Illuminance calculation for an arbitrarily shaped flat surface source

— Modification of the contour integration method —
(Follow-up paper)

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My preceding paper disclosed a simple method for calculating the illuminance of a flat surface source of arbitrary shape which could be expressed by x and y using the following new formulae:

(1) When the surface source is parallel to the illuminated plane:

$$E'=\frac{L}{2}\int_a^b \frac{(\text{intercept on } y \text{ axis})}{x^2+y^2+z^2}dx$$

(2) When the surface source is inclined to the illuminated plane by $\angle\beta$:

$$(E')=\frac{L}{2}\cos\beta\int_a^b \frac{(\text{intercept on } y \text{ axis})}{x^2+y^2+2yz\sin\beta+z^2}dx$$

(3) When the surface source is perpendicular to the illuminated plane:

$$《E'》=-\frac{L}{2}z\int_a^b \frac{1}{x^2+y^2+z^2}dx$$

where E'=the illuminance component for the interval A to B on the boundary of the flat surface source, L=luminance of the source, z=the distance from the illuminated point to the origin located just above the illuminated point for cases (1) and (2), and the distance from the origin to the illuminated point located on the normal to the origin for case (3).

This note provides some calculation examples to find the illuminance of flat surface sources of various shapes by means of this new method, and discusses the case having the primitive function $F(x)$.

1. Preface

My previous paper[1)] demonstrated a simple method for calculating the illuminance of any type of flat surface source by modifying the contour integration method expressed by the product of $d\omega$ and $\cos\delta$ into a new formula expressed by x, y and z. The present note describes some concrete examples of calculating the illuminance of a flat surface of any shape and at any position, using the new calculating methode.

2. General description of the new calculating method

Figure 1 shows the cases in which a uniformly diffused flat surface suorce is:

(1) parallel to the illuminated plane,

(2) inclined to the illuminated plane, and

(3) perpendicular to the illuminated plane,

and the axes x and y are on the plane including a flat surface source S, the origin 0 is positioned just above the illuminated point P in cases (1) and (2), and at the foot of the perpendicular from the illuminated point P in case (3), and the axes x and y are defined as being positive in the direction of the arrows.

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A part of this noto was described in J. Illum. Engng. Inst. Jpn. 67, 3 (1983).

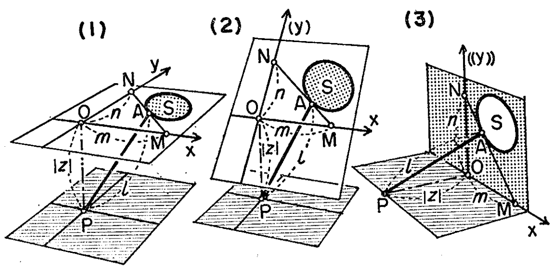

Fig. 1 *Three positions of the co-ordinate plane including a source.*

If the distance between the illuminated point P and the origin O is taken as $|z|$, the length of the intercept made by the tangent at point A on the boundary intersecting the y axis is taken as n, and the distance between point A and the illuminated point P is taken as l, the illuminance component of the minute segment AB on the boundary is as follows:

For case (1):

$$E'=\frac{L}{2}\int_a^b \frac{n}{l^2}dx \quad \cdots\cdots(1)$$

For case (2):

$$(E')=\frac{L}{2}\cos\beta\int_a^b \frac{n}{l^2}dx \quad \cdots\cdots(2)$$

For case (3):

$$《E'》=-\frac{L}{2}z\int_a^b \frac{1}{l^2}dx \quad \cdots\cdots(3)$$

The boundary itself has no illuminance, and each of the values obtained from formulae (1) to (3) is no more than a component in the calculation process, and thus, they are defined as illuminance components and given a prime symbol, and the distinction among cases (1) to (3) is made by adding parentheses and double parentheses.

The minus sign in formula (3) is to compensate for the fact that the luminous surface illuminating the illuminated point P in case (3) faces the direction opposite to those in cases (1) and (2), as shown in Fig. 1.

The length of $\overline{AP}=l$ is:

$$\left.\begin{array}{ll}\text{For cases (1) and (3):} & l=\sqrt{x^2+y^2+z^2}\cdots \\ \text{For case (2):} & l=\sqrt{x^2+y^2+2yz\sin\beta+z^2}\cdots\end{array}\right\} \quad \cdots\cdots(4)$$

If the rectilinear part of the boundary is parallel to the y axis, the length m of the intercept on the x axis is used, and the following formulae are applied using the length m of the intercept on the x axis and substituting the definite integral relative to y according to $dx=-\frac{m}{n}dy$,

For case (1):

$$E'=\frac{L}{2}\int_{b'}^{a'}\frac{m}{l^2}dy \quad \cdots\cdots(5)$$

For case (2):

$$(E')=\frac{L}{2}\cos\beta\int_{b'}^{a'}\frac{m}{l^2}dy \quad \cdots\cdots(6)$$

For case (3):

$$《E'》=-\frac{L}{2}z\int_{b'}^{a'}\frac{m}{n}\cdot\frac{1}{l^2}dy \quad \cdots\cdots(7)$$

3. Routine procedure for calculation

The procedure for obtaining the illuminance by means of the definite integral relative to x is as follows:

(1) Obtain equation $y=f(x)$ and its derivative y'.

(2) Obtain the length of the intercept n on the y axis from the equation $n=y-y'\cdot x$.

(3) Obtain l^2 from formula (4).

(4) Substitute the above values into formulae (1) to (3).

(5) Set the lower limit of the interval of the definite integral at the point where x is smaller.

(6) Obtain the values of the illuminance components using a computer.

(7) The arithmetic sum of the illuminance components belonging to the lower half part of the boundary may be deducted from the arithmetic sum of the illuminance components belonging to the upper half part.

For the definite integral relative to y, equation $x=f(y)$, its derivative x' and the intercept m $(=x-x'\cdot y)$ on x axis may be obtained and substituted into formulae (5) to (7), with the lower limit of the interval set at the point where y is smaller.

The definite integrals produced by the above-mentioned procedures, can be classified into two types, (A) where the primitive function $F(x)$ of $f(x)$ can be obtained and (B) where such a primitive function $F(x)$ cannot be obtained. The method of solution is the same for both types.

4. Examples for calculating the illuminance of flat surface sources of various shapes

In each example calculation given below, the unit of the co-ordinate values is taken as meters, with $|z|=6$(m); the source luminance $L=1{,}000$(nt); and for the inclination angle β in a case where the surface source is inclined to the illuminated plane, $\angle\beta=30°$.

4.1 In the case of a polygonal surface source

For the polygonal source shown in **Fig. 2**, calculations are described for in which each side of the boundary is parallel to, inclined to, or perpendicular to the co-ordinate axis.

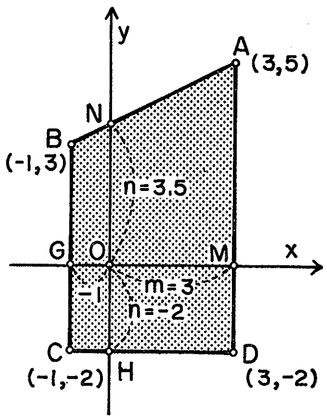

Fig. 2 *Co-ordinates for a polygonal source.*

4.1.1 The case in which the source is parallel to the illuminated plane

Substituting $y=0.5x+3.5$ and $n=3.5$ into formula (1) gives the illuminance component of the oblique side AB:

$$E'_{AB}=\frac{1000}{2}\int_{-1}^{3}\frac{3.5}{x^2+(0.5x+3.5)^2+6^2}dx$$
$$=130.11$$

Substituting $x=3$ and $m=3$ into formula (5) gives the illuminance component of the perpendicular side AD:

$$E'_{AD}=\frac{1000}{2}\int_{-2}^{5}\frac{3}{3^2+y^2+6^2}dy=208.02$$

For the perpendicular side BC, substituting $x=-1$ and $m=-1$ into formula (5) and setting the interval of the definite integral at from -2 to 3, gives:

$$E'_{BC}=-63.77$$

For the horizontal side CD, substituting $y=-2$ and $n=-2$ into formula (1) and setting the interval of the definite integral at from -1 to 3, gives:

$E'_{CD}=-94.83$

$\therefore\ E=E'_{AB}+E'_{AD}-E'_{CD}-E'_{BC}=496.73(\mathrm{lx})$

Since the above-mentioned definite integrals belongs to group (A), the primitive function of the definite integral for obtaining the illuminance component, for example, of the side of HD (a right half of side CD), is:

$$E'_{HD}=\frac{L}{2}\int_{o}^{m}\frac{n}{x^2+(n^2+z^2)}dx$$
$$=\frac{L}{2}\left[\frac{n}{\sqrt{n^2+z^2}}\tan^{-1}\frac{x}{\sqrt{n^2+z^2}}\right]_{o}^{m}$$
$$=\frac{L}{2}\left(\frac{n}{\sqrt{n^2+z^2}}\tan^{-1}\frac{m}{\sqrt{n^2+z^2}}\right)\quad\cdots\cdots\cdots\cdots(8)$$

Thus, the known formula for a rectangular source can be obtained.

4.1.2 The case in which the source is inclined to the illuminated plane by ∠30°

For an inclined source, the calculation should be made according to the above-mentioned procedure by the definite integration with $2yz\sin\beta$ added to case (2) of formula (4) and with $\cos\beta$ added to formulae (2) and (6):

$(E'_{AB})=78.41,\ (E'_{AD})=165.01,$

$(E'_{CD})=-114.98,\ (E'_{BC})=-54.11$

$\therefore\ (E)=(E'_{AB})+(E'_{AD})-(E'_{CD})-(E'_{BC})$

$=412.51(\mathrm{lx})$

These definite integrals will belong to group (A), but the formula derived from a primitive function turns out to be very intricate as shown previously with examples[2] for calculating the illuminance of an inclined right-angled triangular source.

4.1.3 The case in which the source is perpendicular to the illuminated plane

The rectangle MDCG of the part submerged beneath the illuminated plane contributes nothing to the illuminance in the upward direction from the illuminated plane.

If sides AM and BG are parallel to the y axis, the length of the intercept on the y axis is considered to be $n=\infty$, then substituting $n=\infty$ into formula (7) gives:

$$(\!(E'_{AM})\!)=(\!(E'_{BG})\!)=-\frac{L}{2}z\int_{b'}^{a'}\frac{m}{\infty}\cdot\frac{1}{l^2}dy=0\cdots\cdots\cdots(9)$$

Consequently, calculation of the illuminance with the illuminance components of the remaining two sides only is as follows:

$(\!(E'_{AB})\!)=-223.05,\ (\!(E'_{MG})\!)=-314.40$

$\therefore\ (\!(E)\!)=(\!(E'_{AB})\!)-(\!(E'_{MG})\!)=91.35(\mathrm{lx})$

The above-mentioned definite integral belongs to group (A), and the known formula for the sides parallel to the x axis is given by substituting z for n in formula (8).

For the side AB inclined to the x axis, the following formula can be derived, taking the tangent to the inclination angle as t:

$$(\!(E'_{AB})\!)=-\frac{L}{2}z\int_{-b}^{a}\frac{1}{x^2+(tx+n)^2+z^2}dx$$
$$=-\frac{Lz}{2\sqrt{n^2+z^2+t^2z^2}}\left[\tan^{-1}\frac{tn+(1+t^2)x}{\sqrt{n^2+z^2+t^2z^2}}\right]_{-b}^{a}$$
$$=-\frac{Lz}{2\sqrt{n^2+z^2+t^2z^2}}\left(\tan^{-1}\frac{tn+(1+t^2)a}{\sqrt{n^2+z^2+t^2z^2}}-\tan^{-1}\frac{tn-(1+t^2)b}{\sqrt{n^2+z^2+t^2z^2}}\right)\quad\cdots\cdots\cdots\cdots(10)$$

4.2 In the case of a circular surface source

As shown in **Fig. 3**, if the center Q of a circular surface source with radius r is put in an arbitrary position (c, q):

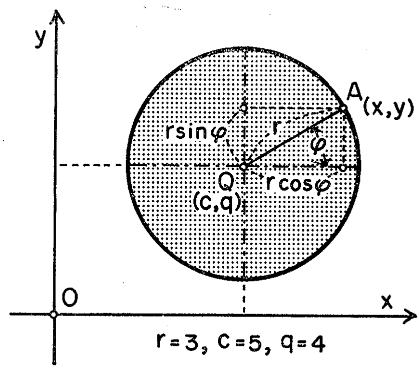

Fig. 3 *Co-ordinates for a circular source in an arbitrary position.*

$$x=c+r\cos\varphi,\quad y=q+r\sin\varphi,\quad \frac{dx}{d\varphi}=-r\sin\varphi,$$
$$n=y-y'\cdot x=\frac{q\sin\varphi+r+c\cos\varphi}{\sin\varphi}\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

4.2.1 The case in which the source is parallel to the illuminated plane

The illuminance component of the upper semicircle may be obtained by substituting expression (11) into formula (1) as follows:

$$E'_{\cap}=\frac{L}{2}\int_{c-r}^{c+r}\frac{n}{l^2}dx$$
$$=-\frac{L}{2}r\int_{\pi}^{o}\frac{q\sin\varphi+r+c\cos\varphi}{(c+r\cos\varphi)^2+(q+r\sin\varphi)^2+z^2}d\varphi\quad\cdots\cdots\cdots\cdots(12)$$

The interval of the definite integral for obtaining $E'_{\cup}$ of the lower semicircle is between π to 2π, but instead of using $E'_{\cap}-E'_{\cup}$ to obtain E_o, if the latter interval which is reversed can be added and made continuous from 2π to 0, E_o (=182.55 lx) in which the illuminance component of the total circle can be obtained by a single process.

The previous paper[1] demonstrated that the definite integral for a circular surface source in any case belongs to group (A), and if formula (12) is modified so as to give $U=c^2+q^2+r^2+z^2$, $V=2qr$ and $W=2cr$,

$$E_O=\pi\frac{L}{2}\left(1+\frac{2r^2-U}{\sqrt{U^2-V^2-W^2}}\right)\quad\cdots\cdots(13)$$

Particularly, if the center of a circular surface source is on the origin 0, the known formula shown below can be obtained by substituting $U=r^2+z^2$ and $V=W=0$ into formula (13):

$$E_O=\pi L\frac{r^2}{r^2+z^2}\quad\cdots\cdots(14)$$

4.2.2 The case in which the source is inclined to the illuminated plane

The calculation shall be made according to the above-mentioned formula to which $\cos\pi/6$ and $2(4+3\sin\varphi)\times 6\sin\pi/6$ are added to the equation for a parallel source:

$(E_O)=123.08(\mathrm{lx})$

The formula based on the primitive function $F(x)$ is, with $U=c^2+q^2+r^2+z^2+2qz\sin\beta$, $V=2r(q+z\sin\beta)$ and $W=2cr$:

$$(E_O)=\pi Lr\left\{\frac{r}{\sqrt{U^2-V^2-W^2}}+\frac{qV+cW}{V^2+W^2}\left(1-\frac{U}{\sqrt{U^2-V^2-W^2}}\right)\right\}\cos\beta\cdots(15)$$

As shown in **Fig. 4**, if the illuminated point is on the line normal to the center Q of a circular surface source inclined to the illuminated plane by $\angle\beta$, then $c=0$ and $q=-z\sin\beta$, hence $V=0$ and $W=0$, and thus the denominator in formula (15) becomes zero, thereby making it impossible to perform any operation.

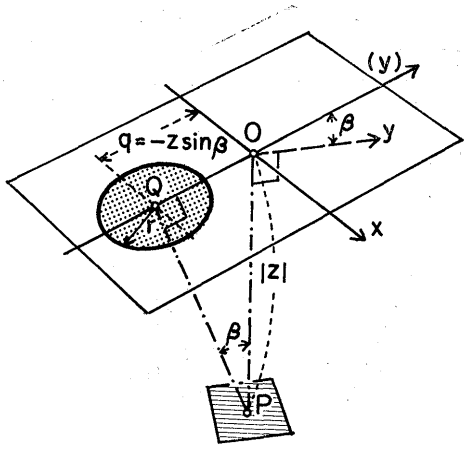

Fig. 4 *The case in which $q=-z\sin\beta$.*

Consequently, $F(x)$ can be obtained once again after substituting $c=0$, $q=-z\sin\beta$ into the original formula (12):

$$(E_O)=\frac{L}{2}r\left(\frac{z\sin\beta\cos\beta}{r^2+z^2\cos^2\beta}\Big[-\cos\varphi\Big]_{2\pi}^{0}-\frac{r\cos\beta}{r^2+z^2\cos^2\beta}\Big[\varphi\Big]_{2\pi}^{0}\right)=\pi L\frac{r^2\cos\beta}{r^2+z^2\cos^2\beta}\cdots(16)$$

4.2.3 The case in which the source is perpendicular to the illuminated plane

If formula (3) is substituted by the integral of φ according to equation (11), we have:

$$《E_O》=-\frac{L}{2}z\int_{2\pi}^{0}\frac{-r\sin\varphi}{(c+r\cos\varphi)^2+(q+r\sin\varphi)^2+z^2}d\varphi\quad\cdots\cdots(17)$$

$=108.25(\mathrm{lx})$

The formula based on the primitive function $F(x)$, which is expressed using U, V and W as in case of a parallel source, is:

$$《E》=\pi Lrz\frac{V}{V^2+W^2}\left(\frac{U}{\sqrt{U^2-V^2-W^2}}-1\right)\cdots\cdots(18)$$

4.3 In the case of an elliptical surface source

If the Center Q of an elliptical surface source with a major axis $2a$ and a minor axis $2b$, as shown in **Fig. 5**, is at the arbitrary position (c, q), then:

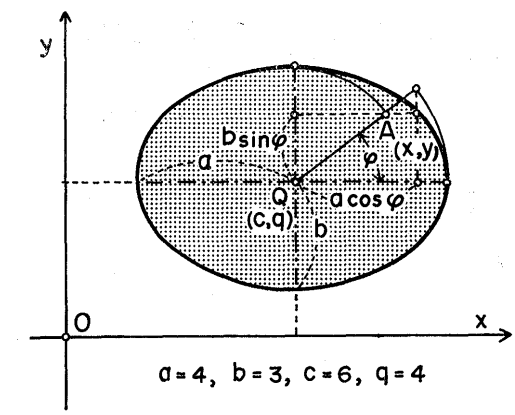

Fig. 5 *Co-ordinates for an elliptical source in an arbitrary position.*

$x=c+a\cos\varphi$. $y=q+b\sin$ and

$$\frac{dx}{d\varphi}=-a\sin\varphi\quad\cdots\cdots(19)$$

$$n=y-y'\cdot x=\frac{aq\sin\varphi+ab+bc\cos\varphi}{a\sin\varphi}\quad\cdots\cdots(20)$$

4.3.1 The case in which the source is parallel to the illuminated plane

If formula (20) is substituted into formula (1) as in the case of the above-mentioned circular surface source, we have:

$$E=-\frac{L}{2}\int_{2\pi}^{0}\frac{aq\sin\varphi+ab+bc\cos\varphi}{(c+a\cos\varphi)^2+(q+b\sin\varphi)^2+z^2}d\varphi\quad\cdots\cdots(21)$$

$=197.57(\mathrm{lx})$

For the formula based on the primitive function $F(x)$, when the center of an elliptical surface source is on the origin 0, if $c=0$ and $q=0$ are substituted into formula (21) to obtain the illuminance component of a quarter of the ellipse located at the first quadrant, then it follows that:

$$E'_1=-\frac{L}{2}\int_{\pi/2}^{0}\frac{ab}{a^2\cos^2\varphi+b^2\sin^2\varphi+z^2}d\varphi$$

$$=-\frac{Lab}{2\sqrt{b^2+z^2}}\left[\sqrt{\frac{b^2+z^2}{a^2+z^2}}\right.$$

$$\left.\tan^{-1}\left(\sqrt{\frac{b^2+z^2}{a^2+z^2}}\tan\varphi\right)\right]_{\pi/2}^{0}$$

$$=\frac{\pi}{4}L\frac{ab}{\sqrt{a^2+z^2}\cdot\sqrt{b^2+z^2}}$$

$$\therefore\quad E_0=4\times E'_1=\pi L\frac{ab}{\sqrt{a^2+z^2}\cdot\sqrt{b^2+z^2}}\cdots\cdots\cdots(22)$$

4.3.2 The case in which the source is inclined to the illuminated plane

Since the terms $\cos-\pi/6$ and $2(4+3\sin\varphi)\times 6\sin-\pi/6$ may be added to the above-mentioned expression (21), we obtain:

$(E)=136.95\,(\mathrm{lx})$

4.3.3 The case in which the source is perpendicular to the illuminated plane

If formula (3) is substituted by the integral of φ according to equation (19), then:

$$《E》=-\frac{L}{2}z\int_{2\pi}^{0}\frac{-a\sin\varphi}{l^2}d\varphi=118.45(\mathrm{lx})$$

4.4 In the case of a finite parabolic surface source

If the co-ordinates for the vertex of the parabolic surface source shown in **Fig. 6** are set at $(c,\ q)$ then the side BC which closes the finite end will be parallel to the x axis, and it follows that:

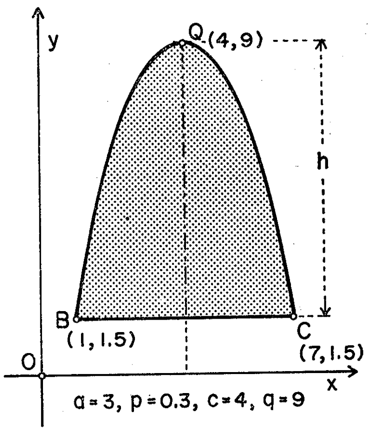

Fig. 6 *Co-ordinates for a parabolic source in an arbitrary position.*

$$y=q-\frac{(x-c)^2}{4p},\quad y'=-\frac{(x-c)}{2p},$$

$$n=y-y'\cdot x=\frac{4pq-c^2+x^2}{4p}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(23)$$

4.4.1 The case in which the source is parallel to the illuminated plane

Substituting equation (23) in to formula (1) and letting $BC=2a$, $a=3$, $p=0.3$, $c=4$ and $q=9$, gives:

$$E'_Q=\frac{L}{2}\int_{C-a}^{c+a}\frac{4pq-c^2+x^2}{4p}$$

$$\times\frac{1}{x^2+\{q-(x-c)^2/4p\}^2+z^2}dx=308.81(\mathrm{lx})$$

As the distance h from the vertex Q to the side BC is $h=a^2/4p=7.5$, the illuminance component of the side BC obtained by formula (1) by taking $n=(9-7.5)=1.5$ is:

$E'_{BC}=83.29,$

$\therefore\quad E=E'_Q-E'_{BC}=225.52\ (\mathrm{lx})$

4.4.2 The case in which the source is inclined to the illuminated plane

Since the terms $\cos\beta$ and $2yz\sin\beta$ are added to the above-mentioned expression, it follows:

$(E'_Q)=207.30,\ (E'_{BC})=61.42$

$\therefore\quad (E)=(E'_Q)-(E'_{BC})=145.88\ (\mathrm{lx})$

4.4.3 The case in which the source is perpendicular to the illuminated plane

By similar calculations,

$《E'_Q》=-162.88,\ 《E'_{BC}》=-333.17$

$\therefore\quad 《E》=《E'_Q》-《E'_{BC}》=170.29\ (\mathrm{lx})$

4.5 In the case of an infinite parabolic surface source

As shown in **Fig. 7**, if the vertex of an infinite parabolic surface source is set at the origin O, the distance between point A $(x,\ y)$ on the parabola and the directrix (shown on the d axis) is equal to the distance between point A and focus F, and $FA=x+p$.

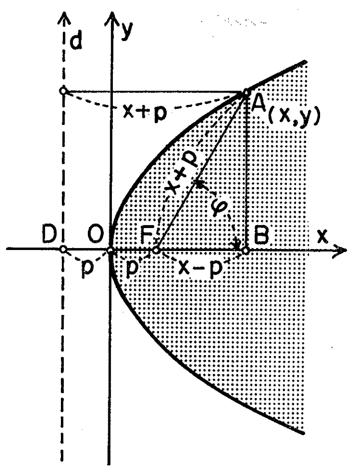

Fig. 7 *Parameters in the case of an infinite parabolic source.*

As the orthogonal projection FB of the vector FA on the x axis is, $FB=x-p$, it follows that:

$$\cos\varphi=\frac{x-p}{x+p},\quad x=p\frac{1+\cos\varphi}{1-\cos\varphi},$$

$$\frac{dx}{d\varphi}=-2p\frac{\sin\varphi}{(1-\cos\varphi)^2}\cdots\cdots(24)$$

$$y=2\sqrt{px}=2\frac{p\sin\varphi}{1-\cos\varphi}$$

$$n=y-y'\cdot x=\frac{p\sin\varphi}{1-\cos\varphi}=\sqrt{px}\cdots\cdots(25)$$

Substituting the above-mentioned expressions into formula (4) gives:

$$l^2=\frac{5p^2-2(z^2-p^2)\cos\varphi+(z^2-3p^2)\cos^2\varphi+z^2}{(1-\cos\varphi)^2}\cdots\cdots(26)$$

Substituting expressions (24) to (26) in formula (1) for a source parallel to the illuminated plane, and letting $p=0.3$ gives the illuminance component of the parabola within the first quadrant:

$$E'_D=\frac{L}{2}\int_0^\infty\frac{\sqrt{px}}{l^2}dx=\frac{L}{2}\int_\pi^0\frac{p\sin\varphi}{(1-\cos\varphi)}\times\frac{1}{\text{exp. (26)}}\times\frac{-2p\sin\varphi}{(1-\cos\varphi)^2}d\varphi$$

$$=-Lp^2\int_\pi^0\frac{1+\cos\varphi}{(5p^2+z^2)-2(z^2-p^2)\cos\varphi+(z^2-3p^2)\cos^2\varphi}d\varphi=236.81\ (\text{lx})$$

Hence, $E=2\times E'_1=473.62$ (lx)

Moreover, the straight lines connecting the illuminated point P to both infinite ends become one on the illuminated plane with the included angle being equal to zero, which thus makes no contribution to the illuminance.

As the definite integral in this case belongs to group (A), it follows that:

$$E'_1=\frac{L}{2}\int_0^\infty\frac{\sqrt{px}}{l^2}dx=\frac{L}{2}\int_0^\infty\frac{\sqrt{px}}{x^2+4px+z^2}dx$$

$$=\frac{L}{2}\left\{\frac{\sqrt{p}\sqrt{2p+A}}{A}\left[\tan^{-1}\sqrt{\frac{x}{2p+A}}\right]_0^\infty-\frac{\sqrt{p}\sqrt{2p-A}}{A}\left[\tan^{-1}\sqrt{\frac{x}{2p-A}}\right]_0^\infty\right\}$$

$$=-\frac{1}{4}\pi L\sqrt{\frac{4p^2-2pz}{4p^2-z^2}}\quad(\text{where } A=\sqrt{4p^2-z^2})$$

$$\therefore\quad E=2\times E'_1=\frac{1}{2}\pi L\sqrt{\frac{4p^2-2pz}{4p^2-z^2}}\cdots\cdots(27)$$

4.6 In the case of a finite hyperbolic surface source

In **Fig.** 8, if the distance between both vertexes H and H' is taken as $2a$, the length of the vector OG connecting the point G on the line normal to the vertex H to the center O is taken as x, and the angle between the vector and the original line OH is taken as φ, then $r=a\sec\varphi$.

$$\overline{GH}=a\tan\varphi=\sqrt{x^2-a^2},$$

$$y=\frac{b}{a}\sqrt{x^2-a^2}=b\tan\varphi\cdots\cdots(28)$$

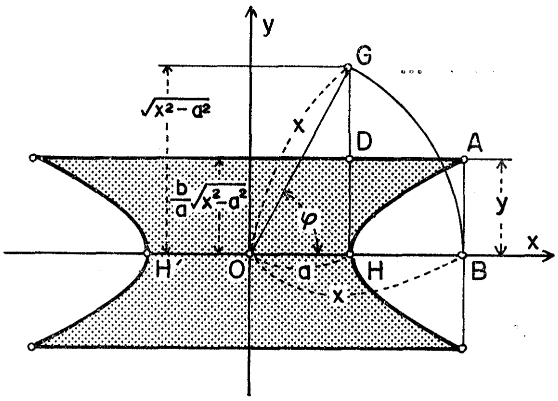

Fig. 8 *Parameters for a hyperbola.*

The rotating angle φ of the vector OG ranges from O to $\pi/2$ on the first quadrant, from π to $3\pi/2$ fi on the second quadrant, from π to $\pi/2$ on the third quadrant, and from O to $-\pi/2$ on the fourth quadrant.

4.6.1 The case in which the source is parallel to the illuminated plane

As shown in **Fig.** 9, if the center of a hyperbolic surface source is at (c, q), then:

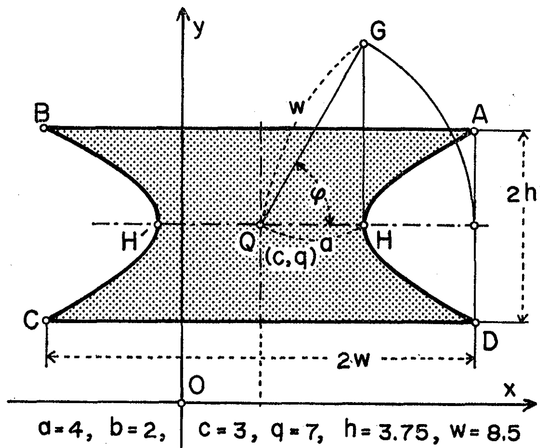

Fig. 9 *Co-ordinates for a finite hyperbolic source in an arbitrary position.*

$$x=c+a\sec\varphi,\quad\frac{dx}{d\varphi}=a\sin\varphi\sec^2\varphi\cdots\cdots(29)$$

$$y=q+b\tan\varphi,\quad y'=\frac{b}{a}cO\sec\varphi$$

$$n=y-y'\cdot x=\frac{aq\sin\varphi-ab\cos\varphi-bc}{a\sin\varphi}\cdots\cdots(30)$$

In a symmetrical hyperbolic surface source, $AB=2w$, and $AD=2h$, thus φ corresponds to $\cos^{-1}a/w$ for point A, to $-\cos^{-1}a/w$ for point D, to $(\pi+\cos^{-1}a/w)$ for point B, and to $(\pi-\cos^{-1}a/w)$ for point C.

Hence, substituting expressions (29) and (30) into formula (1) gives:

$$E'_R=\frac{L}{2}\int_{\cos^{-1}a/w}^{\cos^{-1}a/w}\frac{(aq\sin\varphi-ab\cos\varphi-bc)\sec^2\varphi}{(c+a\sec\varphi)^2+(q+b\tan\varphi)^2+z^2}d\varphi \quad \cdots\cdots(31)$$

$$=-173.86(\text{lx})$$

As the integral interval for the left-side curve ranges from 4.222432 to 2.060754, $E'_L=-101.48$.

The illuminance components of the two sides AB and CD closing the top and bottom are $E'_{AB}=511.47$ and $E'_{CD}=408.10$ according to the type of polygonal source.

$$\therefore \quad E_{\text{TOTAL}}=E'_{AB}-E'_R-E'_L-E'_{CD}$$
$$=378.71(\text{lx})$$

4.6.2 The case in which the source is inclined to the illuminated plane

The terms $\cos\beta$ and $2yz\sin\beta$ are added to the above, giving: $(E'_R)=-123.99$, $(E'_L)=-64.10$, $(E'_{AB})=-323.63$ and $(E'_{CD})=268.45$, and consequently, $(E)=243.27$ (lx).

4.6.3 The case in which the source is perpendicular to the illuminated plane

Substituting expressions (29) and (30) into formula (3) gives:

$$(\!(E'_R)\!)=-\frac{L}{2}z\int_{\cos^{-1}a/w}^{\cos^{-1}a/w}\frac{a\sin\varphi\sec^2\varphi}{l^2}d\varphi=26.78(\text{lx})$$

In the same way, $(\!(E'_L)\!)=87.12$, $(\!(E'_{AB})\!)=-285.47$, and $(\!(E'_{CD})\!)=-753.41$.

$$\therefore \quad (\!(E)\!)=(\!(E_A{}'_B)\!)-(\!(E'_R)\!)-(\!(E'_L)\!)-(\!(E'_{CD})\!)$$
$$=354.04(\text{lx})$$

4.7 In the case of an infinite hyperbolic surface source

As shown in **Fig. 10**, in the first quadrant with the center of a hyperbola located at the origin O, $x=\infty$, hence $\varphi=\pi/2$.

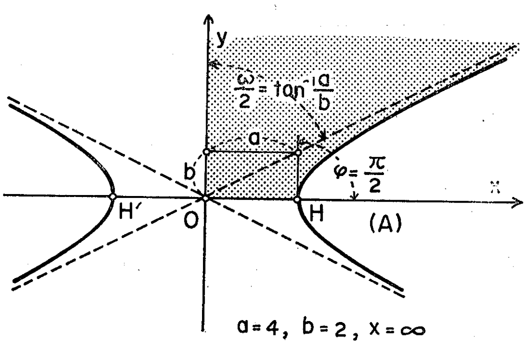

Fig. 10 *Explanation of the calculation for an infinite hyperbolic source.*

$$\therefore \quad E'_1=\frac{L}{2}\int_0^{\pi/2}\frac{-ab\sec\varphi}{a^2\sec^2\varphi+b^2\tan^2\varphi+z^2}d\varphi \quad \cdots(32)$$
$$=-103.635(\text{lx})$$

Since the infinite ends of a hyperbola are regarded to be on the asymptote, it follows that the subtending angle ω from the illuminated point P to which the boundary that closes the infinite ends is $\omega=2\tan^{-1}a/b$ and is included in the illuminated plane. Thus, the illuminance component of that for the first quadrant is:

$$E'_\infty=\frac{L}{2}\tan^{-1}\frac{a}{b}=\frac{1,000}{2}\tan^{-1}\frac{4}{2}=553.574$$

$$\therefore \quad E=4\times(E'_\infty-E'_1)=2628.84(\text{lx})$$

Moreover, the illuminance E_A in a case in which the section shown by (A) at the outer right side of the hyperbola is the source, is:

$$E_A=(1,000\pi-\text{above-mentioned } E)\div 2$$
$$=512.75(\text{lx})$$

This definite integral belongs to group (A), but depending on the relative extent of the distance z to the illuminated plane, the formula is changes remarkably as shown below, with $F(x)$ turning into a formula including a logarithm or an inverse tangent.

In case of $z>b$:

Formula (32)

$$=\frac{L\,ab}{4\sqrt{a^2+z^2}\sqrt{z^2-b^2}}\left(\log\left|\frac{\sin\varphi-\sqrt{\frac{a^2+z^2}{z^2-b^2}}}{\sin\varphi+\sqrt{\frac{a^2+z^2}{z^2-b^2}}}\right|\right)_0^{\pi/2}$$

$$=\frac{L\,ab}{4\sqrt{a^2+z^2}\sqrt{z^2-b^2}}\log\left|\frac{(\sqrt{a^2+b^2}-\sqrt{z^2-b^2})^2}{a^2+z^2}\right| \quad \cdots\cdots(33)$$

In case of $z=b$:

Formula (32)

$$=-\frac{L\,ab}{2(a^2+z^2)}\Big[\sin\varphi\Big]_0^{1/2}=-\frac{L\,ab}{2(a^2+z^2)} \quad \cdots\cdots(34)$$

In case of $z<b$:

Formula (32)

$$=-\frac{L\,ab}{2\sqrt{a^2+z^2}\sqrt{b^2-z^2}}\left[\tan^{-1}\left(\sqrt{\frac{b^2-z^2}{a^2+z^2}}\sin\varphi\right)\right]_0^{\pi/2}$$

$$=-\frac{L\,ab}{2\sqrt{a^2+z^2}\sqrt{b^2-z^2}}\tan^{-1}\sqrt{\frac{b^2-z^2}{a^2+z^2}} \quad \cdots\cdots(35)$$

4.8 In the case of a four-leaf-shaped source

This method of calculation can also be applied to flat surface sources of various shapes, such as a four-leaf or a heart shape, which are expressed by polar co-ordinates.

For a four-leaf-shaped source as shown in **Fig. 11**, $r=a\cos 2\theta$, therefore if the center Q is at an arbitrary position (c, q) it follows that:

$$x=c+a\cos 2\theta\cos\theta,\quad y=q+a\cos 2\theta\sin\theta$$

$$\frac{dx}{d\theta}=a\sin\theta(1-6\cos^2\theta)$$

$$\frac{dy}{dx}=\frac{\cos\theta(1-6\sin^2\theta)}{\sin\theta(1-6\cos^2\theta)} \quad \cdots\cdots(36)$$

$$n=\frac{q\sin\theta(1-6\cos^2\theta)-a\cos^2 2\theta-c\cos\theta(1-6\sin^2\theta)}{\sin\theta(1-6\cos^2\theta)} \quad \cdots\cdots(37)$$

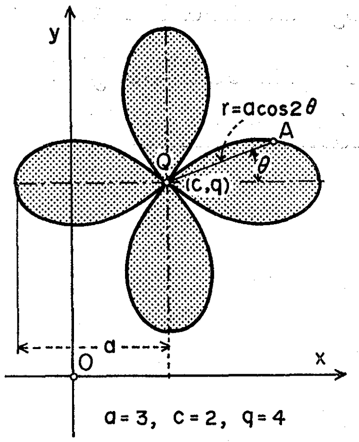

Fig. 11 *Co-ordinates for a four-leaf-shaped source in an arbitrary position.*

4.8.1 The case in which the source is parallel to the illuminated plane

By substituting expressions (36) and (37) into formula (1) and making the integral intervals of each respective leaf continuous, it follows that:

$$E' = \frac{L}{2} a \int_{2\pi}^{0} \frac{q \sin\theta(1-6\cos^2\theta) - a\cos^2 2\theta - c\cos\theta(1-6\sin^2\theta)}{(c+a\cos 2\theta\cos\theta)^2 + (q+a\cos 2\theta\sin\theta)^2 + z^2} d\theta \quad \cdots\cdots (38)$$

$$= 162.09(\text{lx})$$

In particular, the definite integral in a case in which the center is located at the origin 0 belongs to group (A), consequently, with $c=0$ and $q=0$ and with the primitive function $F(x)$ for the first leaf extending from the first to the fourth quadrant, it follows that:

$$E'_1 = \frac{L}{2} \int_{\pi/2}^{-\pi/2} \frac{-a^2\cos^2 2\theta}{a^2\cos^2 2\theta + z^2} d\theta$$

$$= \frac{L}{4}\left\{\left[\frac{z}{\sqrt{a^2+z^2}} \tan^{-1}\left(\frac{z}{\sqrt{a^2+z^2}} \tan 2\theta\right)\right]_{\pi/2}^{-\pi/2} - \left[2\theta\right]_{\pi/2}^{-\pi/2}\right\} = \frac{L\pi}{4}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

$$\therefore \quad E = 4 \times E'_1 = \pi L\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) \quad \cdots\cdots (39)$$

4.8.2 The case in which the source is inclined to the illuminated plane

Adding $\cos \pi/6$ and $2(4+3\cos 2\theta \sin\theta) \times 6 \sin \pi/6$ to expression (39) gives $(E) = 39.10$ (lx).

4.8.3 The case in which the source is perpendicular to the illuminated plane

By similar calculations, we can obtain:
$(E) = 96.03$ (lx)

5. Conclusion

According to this new method of calculation, the definite integrals for a flat surface source of any shape and any condition that can be expressed by $f(x)$ can be obtained easily using memorizable formulae and procedures, and the calculation can be done by computer.

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