THE FACTOR OF SAFETY AGAINST UNDRAINED FAILURE OF A SLOPE

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SUMMARY

In the application of the Fellenius, the Bishop and the friction circle methods of slope stability analysis the factor of safety is defined in terms of the failure shear stress that may be mobilized at the existing in-situ value of effective normal stress. No account is taken of the additional pore pressure change (positive or negative) that may occur between the in-situ state and the failure state when the possibility of undrained failure is being examined. Expressions for all three methods in which allowance is made for the development of these additional pore pressures, have been derived. It is claimed that the resulting redefinition of the factor of safety is more logical and more valid than the conventional definition. The use of these expressions is illustrated by their application to the analysis of the failure of the Seven Sisters embankment in Canada.

1. INTRODUCTION

In many of the methods of slope stability analysis the safety factor is defined as the ratio of the available shear strength of the soil to the shear stress required to maintain equilibrium. The available shear strength is considered to be the shear stress that is mobilized at the same value of effective normal stress as for the in-situ soil mass that is in equilibrium. Largely because of the way in which the slope stability methods are described it is often incorrectly assumed that the effective normal stress remains constant during the mobilization of the available shear strength. For the case of drained loading or unloading such an assumption cannot be generally applicable. For undrained loading, the additional pore pressures that may be developed as the soil approaches failure are not taken into account in such a definition of safety factor.

The pore pressure development for a potential undrained failure situation should preferably be allowed for by the inclusion of an appropriate term in the expression for safety factor. In the Bishop (1954) method of slices the pore pressures developed by undrained loading are allowed for by the introduction of a pore pressure parameter $\bar{B}$. These pore pressures are used to calculate the effective normal stress at which the factor of safety calculation is carried out. Further development of pore pressures between this equilibrium state (following undrained loading) and failure are not taken into consideration. To allow for this further development of pore pressures modified expressions have been

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derived for the Fellenius (1936) and Bishop methods of slices and for the friction circle method (Taylor, 1948).

DETERMINATION OF THE SHEAR STRESS ON THE FAILURE PLANE AT FAILURE

At any point along a potential failure arc the stresses may be represented by the Mohr circle *I* in Fig. 1. By means of two different procedures in the simplified Bishop and ordinary (Fellenius) methods of slices the effective normal stress σ_n' on the failure surface is found. The factor of safety (F_c) is then defined in terms of the ratio of the failure shear stress $\tau_f (=EH)$, at a normal stress of σ_n' to the initial shear stress $\tau (=GH)$ summed over the entire failure arc. It may be shown that this conventional factor of safety is

$$F_c = \frac{\Sigma\,[(\sigma_1' + \sigma_3') - (\sigma_1' - \sigma_3')\sin\phi']\tan\phi' + 2c'}{\Sigma\,(\sigma_1' - \sigma_3')\cos\phi'} \tag{1}$$

in terms of the initial major and minor principal effective stresses. Such a definition of the factor of safety infers that the effective stress path would follow the line *DC* to reach failure. A stress path is here defined as the locus of the tops of the Mohr circles

$$[(\sigma_1' + \sigma_3')/2, \quad (\sigma_1' - \sigma_3')/2]$$

However the stress path *DC* is only one of an infinite number of possible stress paths to failure and account should be taken of this in examining the possibility of undrained failure.

As indicated above the factor of safety is a measure of the amount by which the failure shear stress exceeds the existing equilibrium shear stress. For example let *GH* in Fig. 1 represent the existing shear stress and assume that the actual undrained failure point is represented by point *K*, that is, the shear stress on the failure plane at failure (τ_{ff}) is equal to *KL*. For purposes of illustration assume that the shear stresses *GH*, *EH* and *KL* are 1.0 2.0 and 1.6 kg/cm² respectively. The conventional factor of safety (F_c) is equal

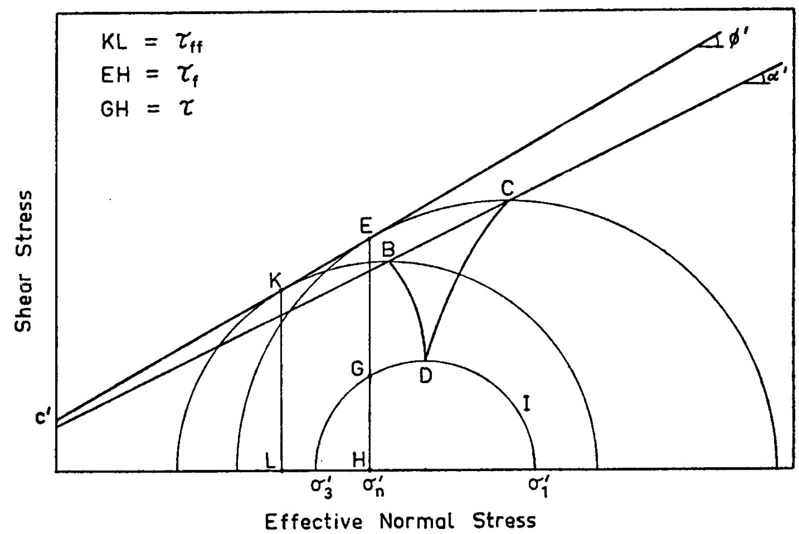

Fig. 1. Relationship between failure circles and stress paths

to 2.0. However this does not mean that the soil could sustain a doubling of the shear stress, failure would in fact occur at a lower stress of 1.6 kg/cm². For undrained failure some allowance should be made for possible changes in the effective normal stress as the soil approaches failure. In the example it would be more logical to indicate the safety factor as 1.6 and not 2.0. That is, the factor of safety (F) should be defined as the ratio of τ_{ff} to τ when summed over the failure arc.

$$F=\frac{\Sigma\, 2\{c'\cos\phi'+\sin\phi'[\sigma_3'+A_f(\sigma_1'-\sigma_3')]\}}{\Sigma[1-\sin\phi'(1-2A_f)](\sigma_1'-\sigma_3')} \tag{2}$$

This definition of safety factor indicates the role played by the pore pressure parameter A_f in the evaluation of undrained failure. The factors of safety defined in equations (1) and (2) become equal only in the particular case where the stress path to failure is along the line DC, that is, for one particular value of A_f; or when they are each equal to unity.

The shear stress τ_{ff} is related to the initial principal effective stresses as follows

$$\tau_{ff}=\frac{\cos\phi'\{c'\cos\phi'+\sin\phi'[\sigma_3'+A_f(\sigma_1'-\sigma_3')]\}}{1-\sin\phi'(1-2A_f)} \tag{3}$$

but for sloep stability analyses it is more convenient to express τ_{ff} in terms of the initial shear stress τ and initial effective normal stress σ_n'.

$$\tau_{ff}=\cos\phi'\left\{\frac{c'\cos\phi'+\sigma_n'\sin\phi'+\tau\tan\phi'(2A_f-1+\sin\phi')}{1-\sin\phi'(1-2A_f)}\right\} \tag{4}$$

MODIFICATION TO THE ORDINARY (FELLENIUS) METHOD OF SLICES

Referring to a typical slice in Fig. 2 the effective normal stress (σ_n') and shear stress (τ) may be related to the weight (W) of the slice by resolving forces normal and tangential to the failure arc.

$$\sigma_n'=\frac{W}{\Delta X}\cos^2\alpha-u_i \tag{5}$$

$$\tau=\frac{W}{\Delta X}\sin\alpha\cos\alpha \tag{6}$$

Substitution of equations (5) and (6) into equation (4) yields a further expression for the shear stress τ_{ff},

$$\tau_{ff}=\left[\frac{\cos\phi'}{1-\sin\phi'(1-2A_f)}\right]\left[c'\cos\phi'+\sin\phi\left(\frac{W}{\Delta X}\cos^2\alpha-u_i\right)\right.$$
$$\left.+\frac{W}{\Delta X}\sin\alpha\cos\alpha\tan\phi'(2A_f-1+\sin\phi')\right] \tag{7}$$

This expression could then be used in conjunction with the definition of the factor of safety

$$F=\frac{\Sigma\,\tau_{ff}\Delta X\sec\alpha}{\Sigma\, W\sin\alpha} \tag{8}$$

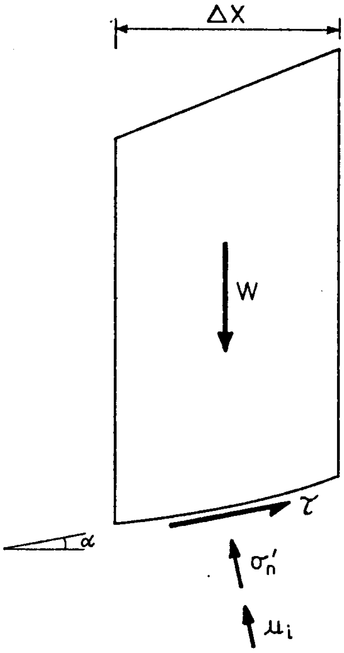

Fig. 2. Typical slice

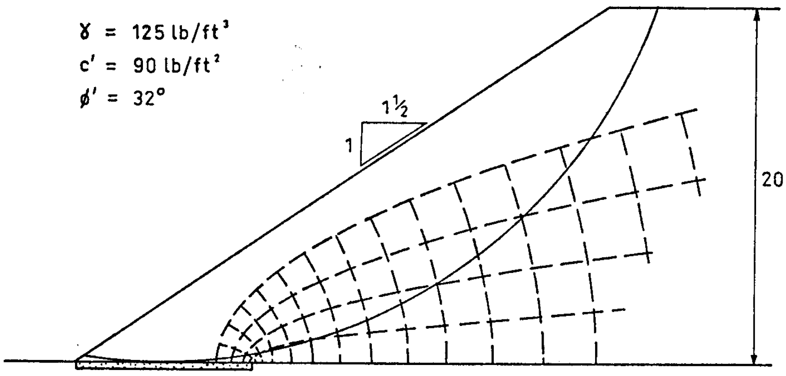

Fig. 3. Embankment for slope stability analyses

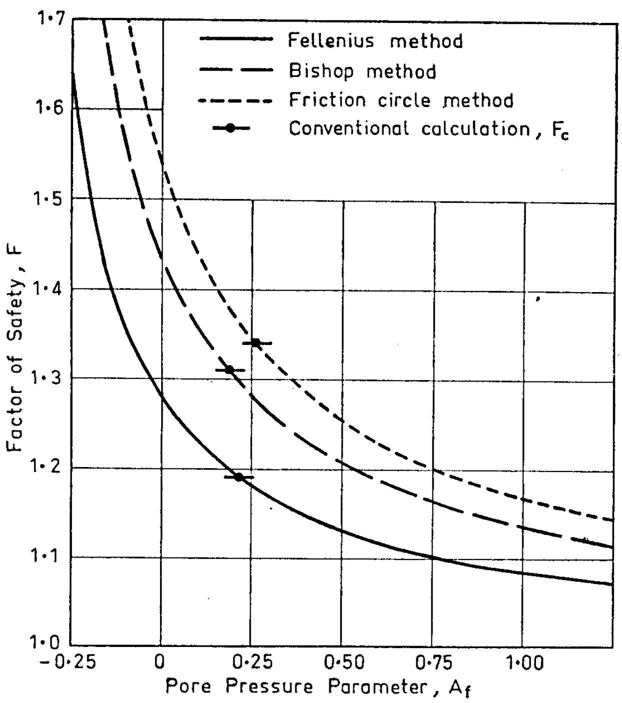

Fig. 4. Results of slope stability analyses for embankment in Fig. 3

to evaluate the stability of a slope against undrained failure. Such an evaluation was carried out for the potential failure arc and slope in Fig. 3. This is the same slope that has been subjected to detailed analyses by Whitman and Moore (1963). The results of the calculations using equations (7) and (8) are shown by the lowest curve in Fig. 4. The dominant influence of the pore pressure parameter A_f on the factor of safety against undrained failure is evident. The use of the conventional calculation infers that the soil possesses one particular value of the parameter A_f. Generally this cannot be true. It should be emphasised that the modification contained in equations (7) and (8) do not remove the shortcomings of the Fellenius method which have been discussed by Taylor (1948) and by Whitman and Moore (1963).

2. MODIFICATION TO THE FRICTION CIRCLE METHOD

The conventional graphical construction for the solution of slope stability problems by means of the friction circle method is illustrated in Fig. 5. The effective normal force N' per unit length may be considered to be equal to an average normal enective stress (σ_n') acting over a length equal to the length of the chord (L_c) connecting the extremities of the potential failure arc.

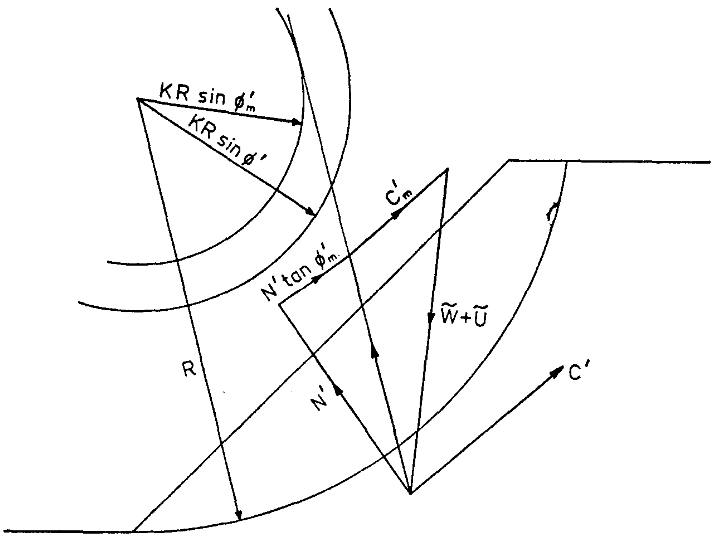

Fig. 5. Construction for friction circle method of analysis

$$N' = \sigma_n' L_c \tag{9}$$

Similarly the shear force (T) may be put equal to the shear stress (τ) acting over the length L_c.

$$T = \tau L_c \tag{10}$$

In Fig. 5 the shear force T would be equal to the vector sum of the initially mobilized forces N' tan ϕ_m' and C_m'. Following substitution of equations (9) and (10) into equation (4) an expression for safety factor in terms of the forces N' and T is obtained.

$$F=\frac{\tau_{ff}}{\tau}=\frac{c'L_c\cos^2\phi'+\sin\phi'[N'\cos\phi'+T(2A_f-1+\sin\phi')]}{T[1-\sin\phi'(1-2A_f)]} \tag{11}$$

This expression was used to find the factors of safety for the arc in Fig. 3, the results being indicated by the dotted curve in Fig. 4. These safety factors exceeded those found by the Fellenius method for all values of A_f.

MODIFICATION TO THE BISHOP METHOD OF SLICES

From Fig. 2 the effective normal stress is determined by resolving forces in the vertical direction giving

$$\sigma_n'=\frac{W}{\Delta X}-\tau\tan\alpha-u_i \tag{12}$$

The shear stress τ is assumed to be related to the failure shear stress τ_{ff} through the factor of safety

$$\tau=\frac{\tau_{ff}}{F} \tag{13}$$

It should be noted that this is not the same as the assumption made in the original Bishop method where the shear stress was taken as

$$\tau=\frac{1}{F_c}(c'+\sigma_n'\tan\phi') \tag{14}$$

Bishop assumed that the factor of safety was also equal to the ratio of resisting to overturning moments about the centre of the potential failure arc. Substitution of equations (12) and (13) into equation (4) yields an expression for the failure shear stress

$$\tau_{ff}=\frac{c'\cos\phi'+\sin\phi'\left(\frac{W}{\Delta X}-u_i\right)}{\frac{1+\sin\phi'(2A_f-1)}{\cos\phi'}+\frac{\tan\alpha\sin\phi'}{F}-\frac{\tan\phi'(2A_f-1+\sin\phi')}{F}} \tag{15}$$

This equation in conjuction with equation (8) has been applied to the solution of the problem in Fig. 3 for a condition of undrained failure. The results are shown in Fig. 4. The pattern of variation of safety factor for the Bishop method is quite similar to those for the Fellenius and friction circle methods. As expected from a study of Fig. 1 the conventionally calculated safety factors by all three methods infer that the parameter A_f is in the region of 0.2 to 0.3. For values of A_f outside this range the conventional calculations convery a misleading impression of the factor of safety. For A_f values in the vicinity of zero or below the soil can sustain a much greater increase in shear stress before failure occurs than would be inferred from the numerical value of the conventional factor of safety. On the other hand for A_f values in the region of unity the soil can sustain a smaller increase in shear stress prior to failure than would be inferred from the magnitude of F_c.

Based upon the foregoing discussion it is maintained that the factor of safety, on logical ground, should be defined in terms of the ratio of τ_{ff} to τ.

3 APPLICATION TO AN ACTUAL EMBANKMENT FAILURE

The failure of the Seven Sisters embankment in Canada has been described by Peterson, Iverson and Rivard (1957). This small embankment was located on a foundation of saturated highly plastic clay. Following construction of the embankment a series of slides occurred in spite of an adequate calculated factor of safety. The authors were unable to explain the discrepancy between the computed and actual safety factors. In discussing this paper Walker (1957) concluded that in designing for such conditions (incompetent foundations) either the safety facotr requirements must be greater than 2 or strength factors representative of the worst conditions found should be used.

The reconciliation of calculated and actual factors of safety becomes possible only when consideration is given to the construction pore pressures developed in the foundation clay by the weight of the embankment. Since the problem is obviously one of undrained failure it provides an excellent application of the equations developed earlier in this paper.

A typical cross section of the embankment together with the previously found critical failure arc is shown in Fig. 6. The factor of safety according to the Fellenius method of slices and in which the pore pressures are extracted from a flow net is 1.40. But the factor of safety obtained by the more correct Bishop method of slices is 1.74.

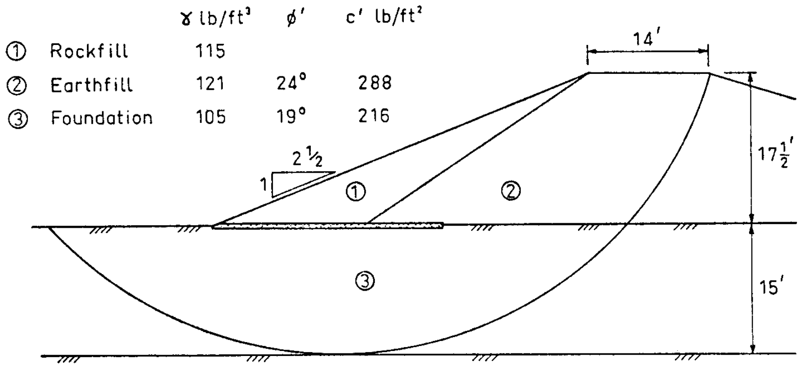

Fig. 6. Typical cross section of Seven Sisters embankment

A determination of safety factors using equations (8) and (15), in which the initial pore pressure u_i, was determined from steady seepage considerations was carried out. The results are shown by the full line in Fig. 7. The fact that this curve does not yield safety factors in the vicinity of unity supports the belief that it is unrealistic to consider the initial pore pressures determinable from steady seepage considerations.

A re-analysis was carred out with the intial pore pressures considered as construction pore pressures. These pore pressures were calculated from the expression of Skempton (1954) for a saturated soil,

$$\Delta u = \Delta\sigma_3 + A(\Delta\sigma_1 - \Delta\sigma_3) \tag{16}$$

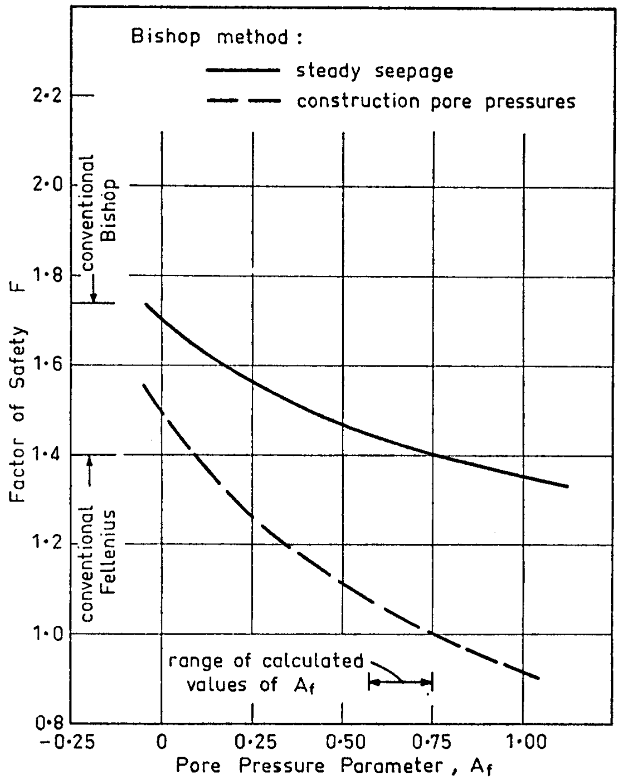

Fig. 7. Safety factors for Seven Sisters embankment

in which the changes in major and minor principal stresses were determined from Bishops (1952) relaxation solution for stresses in an earth dam after construction. This solution is applicable to a plane strain situation in which the Posissons ratio of the soil is 0.5.

The results of the analysis are shown by the dashed line in Fig. 7. It is seen that failure would be predicted for values of the pore pressure parameter A_f in the vicinity of 0.75. Laboratory determined values of this parameter have been reported by Casagrande and Rivard (1959) and by Simons (1959). From triaxial compression tests values of A_f varying from 0.57 to 0.75 were measured. Based upon these findings it seems that failure of the embankment could have been expected. It may be argued that fissuring of the foundation clay and cracking of the embankment contributed to the failure. Nevertheless it must be emphasized that failure would have been imminent even with the two proposed mechanisms inoperative.

4. APPROXIMATE DETERMINATION OF SAFETY FACTOR

The analyses described above can be quite laborious and time consuming if they are not carried out with the help of a computer. Another alternative, however, is available, namely, the use of an approximate technique which yields answers close to those obtained by the longer method.

Referring to Fig. 1 an expression relating the shear stresses τ_{ff}, τ_f and τ may be obtained to yield

$$\tau_{ff} = \frac{\cos \phi'}{1 - \sin \phi' (1 - 2A_f)} [\tau_f \cos \phi' + \tau \tan \phi' (2A_f + \sin \phi' - 1)] \tag{20}$$

If the ratio of τ_{ff} to τ is put equal to the approximate safety factor $F_{\text{approx.}}$ and the ratio of τ_f to τ equal to the conventional factor of safety F_c, then equation (20) can be expressed as

$$F_{\text{approx.}} = \frac{F_c \cos^2 \phi' + \sin \phi' (2A_f \sin \phi' - 1)}{1 - \sin \phi' (1 - 2A_f)} \tag{21}$$

This equation was used in conjunction with the embankment outlined tin Fig. 3 and for which the safety factors determined by the previously described techniques are shown in Fig. 4. The extent to which $F_{\text{approx.}}$ approximates the safety Factors F obtained by the longer methods is indicated in Fig. 8. For the Bishop method it is seen that the error is no more than a few percent.

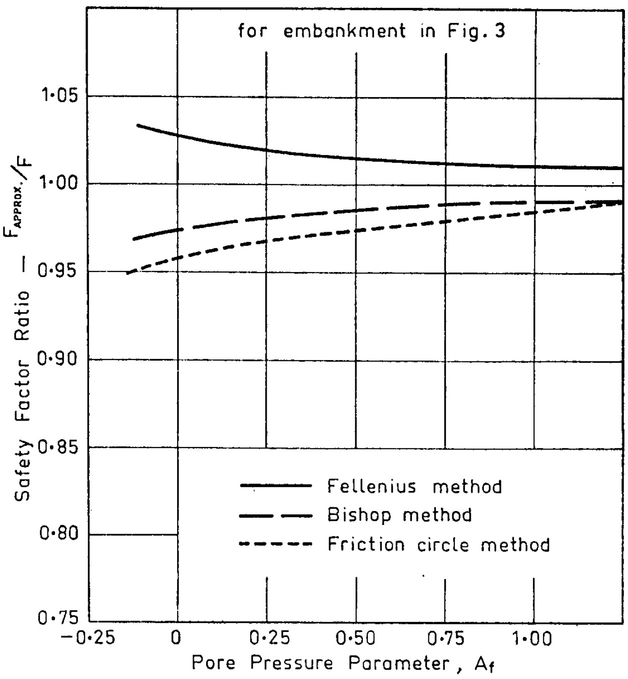

Fig. 8. Variations in safety factor introduced by approximate analysis

The magnitudes of the differences that can exist between the approximate and conventional safety factors are illustrated for two values of the effective angle of shearing resistance in Fig. 9. For values of the pore pressure parameter A_f in the vicinity of unity the conventional definition yields an unconservative estimate of the factor of safety. With negative A_f parameters the conventional safety factor is significantly lower than the logically more correct value. The two estimates of safety factor are in approximate agreement if the A parameter is in the vicinity of 0.2 to 0.3.

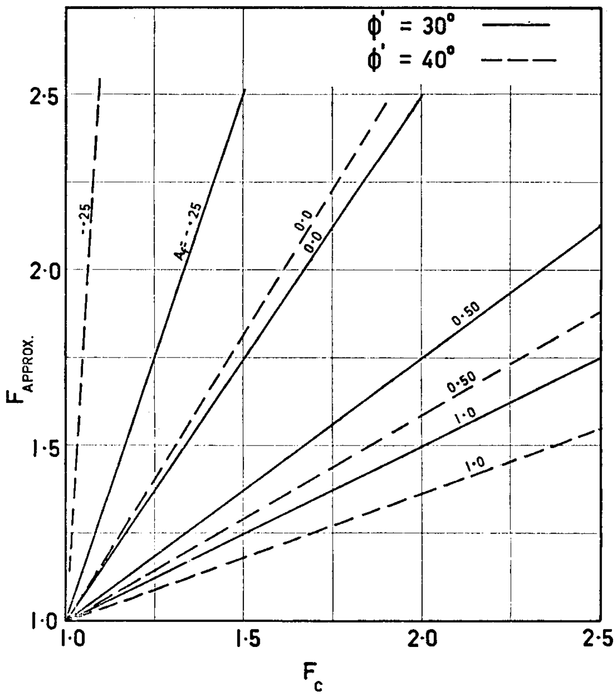

Fig. 9. Comparison of approximate and conventional safety factors

5. CONCLUDING COMMENTS

An argument has been presented for redefining the safety factor for undrained failure of a slope in terms of the shear stress on the failure plane at failure. This necessitates a determination of the additional pore pressure that may develop between the in-situ and failure states of the soil. For the Fellenius, Bishop and friction circle methods of slope stability analysis, equations, which take these pore pressures into account have been derived.

6. NOTATION

A_f: Skemptons pore pressure parameter at failure
$\bar{B}$: Pore pressure parameter ($\bar{B} = u_i\ \Delta X/W$)
c': effective cohesion intercept
c_m': mobilized value of effective cohesion
F: factor of safety ($F = \tau_{ff}/\tau$)
F_c: factor of safety calculated from conventional techniques ($F_c = \tau_f/\tau$)
$F_{\text{approx.}}$: factor of safety calculated from approximate expression
K: coefficient for use with friction circle method
L_c: length of chord in friction circle method
N': effective normal force in friction circle method
R: radius of potential failure arc
T: shear force in friction circle method
u_i: initial pore pressure

W: weight of slice
α: inclination of base of slice to horizontal
γ: unit weight of soil
Δu: change in pore pressure
ΔX: slice width
$\Delta\sigma_1$, $\Delta\sigma_3$: changes in magnitudes of principal stresses
ϕ': effective angle of shearing resistance
ϕ_m': mobilized angle of shearing resistance
σ_1', σ_3': initaial major and minor principal effective stresses
σ_n': initial normal effective stress on failure plane
τ: initial shear stress on failure plane
τ_f: failure shear stress corresponding to σ_n'
τ_{ff}: shear stress on failure plane at failure

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