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Topology Change in Quantum Gravity[1]

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abstract

It is known that the universe cannot change its topology in classical theory of general relativity. In this talk we show explicit examples of topology changing processes by quantum tunneling in the (2+1)-dimensional Einstein gravity model with negative cosmological constant in the WKB approximation. It turns out that the wave function of the universe exhibits a localization property in the moduli space of the two dimensional universe.

1.Introduction

Probably one of the most fascinating (devastating?) phenomena in Nature will be topology change of the Universe, if it really occurs. It is known,however, that topology cannot change in classical gravity unless one allows singularities or closed time-like curves, as proven by Geroch [1]. However, topology-changing processes may be allowed in quantum mechanics of geometry, e.g. by quantum tunneling. In the path-integral approach to quantum gravity, tunneling phenomenon can be semi-classically described by a transition between Lorentzian (real time) and Euclidean(imaginary time)signature regions of space-time and the tunneling transition amplitudes can be evaluated by looking for solutions of the Einstein equation in Euclidean signature region with appropriate boundary conditions. Unfortunately it is not easy to find such Euclidean 4-geometry which represents topology-changing processes in (3+1)-dimensional quantum gravity.

We are going down to the (2+1)-dimensional Einstein gravity to investigate possible topology changes by quantum tunneling in the WKB approximation. This is a sufficiently simple toy model which contains only global degrees of freedom with no Newtonian forces and no gravitational wave modes. The topology of two dimensional closed orientable surfaces are completely classified by genus so that the topology "change" reduces to change of genus and number of connected components of spatial surfaces.

2. Tunneling as a Transition between Euclidean and Lorentzian Signature Regions

In the path integral approach to quantum gravity, the tunneling phenomena may be described by transitions between Euclidean and Lorentzian signature regions in the WKB approximation. Gibbons and Hartle [2] provided a very restrictive constraint on the boundary surface Σ between the two regions(Fig.1). On the boundary hypersurface Σ, where the Wick rotation $t \to -i\tau$ takes place, we demand that the spatial metric h_{ab} be smooth on Σ at $t = \tau = 0$. Or equivalently, we demand the continuity of the extrinsic curvature K_{ab}, which is the time derivative of spatial metric. We obtain

$$K_{Lab}(\Sigma) = K_{Eab}(\Sigma) = -iK_{Lab}, \qquad (1)$$

from which an important consequence follows

$$K_{ab}(\Sigma) = 0. \qquad (2)$$

That is, all the components of the extrinsic curvature must vanish. The surface satisfying the above condition is called a *totally geodesic* hypersurface.

The totally geodesic property of the boundary surface is easily understood in physical terms.

[1]Based on the work in collaboration with Y. Fujiwara, S. Higuchi T. Mishima and M. Siino.

In general, the dynamical motion momentarily stops at the turning point when the system goes in and out of the tunnel. Therefore our problem reduces to finding a Euclidean signature 3-manifold g with totally geodesic boundaries.

The tunneling amplitude is then given by

$$T(\,i,\,f) \,= N\exp(-\bar{S}_E). \tag{3}$$

with N being a prefactor in the WKB approximation which is in principle calculable and $\bar{S}_E$ is the classical Euclidean action.

3. Euclidean Solutions

Let us consider a simple model of the Einstein gravity with a negative cosmological constant given by the action:

$$S_E = -\frac{1}{16\pi G}\int(R-2\Lambda)\sqrt{g}\; d^3x. \tag{4}$$

From the variation of the action (4) we obtain the (2+1)-dimensional Einstein equation:

$$R_{\alpha\beta} = 2\Lambda g_{\alpha\beta}. \tag{5}$$

We can easily see that the sectional curvature is constant by looking at the identity to the Riemann tensor:

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} \\ -g_{\nu\rho}R_{\mu\sigma} + \tfrac{R}{2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \tag{6}$$

which holds only for the three dimensional space-time. From the Einstein equation the right hand side can be expressed by the metric tensor. We obtain a constant negative curvature space-time:

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{7}$$

with negative sectional curvature Λ.

4. A Double-Torus Universe from Nothing [3]

Our problem is to find a compact, orientable 3-manifold which satisfies the Einstein equation 5 with a totally geodesic boundary surfaces Σ. In three dimensions, as already shown in §3, an Einstein space with negative cosmological constant is a Riemannian manifold with negative constant sectional curvature and therefore a hyperbolic manifold. The three dimensional hyperbolic space can be canonically realized by the Klein model which is equipped with the metric:

$$ds^2 = \frac{(dx^i)^2}{(1-x^2)} + \frac{(x^i dx^i)^2}{(1-x^2)^2}, \quad (i=1,2,3), \tag{8}$$

defined inside the sphere at infinity S_∞,

$$x^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 < 1. \tag{9}$$

In this model the geodesic curve is a Euclidean straight line. The metric (8) is a solution of the Euclidean signature version of the Einstein equation (5) in the unit $\Lambda = -1$. It follows that totally geodesic surfaces in the Kleinian model are Euclidean planes.

As an example, we would like to construct a hyperbolic 3-manifold which has a double-torus, a closed surface with the genus = 2, as the totally geodesic boundary [4] We construct this by appropriately gluing two regular truncated tetrahedra together which are embedded in the Kleinian model D^3. We embed a regular tetrahedron and the Kleinian model D^3 into R^3 so that both of them would center around the origin. We arrange the tetrahedron so that the angle between each pair of the faces of the tetrahedron to $\pi/6$ and truncate each vertex of the tetrahedron by the common perpendiculars to the edges (Fig.2). We prepare two such regular truncated tetrahedra . Then we identify each pair of the faces so as to match the arrows indicated in Fig.3 and identify all the edges. This gives a topological 3-manifold with a boundary. We can easily check the gluing consistency to see that this 3-manifold actually admits a hyperbolic structure with a totally geodesic boundary. We can show that the boundary is topologically a double-torus by doing a 'patch work' as illustrated in Fig.4 which follows from the identification rule.

By using similar tactics we can construct hyperbolic 3-manifolds which can be interpreted as various topology changing processes[5]; a splitting of a double-torus universe, a transition of a double torus universe to a triple-torus universe etc.(Fig.5)

5. Tunneling Amplitude

In this section, we will explicitly calculate the tunneling amplitude in the WKB approximation. We assume that the amplitude of topology change can be described by the Hartle-Hawking path-integral [6]:

$$\Psi(h) = \sum_{M_R} \int_{g=h on \partial M_R} \mathcal{D}g \ \exp(-S_E[g]) \quad (10)$$

where h is the two dimensional metric on a spacelike hypersurface Σ and S_E is the Euclidean action which has been explicitly given in (10) below. The path integral is over smooth 3-metric g on the Riemannian space-time manifold M_R with a boundary ∂M_R, and the summation over M_R means that we should also sum over different topologies of space-time M_R. The Euclidean action is given in (4) in our present model. Due to (5), the classical action $\bar{S}_E$ is simply proportional to the *volume* of M_R:

$$\bar{S}_E = \frac{1}{4\pi G} \frac{V}{\sqrt{|\Lambda|}}, \quad (11)$$

where V is a numerical value representing the volume of M_R in the case of $\Lambda = -1$.

Therefore the amplitude reduces to

$$T(\ i,\ f) \ = N \exp(-\frac{1}{4\pi G} \frac{V}{\sqrt{|\Lambda|}}). \quad (12)$$

with N being a prefactor in the WKB approximation which is in principle calculable.

The tunneling amplitudes are suppressed for large 3-volumes V. This implies that contributions from complicated 3-manifolds are exponentially small.

6. Localizaion of Wave-function of the Universe

One might conclude from the previous section that the 3-manifold of the smallest volume would dominate the path-integral for the wave-function of the universe. However, the life is not so simple in the three dimensional hyperbolic spacetime. The moduli of the totally geodesic boundary surface, which give a shape of the two dimensional universe in the present model, cannot be continuously deformed. They are rigid. Their variety is countably infinite and dense in the moduli space the surface[7]. Somewhat surprising thing is that for some moduli the volume distribution has an infinite number of accumulation points,v_ω,v_{ω^2}...,each of which correspond to a single cusp, two cusps... In other words, there are infinite number of distinct hyperbolic manifolds which have almost the same volume for some moduli of the totally geodesic boundary. So the wave-function of the universe becomes divergent for such values of moduli.This is a kind of localization. A localization of the wave-function of the universe in the moduli space. This aspect of the three dimensional gravity has recently been stressed by S.Carlip [7].

7. Conclusion

We have considered the topology change of universes by quantum tunneling in (2+1)-dimensional Einstein gravity with negative c osmological constant in the WKB approximation. It is also found that the wave-function of the universe exhibits a kind of localization in the moduli space of the spatial hypersurface. This may be physically interpreted in the following way. Only a countable set of varieties of universe can emerge from nothing by quantum tunneling. This statement is a strong restriction to the initial value of classical Einstein equation.

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FIGURE CAPTIONS

Figure.1 Σ is the spacelike surface sandwiched by the Euclidean(M_R) and Lorentzian(M_L) signature regions of space-time manifolds.

Figure.2 A regular truncated tetrahedron with the dihedral angle $\pi/6$ in the Kleinian model of hyperbolic space.

Figure.3 Gluing two identical tetrahedra. Each face (e.g.A) of the one of the tetrahedra is identified with its corresponding face ($A\prime$) of the other one so that the three arrows match.

Figure.4 Patchwork of the boundary pieces. The result is a double torus.

Figure.5 Various topology changing processes

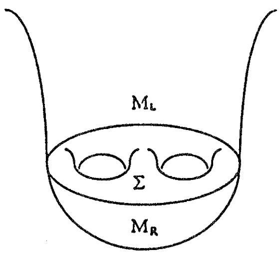

Fig.1

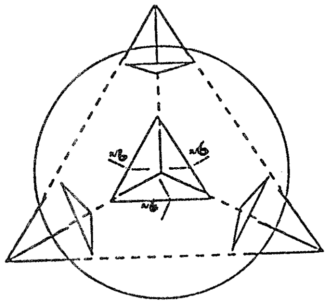

Fig.2

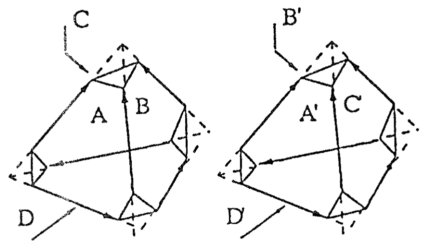

Fig.3

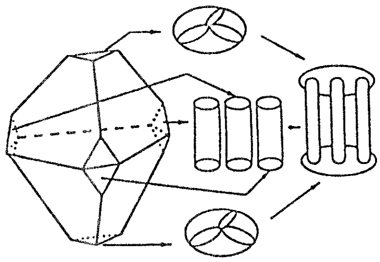

Fig.4

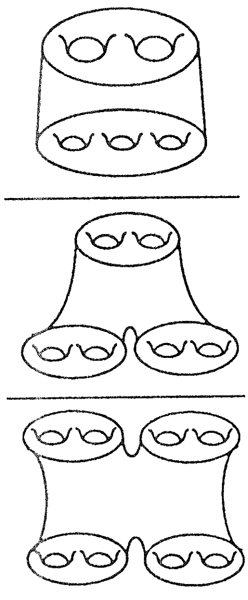

Fig.5