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FINITE ELEMENT ANALYSIS OF CONSOLIDATION FOLLOWING UNDRAINED DEFORMATION

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ABSTRACT

The purpose of this paper is to develop a finite element method for consolidation following undrained deformation under the following assumptions: Soil is inhomogeneous, anisotropically elastic with respect to the effective stress, and saturated by incompressible water; deformation of soil does not depend on pore water pressure but on the effective stress; water flows through soil according to Darcy's law. The paper is of a theoretical nature.

The finite element method previously developed by Sandhu and Wilson (1969) and Yokoo, Yamagata, and Nagaoka (1971) is shown to be inapplicable to problems of consolidation following undrained deformation because of the inherent continuity requirement of water head. A new variational principle for consolidation is derived in which water head may be piecewise continuous, and the finite element technique is applied to the principle in order to develop an effective numerical method for the problems.

Key words: clay, computer application, consolidation, deformation, effective stress, elasticity, pore pressure, saturation, settlement analysis, three dimensional

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INTRODUCTION

Elastic and elastic-plastic problems in soil mechanics have frequently been analyzed numerically by the finite element method which has been developed for the stress analysis of a nonporous solid. Most of the problems were analyzed in terms of total stress and a few in terms of effective stress and pore water pressure. Since the mechanical behavior of a saturated soil can be interpreted more clearly in terms of the effective stress than the total stress, finite element analyses in terms of effective stress are apparently important. The following four analyses in terms of effective stress can be found.

Christian (1968) analyzed undrained elastic deformation of saturated isotropic soil. The variational principle which he developed to derive the governing equation of finite element solution can be obtained by modifying Herrmann's variational principle (1965)

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for elastic deformation of nonporous incompressible solid.

Sandhu and Wilson (1969) and Yokoo, Yamagata, and Nagaoka (1971) analyzed consolidation problems. The variational principle equivalent to the governing equations in consolidation problems was developed and, by application of the direct method to the principle, the governing equation of finite element solution was derived.

Christian and Boehmer (1970) formulated mixed type of finite element and finite difference methods in consolidation problems. The direct method was applied to a variational principle equivalent to the equilibrium equation and the elastic stress-strain relation, but finite difference technique was applied to the relation between volume change of soil and hydraulic gradient. Because of the nature of the finite difference technique, fictitious elements outside the region occupied by soil must be introduced.

In the present paper, the finite element method developed by Sandhu and Wilson (1969) and Yokoo, Yamagata, and Nagaoka (1971) is shown to be inapplicable to problems of consolidation following undrained deformation, i.e., problems of soil deformation under piecewise continuous load with respect to time, and a new finite element method effective in the problems is developed. The former method is effective only when applied load is continuous with respect to time, while the latter method is effective when applied load is either continuous or piecewise continuous with respect to time.

NOTATIONS AND DEFINITIONS

The standard indicial system with respect to the rectangular Cartesian reference frame is employed: Repeated subscripts imply summation, Kronecker's delta is denoted by δ_{ij}, differentiation with respect to space is indicated by subscripts preceded by comma and differentiation with respect to time by superposed dots, no distinction is made between covariant and contravariant components of tensors. Spatial coordinate vector is denoted by $\boldsymbol{x}$, and time by t.

The region occupied by saturated soil is denoted by R and its boundary by B. Surface traction $\bar{T}_i$ and displacement of soil $\bar{u}_i$ are prescribed on B_T and B_u, respectively; normal component of water velocity relative to soil $\bar{v}$ and water head $\bar{h}$ are prescribed on B_v and B_h, respectively, where

$$B = B_T + B_u = B_v + B_h \tag{1}$$

The region R is divided into M elements. The region and the boundary of the m-th element are denoted by R^m and B^m, respectively. The boundaries $B_T{}^m$, $B_u{}^m$, $B_v{}^m$, and $B_h{}^m$ are the portions of B^m belonging, respectively, to B_T, B_u, B_v, and B_h. The interelement boundary of the m-th element is denoted by $B_I{}^m$. The boundaries $B_I{}^m$ and $B_h{}^m$ are divided into N^m subsets $B_I{}^{mn}$ ($n = 1,2,\ldots, N^m$) and P^m subsets $B_h{}^{mp}$ ($p = 1,2,\ldots,P^m$) (cf. Fig. 1).

$$R = \sum_m R^m \tag{2}$$

$$B^m = B_T{}^m + B_u{}^m + B_I{}^m = B_v{}^m + B_h{}^m + B_I{}^m \tag{3}$$

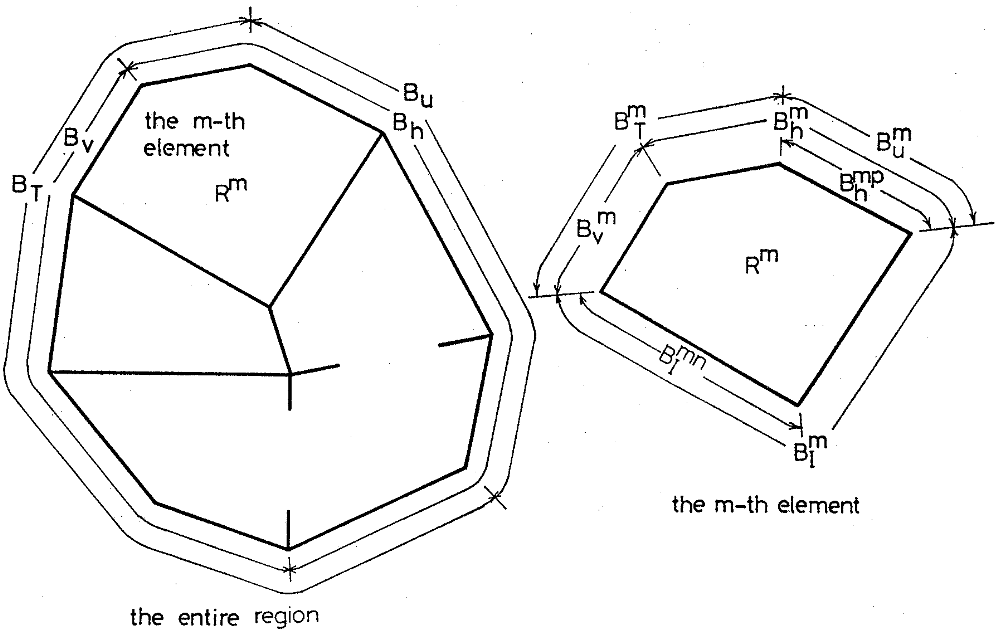

Fig. 1. Division of region into finite elements

$$B_T = \sum_m B_T{}^m \quad (4) \qquad B_u = \sum_m B_u{}^m \quad (5)$$

$$B_v = \sum_m B_v{}^m \quad (6) \qquad B_h = \sum_m B_h{}^m \quad (7)$$

$$B_I{}^m = \sum_n B_I{}^{mn} \quad (8) \qquad B_h{}^m = \sum_p B_h{}^{mp} \quad (9)$$

where $\sum_m$, $\sum_n$, and $\sum_p$ indicate the sum of M, N^m, and P^m terms. When the subboundary $B_I{}^{mn}$ is a common boundary of the m-th and m'-th elements and a quantity A is defined in R^m and $R^{m'}$, the quantity belonging to R^m and determined on $B_I{}^{mn}$ is denoted by $A^{(m)}$ and the quantity belonging to $R^{m'}$ and determined on $B_I{}^{mn}$ by $A^{(m')}$.

The following continuities and differentiabilities with respect to space are required. Displacement vector of soil u_i is continuous in R and its derivatives up to the second order with respect to space are continuous in R^m and may be discontinuous across $B_I{}^m$. Water head h and its derivatives up to the second order with respect to space are continuous in R^m and may be discontinuous across $B_I{}^m$. Normal component of water velocity relative to soil v defined on $B_I{}^m$ and $B_h{}^m$ is continuous on $B_I{}^{mn}$ and $B_h{}^{mp}$, respectively, and piecewise continuous on $B_I{}^m$ and $B_h{}^m$, respectively. When the subboundary $B_I{}^{mn}$ is a common boundary of the m-th and m'-th elements, $v^{(m)}$ belonging to the m-th element and $v^{(m')}$ to the m'-th element satisfy the following continuity equation (cf. Fig. 2).

$$v^{(m)} + v^{(m')} = 0 \qquad \text{on } B_I{}^{mn} \qquad (10)$$

Elasticity tensor of order four with respect to effective stress and permeability coefficient tensor of order two which are functions of x are denoted by E_{ijkl} and k_{ij}, respectively, where Latin subscripts take the values 1, 2, 3 and denote components of tensor with

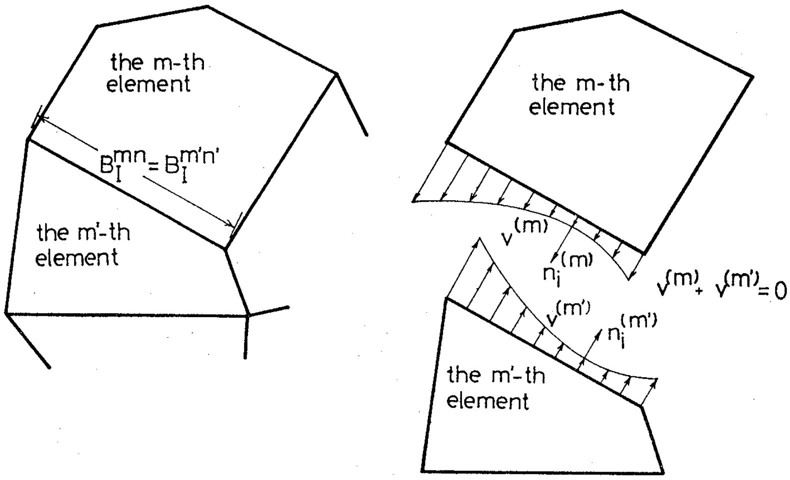

Fig. 2. Normal component of water velocity on interelement boundary

respect to rectangular Cartesian reference frame. The tensors have the following symmetric properties.

$$E_{ijkl} = E_{jikl} = E_{klij} \tag{11}$$

$$k_{ij} = k_{ji} \tag{12}$$

The quantities E_{ijkl}, k_{ij}, and their first partial derivatives are continuous in R^m and may be discontinuous across $B_I{}^m$. The prescribed boundary values $\bar{T}_i$ and $\bar{v}$ are piecewise continuous on $B_T{}^m$ and $B_v{}^m$, respectively, and $\bar{u}_i$ and $\bar{h}$ continuous on B_u and B_h, respectively.

The quantities u_i, h, v, $\bar{T}_i$, $\bar{u}_i$, $\bar{v}$, $\bar{h}$, and the first partial derivative of u_i with respect to time $\dot{u}_i$ are continuous for $t > 0$ and may be discontinuous at $t = 0$.

Constant unit weight of water is γ. The outward unit normal vector on the boundary of an element is denoted by n_i.

An integral $\int_A \ldots d\boldsymbol{x}$ is the integral over the set A with respect to the variable $\boldsymbol{x}$. For example, $\int_R \ldots d\boldsymbol{x}$ is a volume integral and $\int_B \ldots d\boldsymbol{x}$ is a surface integral.

For functions Y and Z of space and time, $Y * Z$ implies convolution and is defined by the following equation (e.g., Churchill, 1958).

$$Y(\boldsymbol{x}, t) * Z(\boldsymbol{x}, t) = \int_0^t Y(\boldsymbol{x}, t - \tau) Z(\boldsymbol{x}, \tau) d\tau \tag{13}$$

VARIATIONAL PRINCIPLE

The following assumptions are made: Soil is inhomogeneous, anisotropically elastic, and saturated by incompressible water; deformation of soil does not depend on pore water pressure but on effective stress; water flows through soil according to Darcy's law.

Relaxing continuity requirements on stress and water velocity, the governing equations for consolidation (Biot, 1941, 1955) can be rewritten as follows. For $t \geqq 0$

$$(E_{ijkl}u_{k,l} - \gamma h\delta_{ij}),_j = 0 \qquad \text{in } R^m \tag{14}$$

$$u_{i,i} = (k_{ij} * h,_j),_i \qquad \text{in } R^m \tag{15}$$

$$\{n_j(E_{ijkl}u_{k,l} - \gamma h\delta_{ij})\}^{(m)} + \{n_j(E_{ijkl}u_{k,l} - \gamma h\delta_{ij})\}^{(m')} = 0 \qquad \text{on } B_I^{mn} \tag{16}$$

$$\bar{u}_i = u_i \qquad \text{on } B_u^m \tag{17}$$

$$\bar{T}_i = n_j(E_{ijkl}u_{k,l} - \gamma h\delta_{ij}) \qquad \text{on } B_T^m \tag{18}$$

and for $t > 0$

$$v^{(m)} + v^{(m')} = 0 \qquad \text{on } B_I^{mn} \tag{19}$$

$$v = -n_i k_{ij} h,_j \qquad \text{on } B_I^{mn},\ B_h^{mp} \tag{20}$$

$$h^{(m)} - h^{(m')} = 0 \qquad \text{on } B_I^{mn} \tag{21}$$

$$\bar{v} = -n_i k_{ij} h,_j \qquad \text{on } B_v^m \tag{22}$$

$$\bar{h} = h \qquad \text{on } B_h^{mp} \tag{23}$$

Equation (14) is the equilibrium equation in terms of displacement of soil and water head, Eq. (15) the relation between volumetric strain of soil and water head (Yokoo, Yamagata, Nagaoka, 1971). On the interelement boundary B_I^{mn}, Eq. (16) expresses equilibrium of surface traction, Eq. (19) continuity of normal component of water velocity relative to soil, Eq. (20) the relation between normal component of water velocity relative to soil and water head, and Eq. (21) continuity of water head. Equations (17), (18), (22), and (23) are the prescribed boundary conditions. Equation (19) which is assumed to hold in the second section is written again here to emphasize continuity of v in the variational principle stated below.

For $t = 0$, Eq. (15) is reduced to

$$u_{i,i} = 0 \qquad \text{in } R^m \tag{24}$$

Equations (14), (16), (17), (18), and (24) are the governing equations for undrained deformation.

Hence, Eqs. (14)–(23) are the governing equations for consolidation following undrained deformation.

The variational principle equivalent to the governing equations for consolidation following undrained deformation is as follows: For the functional Π_1 defined by

$$\Pi_1 = \sum_m \left\{ \int_{R^m} \left(-\frac{1}{2} E_{ijkl} u_{i,j} * u_{k,l} + \frac{1}{2} \gamma k_{ij} * h,_i * h,_j + \gamma h * u_{i,i} \right) dx + \right.$$
$$+ \int_{B_T{}^m} \bar{T}_i * u_i dx + \int_{B_I{}^m} \gamma * h * v dx +$$
$$\left. + \int_{B_v{}^m} \gamma * h * \bar{v} dx + \int_{B_h{}^m} \gamma * (h - \bar{h}) * v dx \right\} \tag{25}$$

with side conditions (17) and (19),

$$\delta \Pi_1 = 0 \tag{26}$$

(*Proof*) The first variation of Π_1 becomes, by the divergence theorem,

$$\begin{aligned}\delta \Pi_1 = \sum_m \Big\langle & \int_{R^m} [(E_{ijkl} u_{k,l} - \gamma h \delta_{ij}),_j * \delta u_i + \gamma \{u_{i,i} - (k_{ij} * h,_j),_i\} * \delta h] d\boldsymbol{x} + \\ & + \int_{B_T{}^m} \{\bar{T}_i - n_j(E_{ijkl} u_{k,l} - \gamma h \delta_{ij})\} * \delta u_i d\boldsymbol{x} + \\ & + \int_{B_v{}^m} \gamma * (\bar{v} + n_i k_{ij} h,_j) * \delta h d\boldsymbol{x} + \\ & + \int_{B_h{}^m} \{\gamma * (h - \bar{h}) * \delta v + \gamma * (v + n_i k_{ij} h,_j) * \delta h\} d\boldsymbol{x} + \\ & + \int_{B_I{}^m} \{- n_j(E_{ijkl} u_{k,l} - \gamma h \delta_{ij}) * \delta u_i + \gamma * h * \delta v + \\ & \quad + \gamma * (v + n_i k_{ij} h,_j) * \delta h\} d\boldsymbol{x} \Big\rangle \\ = 0 \end{aligned} \tag{27}$$

By the fundamental lemma of the calculus of variations which include convolution (Gurtin, 1964), satisfaction of Eq. (27) for each δu_i, δh, and δv which satisfy

$$\delta u_i = 0 \qquad \text{on } B_u{}^m \tag{28}$$

$$\delta v^{(m)} + \delta v^{(m')} = 0 \qquad \text{on } B_I{}^{mn} \tag{29}$$

is equivalent to Eqs. (14) to (16), (18), and (20) to (23).

If, in addition to Eqs. (17) and (19), Eqs. (21) and (23) are set as side conditions, functional (25) is reduced to the following functional developed by Sandhu and Wilson (1969) and Yokoo, Yamagata, and Nagaoka (1971).

$$\begin{aligned}\Pi_2 = \sum_m \Big\{ & \int_{R^m} \Big(- \frac{1}{2} E_{ijkl} u_{i,j} * u_{k,l} + \frac{1}{2} \gamma k_{ij} * h,_i * h,_j + \gamma h * u_{i,i} \Big) d\boldsymbol{x} + \\ & + \int_{B_T{}^m} \bar{T}_i * u_i d\boldsymbol{x} + \int_{B_v{}^m} \gamma * h * \bar{v} d\boldsymbol{x} \Big\} \end{aligned} \tag{30}$$

The admissible function h in functional (25) may be discontinuous across $B_I{}^m$, but h in functional (30) is continuous across $B_I{}^m$. The importance of discontinuity of h across interelement boundary in the finite element analysis of consolidation following undrained deformation is discussed subsequently.

FINITE ELEMENT FORMULATION

To the variational principle developed in the preceding section, the direct method (e.g., Kantorovich and Krylov, 1958) is applied in order to derive the governing equation of approximate solution.

Functional (25) can be rewritten by using matrix representation as follows:

$$\Pi_1 = \sum_m \left\langle \int_{R^m} \left(-\frac{1}{2}\{\varepsilon_i\}^T[E_{ij}] * \{\varepsilon_j\} + \frac{1}{2}\gamma\{h,_i\}^T * [k_{ij}] * \{h,_j\} + \gamma h * u_{i,i} \right) dx + \right.$$
$$+ \int_{B_T{}^m} \{\bar{T}_i\}^T * \{u_i\} dx + \int_{B_v{}^m} \gamma * h * \bar{v} dx +$$
$$\left. + \sum_n \int_{B_I{}^{mn}} \gamma * h * v dx + \sum_p \int_{B_h{}^{mp}} \gamma * (h - \bar{h}) * v dx \right\rangle \quad (31)$$

The column vector $\{u_i\}$ is the displacement vector of soil, $\{\varepsilon_i\}$ the reduced strain tensor determined by appropriate space differentiation of u_i, $\{h,_i\}$ the hydraulic gradient vector, and $\{\bar{T}_i\}$ the prescribed surface traction vector. The symmetric matrix $[E_{ij}]$ is the reduced elasticity tensor with respect to effective stress, i.e., $[E_{ij}]\{\varepsilon_j\}$ is the reduced effective stress tensor, and $[k_{ij}]$ the permeability coefficient tensor. The superscript T denotes transpose of a matrix.

A column vector $\{\phi^m(t)\}$ is the set of components of displacement vectors of soil at all nodal points belonging to the m-th element. A column vector $\{\psi^q(t)\}$ is the set of parameters which determine normal component of water velocity relative to soil on the q-th subset of $\sum_m (B_I{}^m + B_h{}^m)$ (cf. Fig. 2). When the q-th subset is $B_I{}^{mn}$ or $B_h{}^{mp}$ of the m-th element, $\{\psi^q(t)\}$ is written as $\{\psi^{mn}(t)\}$ or $\{\psi^{mp}(t)\}$. When the interelement boundary $B_I{}^{mn}$ is a common boundary of the m-th and m'-th elements, i.e., $B_I{}^{mn} = B_I{}^{m'n'}$,

$$\{\psi^q(t)\} = \{\psi^{mn}(t)\} = \{\psi^{m'n'}(t)\} \quad (32)$$

A column vector $\{\omega^m(t)\}$ is the set of parameters which determine water head in the m-th element. A column vector $\{\phi(t)\}$ is the set of components of displacement vectors of soil at all nodal points belonging to the entire region R. A column vector $\{\psi(t)\}$ is the set of parameters which determine normal component of water velocity relative to soil on $\sum_m (B_I{}^m + B_h{}^m)$. The column vectors $\{\phi^m(t)\}$ and $\{\psi^q(t)\}$ can be expressed by $\{\phi(t)\}$ and $\{\psi(t)\}$ through transformation matrices $[\Phi^m]$ and $[\Psi^q]$ as follows.

$$\{\phi^m(t)\} = [\Phi^m]\{\phi(t)\} \quad (33)$$
$$\{\psi^q(t)\} = [\Psi^q]\{\psi(t)\} \quad \text{or}$$
$$\{\psi^{mn}(t)\} = [\Psi^{mn}]\{\psi(t)\}, \{\psi^{mp}(t)\} = [\Psi^{mp}]\{\psi(t)\} \quad (34)$$

Shape functions of the m-th element for displacement of soil and water head are denoted by $[a^m(\boldsymbol{x})]$ and $\{d^m(\boldsymbol{x})\}$.

$$\{u_i\} = [a^m(\boldsymbol{x})]\{\phi^m(t)\} \qquad \text{in } R^m \quad (35)$$
$$h = \{d^m(\boldsymbol{x})\}^T\{\omega^m(t)\} \qquad \text{in } R^m \quad (36)$$

Shape functions for normal component of water velocity relative to soil on $B_I{}^{mn}$ and $B_h{}^{mp}$ are denoted by $\{c^{mn}(\boldsymbol{x})\}$ and $\{c^{mp}(\boldsymbol{x})\}$, respectively.

$$v = \{c^{mn}(\boldsymbol{x})\}^T\{\psi^{mn}(t)\} \qquad \text{on } B_I{}^{mn} \text{ and}$$
$$v = \{c^{mp}(\boldsymbol{x})\}^T\{\psi^{mp}(t)\} \qquad \text{on } B_h{}^{mp} \quad (37)$$

The shape functions $[a^m(x)]$, $\{c^{mn}(x)\}$, $\{c^{mp}(x)\}$, and $\{d^m(x)\}$ must be such that $\{u_i\}$, h, and v determined by Eqs. (35), (36), and (37) satisfy the continuity and differentiability requirements with respect to space described in the second section. Two requirements are especially emphasized: The displacement $\{u_i\}$ so determined must be continuous across the interelement boundary $B_I{}^{mn}$; when the q-th subset of $\sum_m (B_I{}^m + B_h{}^m)$ is $B_I{}^{mn}$ and $B_I{}^{m'n'}$, i.e., the q-th subset is the common interelement boundary of the neighboring m-th and m'-th elements (cf. Fig. 2), $v^{(m)}$ determined by Eq. (37) and $v^{(m')}$ determined by

$$v = \{c^{m'n'}(x)\}^T\{\psi^{m'n'}(t)\} \qquad \text{on } B_I{}^{m'n'} \tag{38}$$

must satisfy

$$v^{(m)} + v^{(m')} = 0 \qquad \text{on } B_I{}^{mn} \tag{39}$$

i.e., by Eq. (32)

$$\{c^{mn}(x)\} = -\{c^{m'n'}(x)\} \qquad \text{on } B_I{}^{mn} \tag{40}$$

The water head h determined by Eq. (36) may be discontinuous across the interelement boundary $B_I{}^m$ and the components of column vector $\{\omega^m(t)\}$ are only used in the m-th element.

Using Eqs. (33), (34), (35), and (37),

$$\{u_i\} = [A^m(x)]\{\phi(t)\} \qquad \text{in } R^m \tag{41}$$

$$[A^m(x)] = [a^m(x)][\Phi^m] \tag{42}$$

$$v = \{C^{mn}(x)\}^T\{\psi(t)\} \qquad \text{on } B_I{}^{mn} \quad \text{and}$$
$$v = \{C^{mp}(x)\}^T\{\psi(t)\} \qquad \text{on } B_h{}^{mp} \tag{43}$$

$$\{C^{mn}(x)\} = [\Psi^{mn}]^T\{c^{mn}(x)\} \quad \text{and}$$
$$\{C^{mp}(x)\} = [\Psi^{mp}]^T\{c^{mp}(x)\} \tag{44}$$

By appropriate space differentiation of Eqs. (36) and (41),

$$\{\varepsilon_i\} = [F^m(x)]\{\phi(t)\} \qquad \text{in } R^m \tag{45}$$

$$u_{i,i} = \{G^m(x)\}^T\{\phi(t)\} \qquad \text{in } R^m \tag{46}$$

$$\{h,_i\} = [J^m(x)]\{\omega^m(t)\} \qquad \text{in } R^m \tag{47}$$

Using these matrix representations,

$$\sum_m \Big\langle \int_{R^m}\Big(-\frac{1}{2}\{\varepsilon_i\}^T[E_{ij}] * \{\varepsilon_j\} + \frac{1}{2}\gamma\{h,_i\}^T * [k_{ij}] * \{h,_j\} + \gamma h * u_{i,i}\Big)dx +$$
$$+ \int_{B_T{}^m}\{\bar{T}_i\}^T * \{u_i\}dx + \int_{B_v{}^m}\gamma * h * \bar{v}dx +$$
$$+ \sum_n \int_{B_I{}^{mn}}\gamma * h * vdx + \sum_p \int_{B_h{}^{mp}}\gamma * (h - \bar{h}) * vdx\Big\rangle$$

$$= -\frac{1}{2}\{\phi(t)\}^T \sum_m \int_{R^m} [F^m(\boldsymbol{x})]^T [E_{ij}(\boldsymbol{x})][F^m(\boldsymbol{x})]d\boldsymbol{x} * \{\phi(t)\} +$$

$$+ \frac{1}{2}\sum_m \{\omega^m(t)\}^T * \gamma \int_{R^m} [J^m(\boldsymbol{x})]^T [k_{ij}(\boldsymbol{x})][J^m(\boldsymbol{x})]d\boldsymbol{x} * \{\omega^m(t)\} +$$

$$+ \sum_m \{\omega^m(t)\}^T \gamma \int_{R^m} \{d^m(\boldsymbol{x})\}\{G^m(\boldsymbol{x})\}^T d\boldsymbol{x} * \{\phi(t)\} +$$

$$+ \sum_m \int_{B_T{}^m} \{\bar{T}_i(\boldsymbol{x}, t)\}^T [A^m(\boldsymbol{x})]d\boldsymbol{x} * \{\phi(t)\} +$$

$$+ 1 * \sum_m \gamma \int_{B_v{}^m} \bar{v}(\boldsymbol{x}, t)\{d^m(\boldsymbol{x})\}^T d\boldsymbol{x} * \{\omega^m(t)\} +$$

$$+ \sum_m \{\omega^m(t)\}^T * \sum_n \gamma \int_{B_I{}^{mn}} \{d^m(\boldsymbol{x})\}\{C^{mn}(\boldsymbol{x})\}^T d\boldsymbol{x} * \{\phi(t)\} +$$

$$+ \sum_m \{\omega^m(t)\}^T * \sum_p \gamma \int_{B_h{}^{mp}} \{d^m(\boldsymbol{x})\}\{C^{mp}(\boldsymbol{x})\}^T d\boldsymbol{x} * \{\phi(t)\} -$$

$$- 1 * \sum_m \sum_p \gamma \int_{B_h{}^{mp}} \bar{h}(\boldsymbol{x}, t)\{C^{mp}(\boldsymbol{x})\}^T d\boldsymbol{x} * \{\phi(t)\}$$

$$= -\frac{1}{2}\{\phi(t)\}^T [L] * \{\phi(t)\} + \frac{1}{2}\sum_m \{\omega^m(t)\}^T * [Q^m] * \{\omega^m(t)\} +$$

$$+ \sum_m \{\omega^m(t)\}^T [S^m] * \{\phi(t)\} + \{U(t)\}^T * \{\phi(t)\} +$$

$$+ \sum_m \{\omega^m(t)\}^T * [V^m] * \{\phi(t)\} + \sum_m \{\omega^m(t)\}^T * 1 * \{W^m(t)\} -$$

$$- 1 * \{Y(t)\}^T * \{\phi(t)\} \tag{48}$$

where

$$[L] = \sum_m \int_{R^m} [F^m(\boldsymbol{x})]^T [E_{ij}(\boldsymbol{x})][F^m(\boldsymbol{x})]d\boldsymbol{x} \tag{49}$$

$$[Q^m] = \gamma \int_{R^m} [J^m(\boldsymbol{x})]^T [k_{ij}(\boldsymbol{x})][J^m(\boldsymbol{x})]d\boldsymbol{x} \tag{50}$$

$$[S^m] = \gamma \int_{R^m} \{d^m(\boldsymbol{x})\}\{G^m(\boldsymbol{x})\}^T d\boldsymbol{x} \tag{51}$$

$$\{U(t)\} = \sum_m \int_{B_T{}^m} [A^m(\boldsymbol{x})]^T \{\bar{T}_i(\boldsymbol{x}, t)\}d\boldsymbol{x} \tag{52}$$

$$[V^m] = \sum_n \gamma \int_{B_I{}^{mn}} \{d^m(\boldsymbol{x})\}\{C^{mn}(\boldsymbol{x})\}^T d\boldsymbol{x} + \sum_p \gamma \int_{B_h{}^{mp}} \{d^m(\boldsymbol{x})\}\{C^{mp}(\boldsymbol{x})\}^T d\boldsymbol{x} \tag{53}$$

$$\{W^m(t)\} = \gamma \int_{B_v{}^m} \{d^m(\boldsymbol{x})\}\bar{v}(\boldsymbol{x}, t)d\boldsymbol{x} \tag{54}$$

$$\{Y(t)\} = \sum_m \sum_p \gamma \int_{B_h{}^{mp}} \{C^{mp}(\boldsymbol{x})\}\bar{h}(\boldsymbol{x}, t)d\boldsymbol{x} \tag{55}$$

Since $1/2(E_{ijkl}u_{i,j}u_{k,l})$ and $1/2(k_{ij}h_{,i}h_{,j})$ are positive and negative definite, respectively, symmetric matrices $[L]$ and $[Q^m]$ are positive and negative definite, respectively. Substituting the known $\bar{u}_i$ on B_u into $\{\phi(t)\}$ of Eq. (48), functional Π_1 can be obtained. Column vector $\{\phi(t)\}$ can be divided into the unknown column vector $\{\phi_1(t)\}$ and the known column vector $\{\phi_2(t)\}$.

$$\{\phi(t)\} = \begin{Bmatrix} \phi_1(t) \\ \cdots\cdots \\ \phi_2(t) \end{Bmatrix} \tag{56}$$

In accordance with the rearrangement of $\{\phi(t)\}$, matrices $[L]$ and $[S^m]$ and vector $\{U(t)\}$ are rearranged and Eq. (48) is rewritten in the following form which is functional Π_1.

$$\begin{aligned} \Pi_1 = &-\frac{1}{2}\begin{Bmatrix} \phi_1(t) \\ \cdots\cdots \\ \phi_2(t) \end{Bmatrix}^T \begin{bmatrix} L_{11} & \vdots & L_{12} \\ \cdots & & \cdots \\ L_{21} & \vdots & L_{22} \end{bmatrix} * \begin{Bmatrix} \phi_1(t) \\ \cdots\cdots \\ \phi_2(t) \end{Bmatrix} + \\ &+ \frac{1}{2}\sum_m \{\omega^m(t)\}^T * [Q^m] * \{\omega^m(t)\} + \sum_m \{\omega^m(t)\}^T [S_1^m \vdots S_2^m] * \begin{Bmatrix} \phi_1(t) \\ \cdots\cdots \\ \phi_2(t) \end{Bmatrix} + \\ &+ \begin{Bmatrix} U_1(t) \\ \cdots\cdots \\ U_2(t) \end{Bmatrix}^T * \begin{Bmatrix} \phi_1(t) \\ \cdots\cdots \\ \phi_2(t) \end{Bmatrix} + \sum_m \{\omega^m(t)\}^T * [V^m] * \{\psi(t)\} + \\ &+ \sum_m \{\omega^m(t)\}^T * 1 * \{W^m(t)\} - 1 * \{Y(t)\}^T * \{\psi(t)\} \end{aligned} \tag{57}$$

The first variation of Π_1 is

$$\begin{aligned} \delta\Pi_1 = &\{\delta\phi_1(t)\}^T * \left(-[L_{11} \vdots L_{12}] \begin{Bmatrix} \phi_1(t) \\ \cdots\cdots \\ \phi_2(t) \end{Bmatrix} + \sum_m [S_1^m]^T \{\omega^m(t)\} + \{U_1(t)\} \right) + \\ &+ \sum_m \{\delta\omega^m(t)\}^T * \left([Q^m] * \{\omega^m(t)\} + [S_1^m \vdots S_2^m] \begin{Bmatrix} \phi_1(t) \\ \cdots\cdots \\ \phi_2(t) \end{Bmatrix} + \right. \\ &\left. \quad + [V^m] * \{\psi(t)\} + 1 * \{W^m(t)\} \right) + \\ &+ \{\delta\psi(t)\}^T * (\sum_m [V^m]^T * \{\omega^m(t)\} - 1 * \{Y(t)\}) \\ &= 0 \end{aligned} \tag{58}$$

$$\therefore \quad -[L_{11}]\{\phi_1(t)\} + \sum_m [S_1^m]^T \{\omega^m(t)\} + (-[L_{12}]\{\phi_2(t)\} + \{U_1(t)\}) = 0 \tag{59}$$

$$\begin{aligned} &[S_1^m]\{\phi_1(t)\} + [Q^m] * \{\omega^m(t)\} + [V^m] * \{\psi(t)\} + \\ &\quad + ([S_2^m]\{\phi_2(t)\} + 1 * \{W^m(t)\}) = 0 \qquad (m = 1, 2, \ldots, M) \end{aligned} \tag{60}$$

$$\sum_m [V^m]^T * \{\omega^m(t)\} - 1 * \{Y(t)\} = 0 \tag{61}$$

Since the column vector $\{\omega^m(t)\}$ of the m-th element is independent of the column vectors of other elements, the sum sign $\sum_m$ is taken off in Eq. (60). Equations (59), (60), and (61) are the governing equations of finite element solution. For $t=0$, Eqs. (59), (60), and (61) become

$$-[L_{11}]\{\phi_1(0)\}+\sum_m[S_1{}^m]^T\{\omega^m(0)\}+(-[L_{12}]\{\phi_2(0)\}+\{U_1(0)\})=0 \tag{62}$$

$$[S_1{}^m]\{\phi_1(0)\}+[S_2{}^m]\{\phi_2(0)\}=0 \qquad (m=1,2,\ldots,M) \tag{63}$$

For $t>0$, Eqs. (59), (60), and (61) become

$$-[L_{11}]\{\phi_1(t)\}+\sum_m[S_1{}^m]^T\{\omega^m(t)\}+(-[L_{12}]\{\phi_2(t)\}+\{U_1(t)\})=0 \tag{64}$$

$$[S_1{}^m]\{\dot\phi_1(t)\}+[Q^m]\{\omega^m(t)\}+[V^m]\{\psi(t)\}+ \\ +([S_2{}^m]\{\dot\phi_2(t)\}+\{W^m(t)\})=0 \qquad (m=1,2,\ldots,M) \tag{65}$$

$$\sum_m[V^m]^T\{\omega^m(t)\}-\{Y(t)\}=0 \tag{66}$$

One of solution techniques for Eqs. (64), (65), and (66) is as follows:
In the time interval $(t,\ t+\Delta t)$

$$-[L_{11}]\left\{\phi_1\left(t+\frac{1}{2}\Delta t\right)\right\}+\sum_m[S_1{}^m]^T\left\{\omega^m\left(t+\frac{1}{2}\Delta t\right)\right\}+ \\ +\left(-[L_{12}]\left\{\phi_2\left(t+\frac{1}{2}\Delta t\right)\right\}+\left\{U_1\left(t+\frac{1}{2}\Delta t\right)\right\}\right)=0 \tag{67}$$

$$[S_1{}^m]\left\{\dot\phi_1\left(t+\frac{1}{2}\Delta t\right)\right\}+[Q^m]\left\{\omega^m\left(t+\frac{1}{2}\Delta t\right)\right\}+[V^m]\left\{\psi\left(t+\frac{1}{2}\Delta t\right)\right\}+ \\ +\left([S_2{}^m]\left\{\dot\phi_2\left(t+\frac{1}{2}\Delta t\right)\right\}+\left\{W^m\left(t+\frac{1}{2}\Delta t\right)\right\}\right)=0 \quad (m=1,2,\ldots,M) \tag{68}$$

$$\sum_m[V^m]^T\left\{\omega^m\left(t+\frac{1}{2}\Delta t\right)\right\}-\left\{Y\left(t+\frac{1}{2}\Delta t\right)\right\}=0 \tag{69}$$

$$\left\{\phi_1\left(t+\frac{1}{2}\Delta t\right)\right\}=\frac{1}{2}(\{\phi_1(t+\Delta t)\}+\{\phi_1(t)\}) \tag{70}$$

$$\left\{\dot\phi_1\left(t+\frac{1}{2}\Delta t\right)\right\}=\frac{1}{\Delta t}(\{\phi_1(t+\Delta t)\}-\{\phi_1(t)\}) \tag{71}$$

Substitution of Eqs. (70) and (71) into Eqs. (67) and (68) yields

$$-\frac{1}{2}[L_{11}]\{\phi_1(t+\Delta t)+\sum_m[S_1{}^m]^T\left\{\omega^m\left(t+\frac{1}{2}\Delta t\right)\right\}+ \\ +\left(-\frac{1}{2}[L_{11}]\{\phi_1(t)\}-[L_{12}]\left\{\phi_2\left(t+\frac{1}{2}\Delta t\right)\right\}+\left\{U_1\left(t+\frac{1}{2}\Delta t\right)\right\}\right)=0 \tag{72}$$

$$[S_1{}^m]\{\phi_1(t+\Delta t)\}+\Delta t[Q^m]\left\{\omega^m\left(t+\frac{1}{2}\Delta t\right)\right\}+\Delta t[V^m]\left\{\psi\left(t+\frac{1}{2}\Delta t\right)\right\}+ \\ +\left(-[S_1{}^m]\{\phi_1(t)\}+\Delta t[S_2{}^m]\left\{\dot\phi_2\left(t+\frac{1}{2}\Delta t\right)\right\}+\Delta t\left\{W^m\left(t+\frac{1}{2}\Delta t\right)\right\}\right)=0 \\ (m=1,2,\ldots,M) \tag{73}$$

If $\{\phi_1(t)\}$ is known, $\{\phi_1(t+\Delta t)\}$, $\{\phi(t+\frac{1}{2}\Delta t)\}$, and $\{\omega^m(t+\frac{1}{2}\Delta t)\}$ can be obtained by Eqs. (69), (72), and (73).

WATER HEAD FIELD FOR UNDRAINED DEFORMATION

The governing equations for undrained deformation are Eqs. (14), (16), (17), (18), and (24). The variational principle equivalent to the equations developed by Christian (1968) in case of isotropy can be extended to the following variational principle in case of anisotropy: For the functional Π_3 defined by

$$\Pi_3 = \sum_m \left\{\int_{R^m}\left(-\frac{1}{2}E_{ijkl}u_{i,j}u_{k,l} + \gamma h u_{i,i}\right)dx + \int_{B_T{}^m} \bar{T}_i u_i dx\right\} \tag{74}$$

with side condition (17),

$$\delta\Pi_3 = 0 \tag{75}$$

The continuity and differentiability requirements to the admissible function u_i and the given functions E_{ijkl}, γ, $\bar{T}_i$, and $\bar{u}_i$ are the same as given in the second section. Water head h and its first partial derivative with respect to space is continuous in R^m and may be discontinuous across $B_I{}^m$.

(*Proof*) The first variation of Π_3 becomes, by the divergence theorem,

$$\begin{aligned}\delta\Pi_3 = \sum_m \Big[\int_{R^m} &\{(E_{ijkl}u_{k,l} - \gamma h\delta_{ij})_{,j}\delta u_i + \gamma u_{i,i}\delta h\}dx + \\ &+ \int_{B_T{}^m}\{\bar{T}_i - n_j(E_{ijkl}u_{k,l} - \gamma h\delta_{ij})\}\delta u_i dx - \\ &- \int_{B_I{}^m} n_j(E_{ijkl}u_{k,l} - \gamma h\delta_{ij})\delta u_i dx\Big] \\ = 0 &\end{aligned} \tag{76}$$

By the fundamental lemma of the calculus of variations (e.g., Courant and Hilbert, 1953), satisfaction of Eq. (76) for each δu_i and δh which satisfy

$$\delta u_i = 0 \qquad \text{on } B_u{}^m \tag{77}$$

is equivalent to Eqs. (14), (16), (18), and (24).

If $[a^m(x)]$ and $\{d^m(x)\}$ in Eqs. (35) and (36) are employed as the shape functions for u_i and h to analyze undrained deformation and finite element technique is applied to functional (74), Eqs. (62) and (63) can be derived, i.e., the governing equations of finite element solution at $t=0$ derived from functional (25) are the same as those derived from functional (74).

If consolidation following undrained deformation is analyzed by the finite element method developed by Sandhu and Wilson (1969) and Yokoo, Yamagata, and Nagaoka (1971), shape function for water head $\{\tilde{d}^m(x)\}$ must be such that water head determined by

$$h = \{\tilde{d}^m(x)\}^T\{\tilde{\omega}^m(t)\} \qquad \text{in } R^m \tag{78}$$

is continuous across $B_I{}^m$, where $\{\tilde{\omega}^m(t)\}$ is the set of parameters which determine water head in the m-th element. When undrained deformation followed by consolidation is analyzed, the same shape function as employed for consolidation must be employed for undrained deformation, i.e., the shape function $\{\tilde{d}^m(x)\}$ must be employed for analysis of the undrained deformation.

It is explained in the following that, when undrained deformation is analyzed, shape function for water head must not be $\{\tilde{d}^m(x)\}$ but $\{d^m(x)\}$, i.e., shape function for water head such that water head is assured to be continuous across interelement boundary cannot be employed.

If water head is piecewise continuous with respect to space in exact solution, it is apparent that a better finite element solution cannot be obtained by imposing on water head the continuity across interelement boundary. Even if water head is continuous in exact solution, it cannot be insisted that a better finite element solution is obtained by imposing the continuity, since water head determined by the finite element solution on which the continuity is not imposed may be piecewise continuous and imposition of the continuity narrows possible extent of finite element solution. This can be verified by the following numerical examples.

A cylinder of saturated homogeneous isotropic elastic soil of length 4 m and radius 1 m sustains vertical load of 10 ton/m² on the top surface. The bottom surface is smooth and no traction is applied on the cylindrical surface. With respect to effective stress, Young's modulus of soil is 2.0×10^2 ton/m² and Poisson's ratio 1/3. Unit weight of water γ is 1.0 ton/m³. Isoparametric element with eight nodes is employed (Ergatoudis, Irons, and Zienkiewicz, 1968) (cf. Fig. 3). Finite element idealization is shown in Fig. 4.

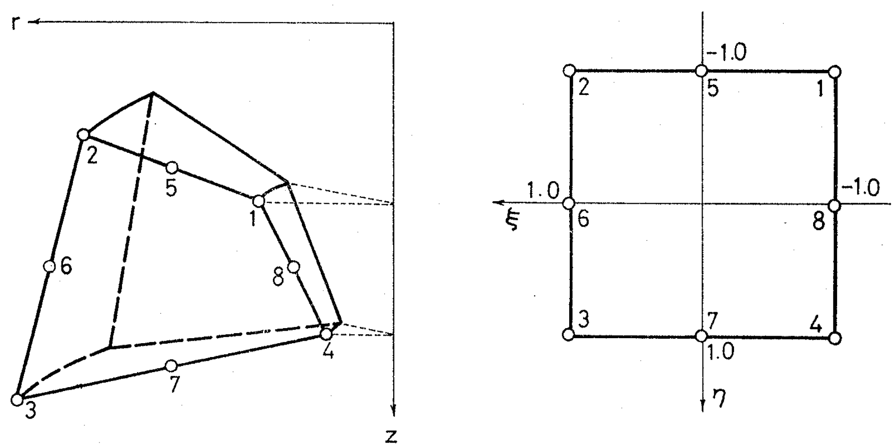

Fig. 3. Axi-symmetric isoparametric element

1. *Example a.*

The shape function for water head which assures continuity of water head across interelement boundary is employed first. The shape functions for displacement of soil and water head in the isoparametric element are as follows:

The sets of coordinates r and z of eight nodes are denoted by $\{r\}$ and $\{z\}$. The sets of

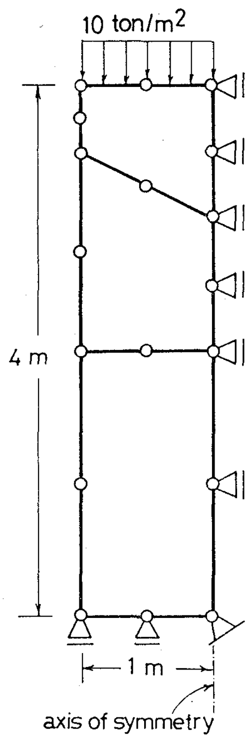

Fig. 4. Finite element idealization of cylinder

horizontal displacement of soil u_r, vertical displacement u_z, and water head h at eight nodes are $\{u_r\}$, $\{u_z\}$, and $\{h\}$. Two coordinates ξ and η are associated with the quadrilateral in Fig. 3 and determined so as to give

$$\eta = -1 \quad \text{on side } 12$$
$$\eta = +1 \quad \text{on side } 43$$
$$\xi = +1 \quad \text{on side } 23$$
$$\xi = -1 \quad \text{on side } 14$$

The relation between the Cartesian coordinates and the $\xi - \eta$ coordinates is

$$r = N_1 r_1 + N_2 r_2 + \dots + N_8 r_8 = \{N\}^T\{r\} \tag{79}$$

$$z = N_1 z_1 + N_2 z_2 + \dots + N_8 z_8 = \{N\}^T\{z\} \tag{80}$$

and displacement of soil and water head are given by

$$u_r = \{N\}^T\{u_r\} \tag{81}$$

$$u_z = \{N\}^T\{u_z\} \tag{82}$$

$$h = \{N\}^T\{h\} \tag{83}$$

where

$$N_i=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)-\frac{1}{4}(1-\xi^2)(1+\eta\eta_i)-$$

$$-\frac{1}{4}(1+\xi\xi_i)(1-\eta^2) \qquad (i=1,2,3,4) \tag{84}$$

$$N_i=\frac{1}{2}(1-\xi^2)(1+\eta\eta_i) \qquad (i=5,7) \tag{85}$$

$$N_i=\frac{1}{2}(1-\eta^2)(1+\xi\xi_i) \qquad (i=6,8) \tag{86}$$

The quantities ξ_i and η_i are nodal values of the $\xi-\eta$ coordinates. If column vector $\{\phi^m(t)\}$ is set as

$$\{\phi^m(t)\}=\begin{Bmatrix}\{u_r\}\\ \cdots\\ \{u_z\}\end{Bmatrix} \tag{87}$$

shape function $[a^m(x)]$ is

$$[a^m(x)]=\begin{bmatrix}N_1,N_2,\ldots,N_8, & 0, & 0, & \ldots, & 0\\ 0, & 0, & \ldots, & 0, & N_1,N_2,\ldots,N_8\end{bmatrix} \tag{88}$$

Column vector $\{\tilde{\omega}^m(t)\}$ is identical with $\{h\}$ and

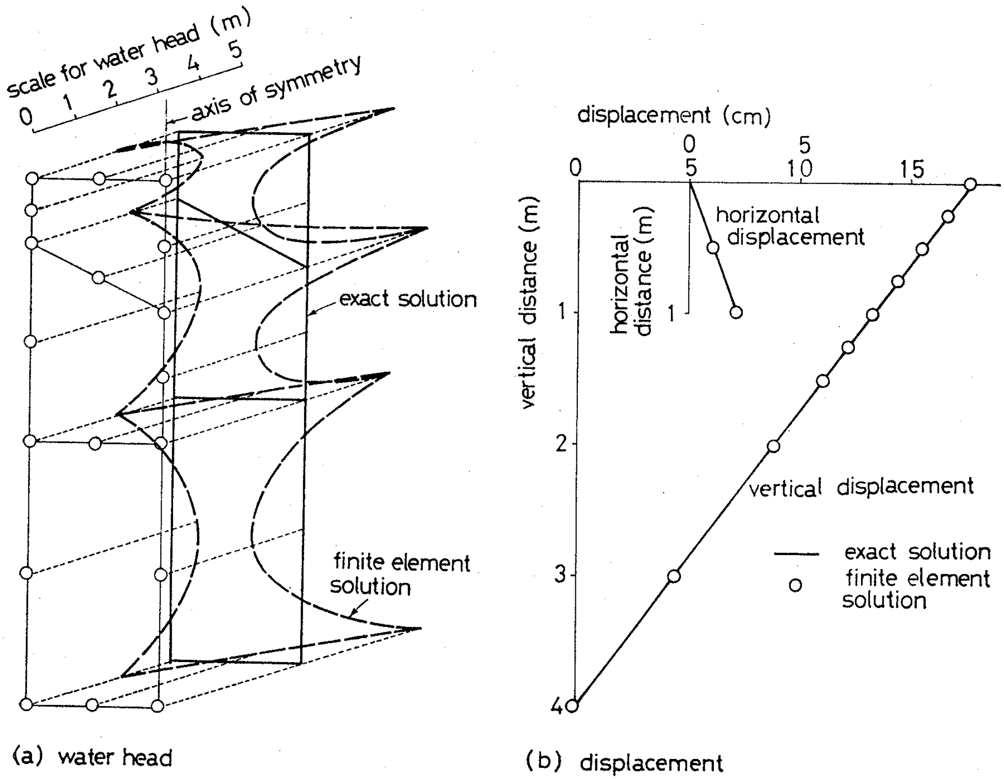

Fig. 5. Solution in continuous water head field

$$\{\tilde{d}^m(x)\} = \{N\} \tag{89}$$

The calculated distributions of water head, vertical and horizontal displacement of soil by shape functions $[a^m(x)]$ and $\{\tilde{d}^m(x)\}$ are shown in Fig. 5. The distribution of water head is far from good approximation, though the distributions of vertical and horizontal displacement of soil are good approximation.

2. *Example b.*

The shape function for water head such that the water head may be discontinuous across interelement boundary is employed. The shape function for displacement of soil employed here is the same as in *Example a.* Column vectors $\{\omega^m(t)\}$ and $\{d^m(x)\}$ are

$$\{\omega^m(t)\}^T = [a_1, a_2, a_3, a_4, a_5, a_6] \tag{90}$$

$$\{d^m(x)\}^T = [1, r, z, r^2, rz, z^2] \tag{91}$$

The calculated distributions of water head, vertical and horizontal displacement of soil by shape functions $[a^m(x)]$ and $\{d^m(x)\}$ are shown in Fig. 6. The distribution of water head as well as those of vertical and horizontal displacement of soil is good approximation.

Hence, the finite element method developed by Sandhu and Wilson (1969) and Yokoo, Yamagata, and Nagaoka (1971) is only applicable to problems of consolidation under continuous load with respect to time and inapplicable to problems of consolidation

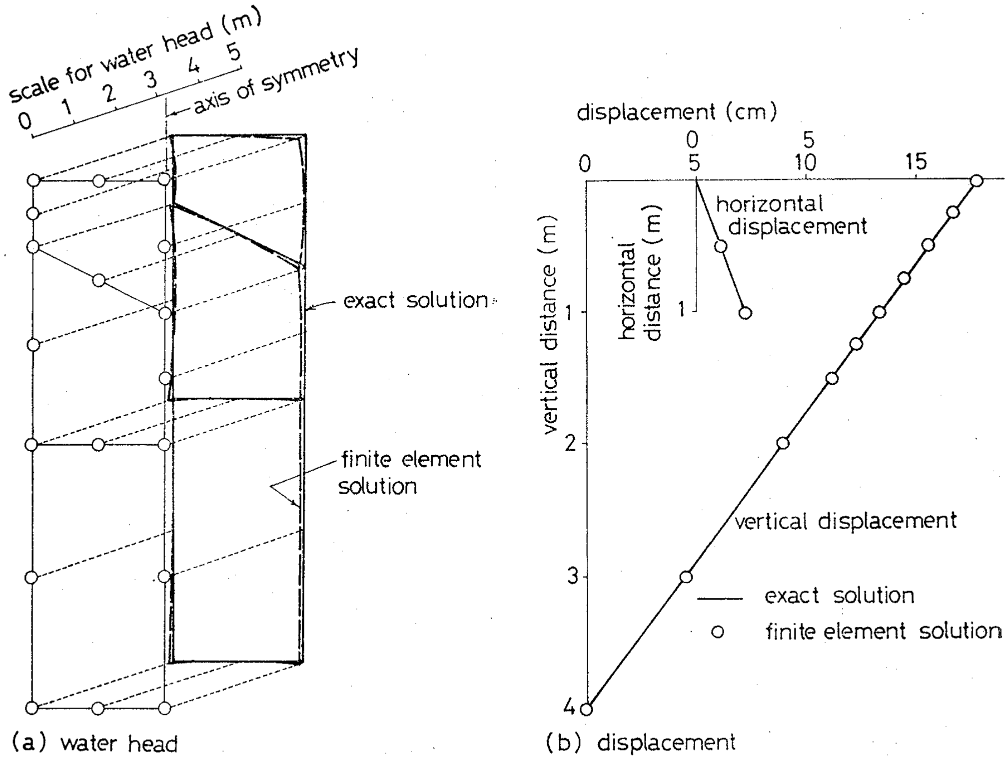

Fig. 6. Solution in piecewise continuous water head field

under piecewise continuous load with respect to time, i.e., problems of consolidation followed by undrained deformation. The finite element method developed in this paper is effective for either continuous load with respect to time or piecewise continuous load with respect to time.

Sandhu and Wilson (1969) analyzed consolidation following undrained deformation by the finite element method employing shape function for water head $\{\tilde{d}^m(x)\}$ such that water head is assured to be continuous across interelement boundary. The numerical result for consolidation a little after undrained deformation was shown but the result for undrained deformation was not.

NUMERICAL EXAMPLE

In order to examine the accuracy of finite element solution by Eqs. (59), (60), and (61), simple one-dimensional consolidation is analyzed. In Fig. 7, the vertical load of 10 ton/m^2 is applied instantaneously on the top surface of a cylinder of saturated homogeneous

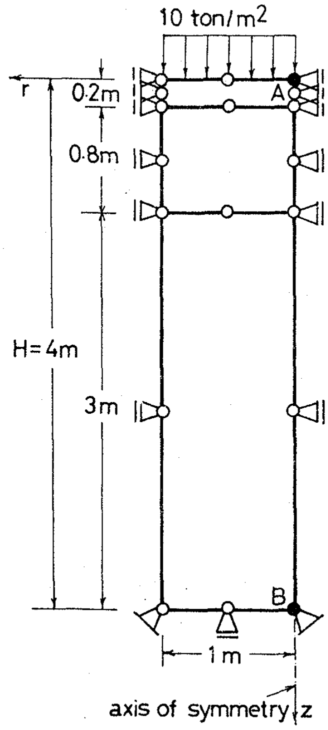

Fig. 7. Finite element idealization of cylinder

isotropic elastic soil of length 4 m and radius 1 m. Vertical displacement of soil on the bottom surface and horizontal displacement on the cylindrical surface are zero. Drainage is allowed only on the top surface. Elasticity coefficients of soil with respect to effective stress and unit weight of water are the same as in the example in the previous section. The permeability coefficient is 1.0×10^{-8} m/sec. The same isoparametric element with eight nodes as in the previous section is employed and the soil cylinder is divided into

three elements. The division is different from that in Fig. 4. Since preliminary analysis based on the division in Fig. 4 indicated poor approximation because of abrupt change of water head near ground surface, the division is modified so as to give good approximation to abrupt change of water head near ground surface.
Shape functions for displacement of soil and water head are given by Eqs. (88) and (91). Column vector $\{\psi^q(t)\}$ employed here consists of one component $b^q(t)$ and shape functions for normal component of water velocity relative to soil on B_I^{mn} and B_h^{mp}, $\{c^{mn}(x)\}$, $\{c^{m'n'}(x)\}$, and $\{c^{mp}(x)\}$, are equal to $+1$ or -1.

$$\{\psi^{mn}(t)\} = \{\psi^{m'n'}(t)\} = b^{mn}(t) \tag{92}$$

$$\{\psi^{mp}(t)\} = b^{mp}(t) \tag{93}$$

$$\{c^{mn}(x)\} = -\{c^{m'n'}(x)\} = 1 \tag{94}$$

$$\{c^{mp}(x)\} = 1 \tag{95}$$

i.e.,

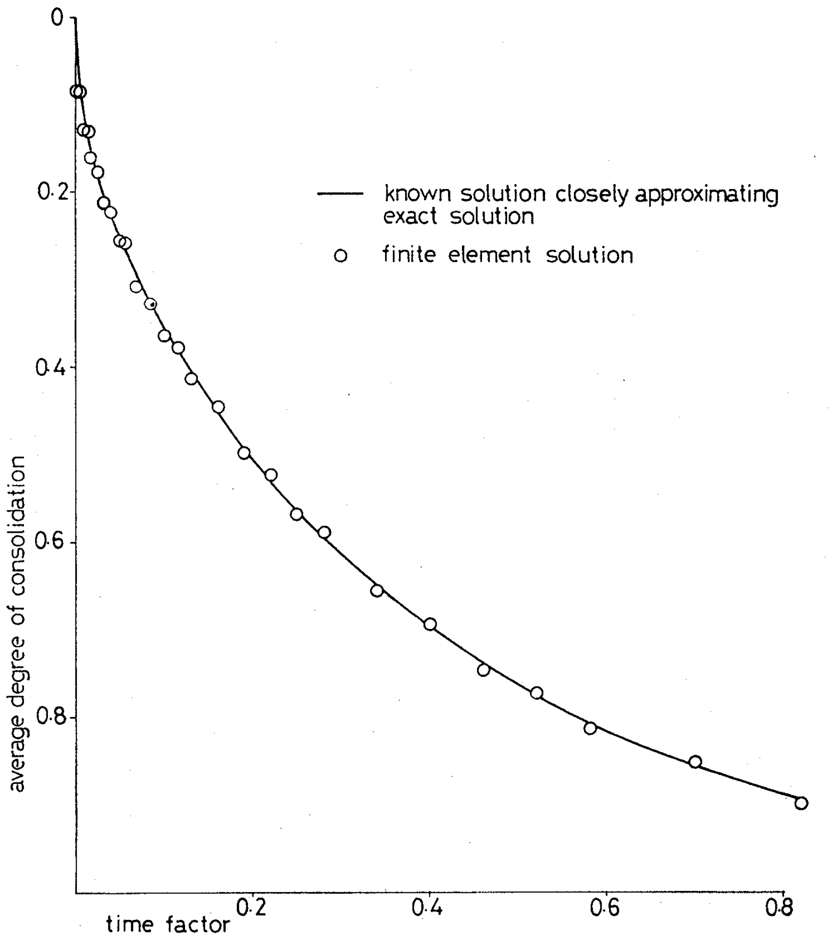

Fig. 8. Average degree of consolidation

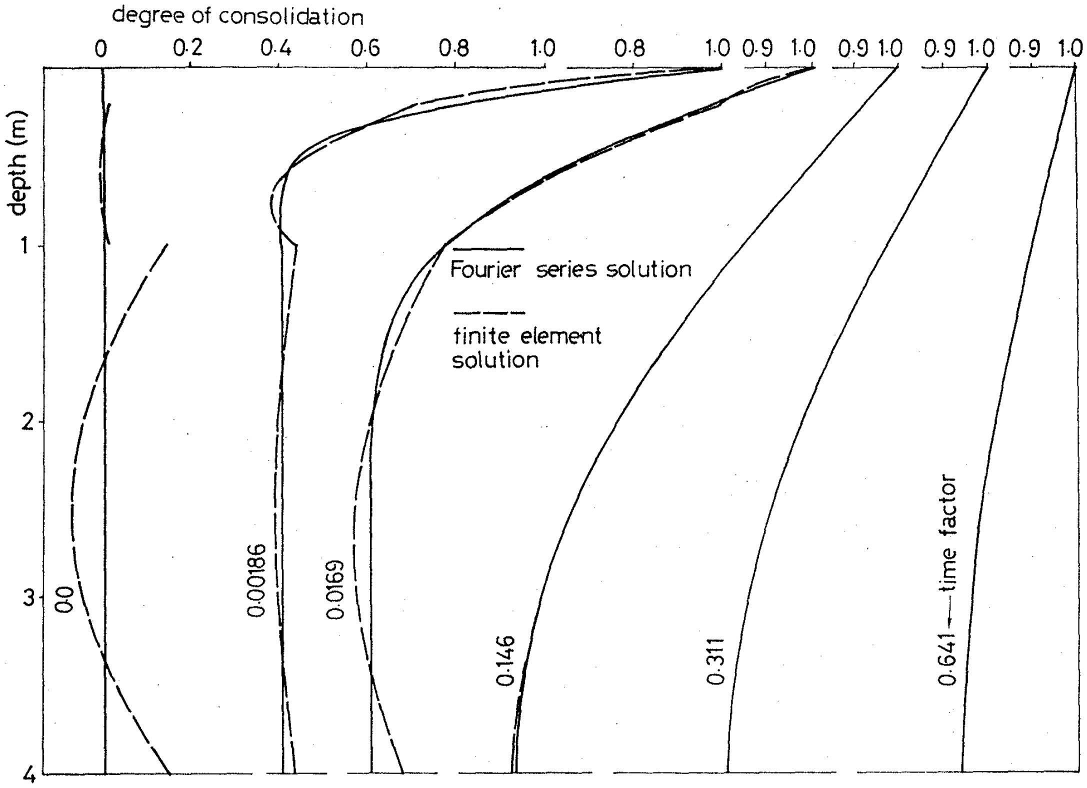

Fig. 9. Degree of consolidation

$$v^{(m)} = -v^{(m')} = b^{mn}(t) \qquad \text{on } B_I^{mn} \tag{96}$$

$$v = b^{mp}(t) \qquad \text{on } B_h^{mp} \tag{97}$$

Equations (59), (60), and (61) are solved by the step-by-step solution technique, i.e., by Eqs. (69), (72), and (73).

The average degree of consolidation is shown in Fig. 8. Dots show the numerical solution obtained by the vertical settlement at point A in Fig. 7 and the solid line shows the following value which closely approximates the exact value (e.g., Scott, 1963).

$$\sqrt{\frac{4T}{\pi}} \qquad (T \leqq 0.2)$$

$$1 - \frac{8}{\pi^2}\exp\left(-\frac{1}{4}\pi^2 T\right) \qquad (T \geqq 0.2)$$

where T is the time factor.

The degree of consolidation is shown in Fig. 9. The solid lines indicate the partial sum of five hundred terms in the exact value (e.g., Scott, 1963)

$$1 - \sum_{m=0}^{\infty} \frac{2}{M} \sin\left(\frac{Mz}{H}\right) \exp\left(-M^2 T\right)$$

$$M = \frac{1}{2}\pi(2m + 1)$$

where H is the length of the cylinder. The broken lines indicate the numerical solution obtained from the water head distribution along AB (axis of symmetry) in Fig. 7. The water head distribution along the other vertical line is almost the same as that along AB. The inaccuracy in the lowest element at time factor zero seems due to fairly rough division of the cylinder into finite elements. For $T > 0.146$, the solid and broken lines coincide.

The numerical solution by Eqs. (69), (72), and (73) agrees fairly well with the exact solution.

CONCLUSIONS

The numerical method developed in this paper is effective to analyze consolidation following undrained deformation, i.e., soil deformation under piecewise continuous load with respect to time.

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NOTATION

B = boundary of region occupied by soil
B_h = boundary on which water head is prescribed
B_T = boundary on which surface traction is prescribed
B_u = boundary on which displacement of soil is prescribed
B_v = boundary on which normal component of water velocity relative to soil is prescribed
B^m = boundary of the m-th element
$B_h{}^m$ = portion of B^m belonging to B_h
$B_I{}^m$ = interelement boundary of the m-th element
$B_T{}^m$ = portion of B^m belonging to B_T
$B_u{}^m$ = portion of B^m belonging to B_u
$B_v{}^m$ = portion of B^m belonging to B_v
$B_I{}^{mn}$ = the n-th subset of $B_I{}^m$
$B_h{}^{mp}$ = the p-th subset of $B_h{}^m$
E_{ij} = the reduced elasticity tensor with respect to effective stress
E_{ijkl} = elasticity tensor with respect to effective stress
R = region occupied by soil
R^m = region of the m-th element

$\bar{T}_i$ = prescribed surface traction
$[a^m(x)]$ = shape function for displacement of the m-th element
$\{c^{mn}(x)\}$ = shape function for normal component of water velocity relative to soil on B_I^{mn}
$\{c^{mp}(x)\}$ = shape function for normal component of water velocity relative to soil on B_h^{mp}
$\{d^m(x)\}$ = shape function for water head of the m-th element in piecewise continuous water head field
$\{\tilde{d}^m(x)\}$ = shape function for water head of the m-th element in continuous water head field
h = water head
$\bar{h}$ = prescribed water head
k_{ij} = permeability coefficient tensor
n_i = outward unit normal vector on boundary of an element
t = time
u_i = displacement of soil
$\bar{u}_i$ = prescribed displacement of soil
v = normal component of water velocity relative to soil on B_I^m and B_h^m
$\bar{v}$ = prescribed normal component of water velocity relative to soil
$\boldsymbol{x}$ = spatial coordinate vector
$[\Phi^m]$ = transformation matrix for displacement of the m-th element
$[\Psi^q]$ = transformation matrix for normal component of water velocity relative to soil of the q-th subset of $\sum_m (B_I^m + B_h^m)$
γ = unit weight of water
δ_{ij} = Kronecker's delta
ε_i = reduced strain tensor of soil
$\{\phi(t)\}$ = set of components of displacement vectors of soil at all nodal points belonging to the entire region
$\{\phi^m(t)\}$ = set of components of displacement vectors of soil at all nodal points belonging to the m-th element
$\{\psi(t)\}$ = set of parameters which determine normal component of water velocity relative to soil on $\sum_m (B_I^m + B_h^m)$
$\{\psi^q(t)\}$ = set of parameters which determine normal component of water velocity relative to soil on the q-th subset of $\sum_m (B_I^m + B_h^m)$
$\{\omega^m(t)\}$ = set of parameters which determine water head of the m-th element in piecewise continuous water head field
$\{\tilde{\omega}^m(t)\}$ = set of parameters which determine water head of the m-th element in continuous water head field

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