int_{\tau_R}[\Delta\sigma_x^T\Delta e_x^T + \Delta\sigma_y^T\Delta e_y^T + 2\Delta\tau_{xy}^T\Delta\gamma_{xy}^T]d\tau$$
$$= -\int_{S_{AB}}[T_x(AB)\Delta u_x^T + T_y(AB)\Delta u_y^T]dS - \int_{S_{BC}}[T_x(BC)\Delta u_x^T + T_y(BC)\Delta u_y^T]dS$$
$$= -\int_{S_{AC}}[T_x(AC)\Delta u_x^T + T_y(AC)\Delta u_y^T]dS \qquad (24)$$

where

$$\Delta\sigma_x^T = \Delta\sigma_x' + \Delta\sigma_x'', \quad \Delta\sigma_y^T = \Delta\sigma_y' + \Delta\sigma_y'', \quad \Delta\tau_{xy}^T = \Delta\tau_{xy}' + \Delta\tau_{xy}'' \qquad (25)$$

$$\left.\begin{aligned} &\Delta e_x^T = \Delta e_x' + \Delta e_x'', \quad \Delta e_y^T = \Delta e_y' + \Delta e_y'', \quad \Delta\gamma_{xy}^T = \Delta\gamma_{xy}' + \Delta\gamma_{xy}'' \\ &\Delta u_x^T = \Delta u_x' + \Delta u_x'', \quad \Delta u_y^T = \Delta u_y' + \Delta u_y'', \end{aligned}\right\} \qquad (26)$$

Eq. (24) is the virtual work equation for the final solutions $\Delta\sigma_x^T$, $\Delta\sigma_y^T$ . . . , Δe_x^T, Δe_y^T . . . which are obtained through two successive steps of cutting.

(2) *One-step cutting*

From the derivation of Eq. (19), it is apparent that Eq. (19) itself is the virtual work equation for the stress and strain changes which are achieved when the volumes τ_1 and τ_2 are removed altogether at a time. The notations $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\tau_{xy}$ and Δe_x, Δe_y, $\Delta\gamma_{xy}$ will be used to indicate the changes occurring as a result of one-step cutting.

Now, before proceeding to the uniqueness proof, it is preferable to have the virtual work equation which combines the statical quantities $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\tau_{xy}$ obtained by one-step cutting with kinematical quantities Δe_x^T, Δe_y^T, $\Delta\gamma_{xy}^T$ obtained by two-step cutting. With reference to Eqs. (19) and (24), it is written as

$$\int_{\tau_R}[\Delta\sigma_x\Delta e_x^T + \Delta\sigma_y\Delta e_y^T + 2\Delta\tau_{xy}\Delta\gamma_{xy}^T]d\tau$$
$$= -\int_{S_{AC}}[T_x(AC)\Delta u_x^T + T_y(AC)\Delta u_y^T]dS \qquad (27)$$

The proof that Eq. (27) is valid can be demonstrated as in the case of Eqs. (13) and (16). Similarly, the virtual equation for the stress system $\Delta\sigma_x^T$, $\Delta\sigma_y^T$, $\Delta\tau_{xy}^T$ and the strain system Δe_x, Δe_y, $\Delta\gamma_{xy}$ is obtained with reference to Eqs. (19) and (24) as follows:

$$\int_{\tau_R}\Delta\sigma_x^T\Delta e_x + \Delta\sigma_y^T\Delta e_y + 2\Delta\tau_{xy}^T\Delta\gamma_{xy}]d\tau$$
$$= -\int_{S_{AC}}[T_x(AC)\Delta u_x + T_y(AC)\Delta u_y]dS \qquad (28)$$

6. UNIQUENESS PROOF

The uniqueness proof that the stresses and strains obtained by one-step cutting are identical with those obtained by two-step cutting will be made in a way similar to that which has been done in the theory of elasticity[14] and incremental plasticity.[15,16] In these theories where no alteration in the boundary shape is involved, the uniqueness proof ordinarily follows a standard pattern as outlined in the following. Two sets of solutions $\Delta\sigma_{ij}$, Δe_{ij} and $\Delta\sigma_{ij}^T$, Δe_{ij}^T are assumed which satisfy the traction boundary conditions ΔT_i and $\Delta T_i{}^T$ on S_T, and the displacement boundary conditions Δu_i and $\Delta u_i{}^T$ on S_u, respectively, where $\Delta\sigma_{ij}$ and Δe_{ij} denote stress and strain tensors, respectively. The difference between the two assumed states is substituted in the virtual work equation as follows:

$$\int_\tau (\Delta\sigma_{ij} - \Delta\sigma_{ij}^T)(\Delta e_{ij} - \Delta e_{ij}^T)d\tau$$
$$= \int_{S_T} (\Delta T_i - \Delta T_i{}^T)(\Delta u_i - \Delta u_i{}^T)dS + \int_{S_u} (\Delta T_i - \Delta T_i{}^T)(\Delta u_i - \Delta u_i{}^T)dS \qquad (29)$$

Since the boundary conditions are identical for two solutions, i.e., $\Delta T_i = \Delta T_i{}^T$ on S_T and $\Delta u_i = \Delta u_i{}^T$ on S_u, the right-side of Eq. (29) becomes zero.

$$\int_\tau (\Delta\sigma_{ij} - \Delta\sigma_{ij}^T)(\Delta e_{ij} - \Delta e_{ij}^T)d\tau = 0 \qquad (30)$$

If the material under consideration is assumed to be elastic, it is easily shown that the integrand in the left-side of Eq. (30) is always positive except for the case $\Delta\sigma_{ij}^T = \Delta\sigma_{ij}^T$ and (or) $\Delta e_{ij} = \Delta e_{ij}^T$. Inasmuch as the integral must be zero as indicated by Eq. (30), it necessarily follows that the only solution which satisfies Eq. (30) is $\Delta e_{ij} = \Delta e_{ij}^T$ and $\Delta\sigma_{ij} = \Delta\sigma_{ij}^T$. Thus, two assumed sets of solutions are shown to be identical with each other, indicating the uniqueness of the solutions.

From the arguement outlined above, it is known that the uniqueness proof consists primarily of two steps.

Step 1. The product between the differences in stresses and in strains which are going to be compared with must be shown to be zero as indicated in Eq. (30).

Step 2. For the assumed material properties, if it is shown that the product is always positive except for the case where two solutions are identical, the uniqueness proof is complete. On the contrary, if it is shown that the product can be equal to zero, even when two solutions are not equal, the uniqueness is not guaranteed.

The uniqueness proof for the present problem will also be made in two-steps as outlined below.

Step 1. The stress-strain system $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\tau_{xy}$, Δe_x, Δe_y, $\Delta\gamma_{xy}$ obtained in one-step cutting is going to be compared with the system $\Delta\sigma_x{}^T$, $\Delta\sigma_y{}^T$, $\Delta\tau_{xy}^T$, $\Delta e_x{}^T$, $\Delta e_y{}^T$, $\Delta\gamma_{xy}^T$ which is reached through two-step cutting. To this end, the integral I as follows will be considered.

$$I=\int_{\tau_R}[\Delta\sigma_{xy}-\Delta\sigma_x{}^T)(\Delta e_x-\Delta e_x{}^T)+(\Delta\sigma_y-\Delta\sigma_y{}^T)(\Delta e_y-\Delta e_y{}^T)$$
$$+2(\Delta\tau_{xy}-\Delta\tau^T_{xy})(\Delta\gamma_{xy}-\Delta\gamma^T_{xy})]d\tau \tag{31}$$

If Eqs. (19), (24), (27) and (28) are substituted into (31), it immediately follows that:

$$I=0 \tag{32}$$

Step 2. The above relation was derived without referring to any information regarding the stress-strain characteristics which the material would show during the process of cutting. In other words, the relation can apply for any material properties, whether it is elastic or plastic. Without specifying the behavior of the material here, it is impossible to know further if the two solutions are identical or not. If the information regarding the stress-strain characteristics of the material is taken into account in Eq. (32), it becomes possible to prove the uniqueness of the two solutions. In what follows, consideration will be made for various types of material properties. To make the illustration easier and simpler, let us consider the partiuclar case where uniaxial deformation occurs uniformly throughout the volume τ_R. In this case, $\Delta\sigma_{xy}$, $\Delta\sigma_y{}^T$, and $\Delta\tau_{xy}$, $\Delta\tau^{T*}_{xy}$ are put equal to zero and Eq. (32) becomes

$$I_u=(\Delta\sigma_x-\Delta\sigma_x{}^T)(\Delta e_x-\Delta e_x{}^T)=0 \tag{33}$$

In Eq. (33) the quantity $\Delta\sigma_x{}^T$ is the summation of $\Delta\sigma_x'$ and $\Delta\sigma_x''$ as defined in Eq. (26). Now the partition ratio

$$\frac{\Delta\sigma_x'}{\Delta\sigma_x{}^T}=m \tag{34}$$

will be defined for later convenience. On the basis of Eq. (33) the uniqueness check will be made for several kinds of materials.

(1) *Linear elastic body*

The stress-strain relationship which holds during the process of one-step cutting and two-step cutting is assumed identical throughout. Then, for the uniaxial extension (removal) of the soil, the stress-strain relationship is

$$\Delta\epsilon_x=\frac{1-\nu^2}{E}\Delta\sigma_x,\qquad \Delta\epsilon_x{}^T=\frac{1-\nu^2}{E}\Delta\sigma_x{}^T \tag{35}$$

where E denotes young's modulus and ν is Poisson's ratio.
Introducting Eq. (35) into Eq. (33), it follows that

$$I_u=\frac{1-\nu^2}{E}(\Delta\sigma_x-\Delta\sigma_x{}^T)^2=\frac{E}{1-\nu^2}(\Delta\epsilon_x-\Delta\epsilon_x{}^T)^2\geqq 0 \tag{36}$$

From the inequality of the Expression (36) it follows that the value I_u can satisfy the Relation (33), only when $\Delta\sigma_x=\Delta\sigma_x{}^T$ and $\Delta\epsilon_x=\Delta\epsilon_x{}^T$. It is also possible to show that the same inequality as that shown by the Expression (36) can be obtained for a more general case where $\Delta\sigma_y$ and $\Delta\tau_{xy}$ are no longer equal to zero. Therefore, it can be concluded that the uniqueness is guaranteed for the simple case described above. To be more specific, the solution obtained by two-step cutting is exactly the same as that obtained by one-step cutting, as long as the linear stress-strain characteristics exhibited during one-step cutting hold also throughout the process of two-step cutting. From this conclusion and the reasoning in Sec. 2, it follows that the addition of several incremental solutions obtained by multi-step cutting must be the same as those obtained by carrying out the cut at a time, if the soil response is elastic throughout the cutting process. It can otherwise be stated that the way or the order of cutting is immaterial and the same result can be obtained whichever way is traced in reaching the final configuration of the cut.

(2) *Linear poroelastic body*

When the soil is saturated with water, the change in stress is generally accompanied by the drainage of water. If the deformation is of concern at the time immediately after removal of the soil, the deformation is forced to occur without volume change, because there is not yet enough time for the pore water to drain out. This kind of deformation will be referred to as "undrained deformation", or "short-time deformation". The stress-strain relationship for this condition may be defined by

$$\Delta\epsilon_x = \frac{1-\nu^2}{E}\Delta\sigma_x \tag{37}$$

where ν is Poisson's ratio being close to 1/2, because of no volume change. On the contrary, if the solution is sought for the case a long time after the cut, the pore water will have drained out by then. The situation a long time after the cutting will be referred to as "drained deformation" or "long-term condition". For this condition, the stress-strain relationship may be defined differently as

$$\Delta\epsilon_x = \frac{1-\nu'^2}{E}\Delta\sigma_x \tag{38}$$

Where ν' is less than 1/2. In the above arguement, it was assumed that the Young's modulus for the drained condition is the same as for the undrained condition, but the Poisson's ratio is different for each case. It is possible to make cutting in either drained or undrained condition. Since the general characteristics can be seen by a simple example, the following case will be examined thoroughly.

Suppose that the second-step cutting is made after a long time has elapsed since the first cut was made. If the behavior of the ground is of concern just after the second-step cutting, the stress-strain relationship for the drained condition must be used for the first-step cutting, while the undrained stress-strain relationship has to be used for the second-step cutting. Then, we have

$$\Delta\epsilon_x' = \frac{1-\nu'^2}{E}\Delta\sigma_x', \quad \Delta\epsilon_x'' = \frac{1-\nu^2}{E}\Delta\sigma_x''$$

$$\Delta\epsilon_x^T = \Delta\epsilon_x' + \Delta\epsilon_x'' = \frac{1}{E}[(1-\nu^{2\prime})m + (1-\nu^2)(1-m)]\sigma_x^T \tag{39}$$

Now, if the solution for the case as assumed above is to be compared with those for the one-step cutting which is made undrained all through, the stress and strain for the latter case is related by

$$\Delta\epsilon_x = \frac{1-\nu^2}{E}\Delta\sigma_x \tag{40}$$

Substituting Eqs. (39) and (40) into Eq. (33), we get

$$I_u = \frac{\Delta\sigma_x - \Delta\sigma_x^T}{E}[(1-\nu^2)\Delta\sigma_x - \{(1-\nu^{2\prime})m + (1-\nu^2)(1-m)\}\Delta\sigma_x^T] \tag{41}$$

In order that I_u in Eq. (41) is non-negative, the condition either $\nu = \nu'$ or $m = 0$ must be satisfied, otherwise I_u could be negative. This indicates that there could be a chance, other than $\Delta\sigma_x = \Delta\sigma_x^T$, which can make the value I_u equal to zero. This is in contradiction to the uniqueness of the solution. Therefore, it can be concluded that the solution obtained by the two-step cutting, drained to undrained, is different from those obtained by one-step undrained cutting. Several other mixed program of cutting is also feasible by combining the drained and undrained processes in two successive steps of cutting. From the similar reasoning as above applied to these mixed scheme, it is easy to realize that different solutions are obtained if the soil material exhibits different deformation characteristics in each step of cutting.

Poroelastic consolidation process essentially involves a time-dependent deformation. Complete discussion of the uniqueness for such a deformation process is out of the scope of this paper.

(3) *Incrementally plastic body*

Plastic stress-strain relations for work-hardening materials are strongly path-dependent. According to this theory, material constants which relate linearly further incremental changes in stress with those in strain depend generally upon the current state of stresses.

The stress-strain relationship to be used for future incremental changes in stress and strain changes depending upon the prsent state of stress from which the deformation starts. Hence, in the case of two-step cutting, the stress-strain relation which the material shows at the first cutting is generally different from that at the second step of cutting. Consequently, the excavation involving multi-stage processes of cutting will show a different behavior of the ground if the cutting is made differently even though the final configuration of the cut is the same. It is also said that the exchange of the order of cutting does produce different solutions, because of the difference in the stress-path.

7. CONCLUSIONS

In connection with the stress analysis for the excavation of the ground, the uniqueness of the solution with respect to changing process of cutting was investigated. As the results of the study using the virtual work theorem, it is confirmed that the solutions obtained by the different order of cutting are all equal when the ground consists of a time-independent linear elastic material. On the basis of the uniqueness proof it was further shown that the multi-stage cutting process can be simulated by a single step cutting as long as the material is linearly elastic. On the contrary, if the material exhibits time-dependent, or plastic properties, the uniqueness can not be guaranteed. In this case, the final state of stresses can not be obtained unless the complete history of loading program is specified.

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