J. Acoust. Soc. Jpn. (*E*) **18**, 3 (1997)

Nonlinear threads in the coat of acoustics†

Robert T. Beyer

Department of Physics, Brown University, Providence, RI, 02912

In 1972, I gave an invited paper at the Buffalo meeting of the Acoustical Society of America on nonlinear acoustics. It was a narrowly focused talk, almost exclusively in physical acoustics. At the end of it, however, I borrowed a phrase from a then current Coca-Cola advertisement to claim that nonlinear acoustics was not a sideline, "it was the real thing."

Apparently I had at least one listener in my audience. Professor Akira Nakamura attended and, when he went back to Japan, wrote an article for the Japanese Journal of Acoustics, reviewing the meeting.[1)]

The last paragraphs of that paper appear in Fig. 1. I am told that the article says that I gave an invited paper that was very humorous and interesting, that the recent progress in this scientific field was considerable, and that we must say "that is real thing." In the next instant, Professor Nakamura wrote, a roar of laughter filled the room. I cannot guarantee roars of laughter today, but I can document my belief that many if not all portions of acoustics have their nonlinear chapter, and that these threads, as I have called them, have been around for a long time.

Before beginning the talk proper, I must warn the audience that, when they get an old man to be the lead-off speaker, they will get a bridge to the past, ã la Bob Dole, rather than a bridge to the future. But I prefer to think that there is in our case only one bridge, the bridge that extends from the past to the future, and we are in the middle of that bridge. I shall describe where we have come from in nonlinear acoustics, and all the other papers during the rest of the meeting will tell you where we are going.

† Talk given at the joint meeting of the Acoustical Society of Japan and the Acoustical Society of America, Honolulu, HI, 2 Dec 1996

The basic subjects of vibration and sound are governed by a number of differential equations, of which the wave equation is the most prominent, followed by the equation for forced, damped vibrations. We like to think of these equations as linear equations, *i.e.*, differential equations in which the dependent variable—the displacement or the displacement velocity—does not appear in products of itself or of its derivatives. Thus the wave equation, in one dimension, takes the form

$$c^2\frac{\partial^2 y}{\partial x^2}+R\frac{\partial y}{\partial t}-\frac{\partial^2 y}{\partial t^2}=F(x,t)\,. \qquad (1)$$

Here $R\,\partial y/\partial t$ is the dissipation term, $F(x, t)$ is the forcing or source term. This equation can also be used for forced oscillations by eliminating the first term. It has dominated our thinking in acoustics for two hundred years. But it often does not reflect reality. Nevertheless, somewhat like Procrustes and his infamous bed, acousticians have tried to make the linear wave equation fit all situations, and have tended to conceal those situations for which it is not satisfactory. And yet, throughout the nineteenth century, there was an undercurrent of research on nonlinearity in acoustics, as we shall see later in this talk. Textbooks rarely mentioned anything nonlinear. As an example, the first edition of Kinsler and Frey's book on acoustics, published in 1950, never refers to the subject, while the earlier books by Ricardson and by Arthur Tabor Jones, refer to it only in connection with the description of Tartini tones. And so, let us begin with Tartini.[2)]

In one sense, the subject of nonlinear acoustics begins with Giuseppe Tartini. In the first half of the eighteenth century, this Italian musician (Fig. 2) reported his observation that he could hear

まず，Prof. Beyer が“非直線の世界”と題して非常にユーモラスな講演をしました．彼は物理現象をより正確に理解するためには直線関係だけでは不十分であるとし，特に音響の分野では音波の強度が大きいとき，媒質の弾性に非直線的性質が強いとき，境界条件に非直線性が含まれているときにこの現象が強く現れるといい，最近のこの分野の研究は非常に進歩してきたので，もはや“This is real thing”と云っていいと発言したときは満場爆笑につつまれました．

Fig. 1 Prof. Nakamura's account of my talk at Buffalo, NY, 1972.[1]

Fig. 2 Giuseppe Tartini.[3]

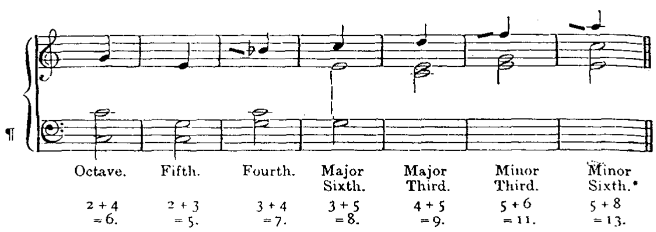


Fig. 3 Difference tones (Helmholtz[4]).

difference tones when he played two musical notes loudly on his violin.[3] As you can see from his portrait, Tartini was nonlinear right down to the tip of his nose. A sample of these difference tones is shown in Fig. 3.[4]

Tartini's work was confirmed by other observers, and the next hundred years saw a succession of arguments as to whether this was a form of beats, and therefore a linear phenomenon, or whether it was due to something nonlinear.

We note that this first discussion of what today we call nonlinear acoustics came in the field of *music*. The second acoustical subfield to be concerned was *hearing*, and in the middle of the nineteenth century Helmholtz assigned the origin of the difference tones to the middle ear. He recognized, correctly that the difference tones were a nonlinear phenomenon, but thought, incorrectly, that the small bones of the middle ear—the ossicles—had a nonlinear response to the sound wave impinging on them. By considering simply a square-law dependence for this response, he was able to justify the appearance, not only of difference tones, but also summation tones, which Helmholtz had already reported hearing. Helmholtz also maintained that he could produce these various combination tones outside the ear, so that they were objective tones, as against the subjective tones perceived in the ear.

Helmholtz's work was apparently not accepted. This is surprising because, by midcentury there had been considerable work on the effect of high amplitude sound on its propagation, in what we would call today *physical acoustics*. The French scientist Poisson had argued in 1808 that the velocity of propagation in a sound wave consisted of the sum of the small-amplitude sound velocity and the displacement or particle velocity of the wave.[5] It was then pointed out in the 1840's by the English scientists Airy and Stokes that Poisson's thesis would lead, in the absence of attenuation, to a distortion of the wave, with one portion of the wavefront becoming steeper and steeper until a discontinuity resulted. These ideas were confirmed (before the work of Helmholtz) by the research of Riemann[6] in the 1850's and by a paper by Samuel Earnshaw in 1860.[7] If two waves of different frequency were introduced into the equation of Earnshaw, it would have been clear that various combination tones would inevitably result. But no one did that. And if this were not enough, Rücker and Edser in England published a paper in 1895, demonstrating conclusively that they could detect the sum and difference frequencies in a laboratory setup.[8)]

Nevertheless, the effect was still not accepted. "None are so deaf as those who won't hear" might have been the motto of the acousticians of the day. It was in fact not until our own times (or, at least, until my own time) that Fubini in 1935 produced the explicit form of Earnshaw's equation[9)] and various observers, beginning with Thuras, Jenkins and O'Neil at the Bell Laboratories detected the sum and difference frequencies in air.[10)]

One should also note in passing that the results of Fubini had actually been obtained earlier by Georg Bessel. David Blackstock pointed out the fact that Bessel, in working on an astronomical problem, invented the Bessel functions to solve Kepler's equation for the eccentricity of an ellipical orbit, an equation that was identical in form to the equation developed by Earnshaw.[11)] Ah, if only we read the literature!

The equation that Earnshaw and Fubini solved is one without dissipation. When dissipation is included, the problem of course becomes much more complex, and for a long time it defied solution. Approximate solutions of the equation proved to be of very limited usefulness, since one soon ran out of the region of convergence of the solutions. What was needed was a new technique. Glimpses of this technique were provided first by Richard Fay with his idea of the 'almost stable waveform' in the 1930's,[12)] and then by Mendousse in the 1950's,[13)] but full exploitation had to await the work of Khokhlov and his students.[14)]

An excellent verification of the existence of these combination tones in the ear was provided by Wever and Lawrence in determination of the response of an intact guinea pig ear due to two exciting tones (Fig. 4[15)]).

Mendousse described the solution of the problem as equivalent to riding along on the crest of a wave and viewing the changes in the wave form from that point. Mathematically, one makes an adroit substitution and converts the problem from the approximate solution of an exact equation to the exact solution of an approximate equation—in this case, Burgers' equation:

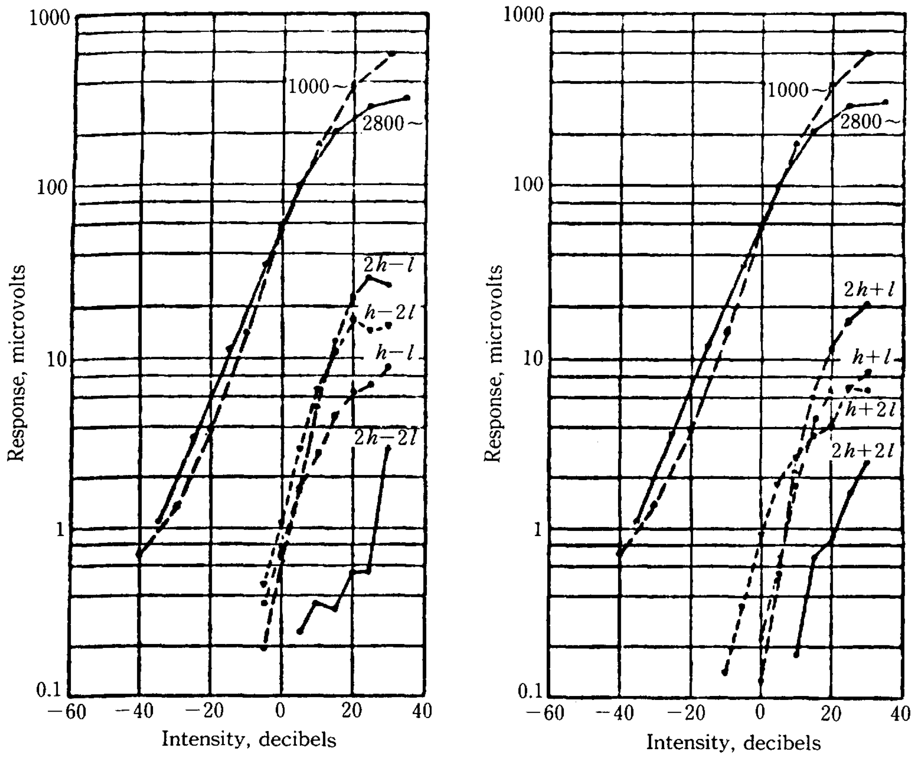


Fig. 4 Difference and combination tone responses of an intact guinea pig ear, stimulated at 1,000 Hz (l) and 2,800 Hz (h) (Wever and Lawrence[15)]).

$$c_0^3 u_z - \beta c_0 u u_\tau = \frac{1}{2}\frac{b\eta}{\rho_0} u_{\tau\tau} = \frac{c_0^3 \alpha}{\omega^2} z_{\tau\tau} .$$

Here u is the displacement velocity, α the absorption coefficient and $u_z = \partial u/\partial z$, *etc.*[16)]

The beauty of this move stems from the fact that another substitution converts Burgers' (nonlinear) equation to a linear one, for which a solution can easily be found. Incidentally, this method of converting a nonlinear equation to a linear equation happens with some frequency in the field of nonlinear equations, but, thus far, no one has found a single guiding principle that makes the discovery of this substitution inevitable.

The work on Burgers' equation was extended by Zabolotskaya and Khokhlov in 1969 to take diffraction effects into account.[17)] This relation, known as the K-Z equation, applied only to a nondissipative medium. However, a modification of this equation was introduced by Kuznetsov a few years later to take approximate account of diffraction effects.[18)] This KZK equation has also been widely studied. Since that time, numerous papers have appeared, both in the U.S. and abroad, applying, with the help of computer programs, these equations to the solution of practical problems.

Another major advance was made in the 1950's by Sir James Lighthill.[19)] He noted that Rayleigh, in studying the scattering effects of small-scale inhomogeneities in fluids, had written out the full wave equation, including terms for the inhomogeneities due to density and sound velocity variations in the medium, and then discarded all the nonlinear terms

$$\Box^2 p = -\frac{2\Delta c}{c_0}\frac{\partial^2 p_s}{\partial t^2} - \frac{\partial}{\partial y}\left(\frac{\Delta\rho}{\rho_0}\right)\frac{\partial p_s}{\partial y}$$

$$\Box^2 p \equiv \nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} .$$

$$\Box^2 p = -\frac{\partial^2}{\partial y_i \partial y_j}(\rho u_i u_j + p_{ij} - c_0^2 \rho \delta_{ij}) ,$$

Lighthill in turn kept the nonlinear terms as forcing terms, but discarded the inhomogeneity terms. He then proceeded to solve the resulting equations

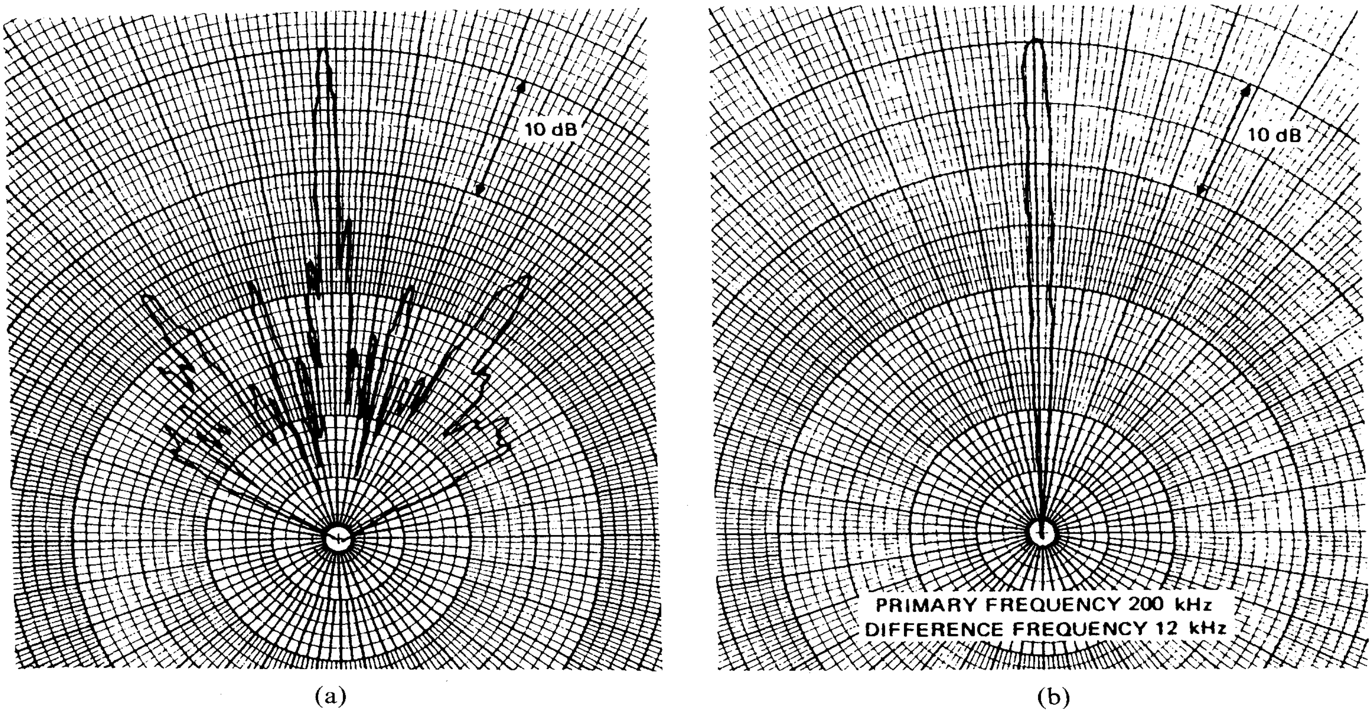
$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + c_0^2 \frac{\partial\rho}{\partial x_i} = -\frac{\partial T_{ij}}{\partial x_j}$$

$$\frac{\partial^2\rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

where the stress T_{ij} is defined by the relation

$$T_{ij} = u_i u_j + p_{ij} - \rho c_0^2 \delta_{ij} .$$



(a) (b)

Fig. 5 Beam patterns of parametric array sonar. Primary beam 200 kHz, difference frequency beam 12 kHz. Source level was 87.6 dB re 1 μbar (Walsh[20)]).

for various cases involving the production of *noise* by jet turbulence.

Applying the theoretical work of Lighthill, Peter Westervelt developed an analysis for the interaction of two collimated sound beams. In this analysis, Westervelt arrived at two conclusions—one widely accepted, and one that has proved to be controversial for forty years. First, it was concluded that the difference frequency would propagate with high directivity and with no side lobes (Figs. 5, 6).[20,21]

From Westervelt's work and later from that of the Russian school of Khokhlov and his colleagues, it became clear that the production of the different frequency had application in *underwater sound*. The employment of two concentric ultrasonic beams of neighboring frequency—say at 100 and 101 kHz, would lead to a difference frequency beam at 1 kHz. This beam had the high directivity of the ultrasonic frequency and the low attenuation of the audio difference frequency. This so-called parametric array sonar has been put to use in shallow water detection of various objects (Fig. 7)[22] and in the search for buried objects, be they mines or treasures (Fig. 8).[23]

The controversial conclusion of Westervelt relates to the possible scattering of sound by sound outside the region of interaction.[24] Westervelt maintained that the intersection of two collimated beams at right angles would lead to no such scattering, and indeed, that such scattering at the sum or difference frequency could exist only if the two beams were

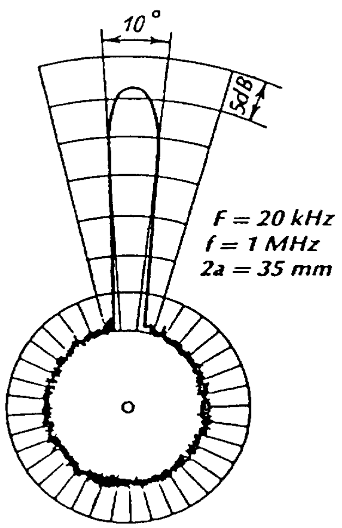


Fig. 6 Beam pattern for a 20-kHz difference frequency, 1-MHz primary (Novikov *et al.*[21]).

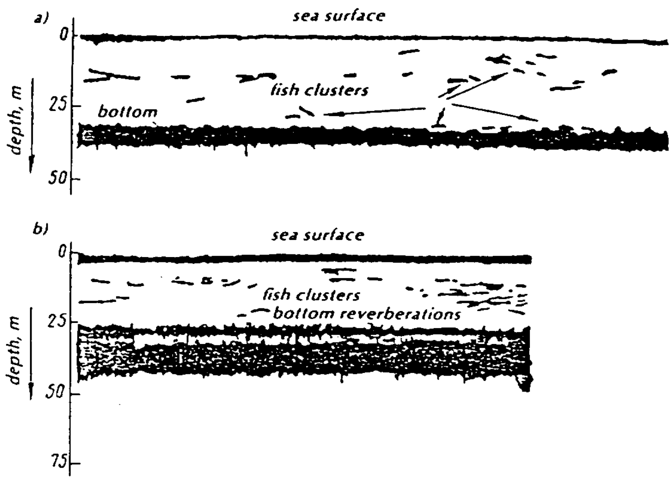


Fig. 7 Parametric array sonar used in search of fish (Novikov *et al.*[22]).

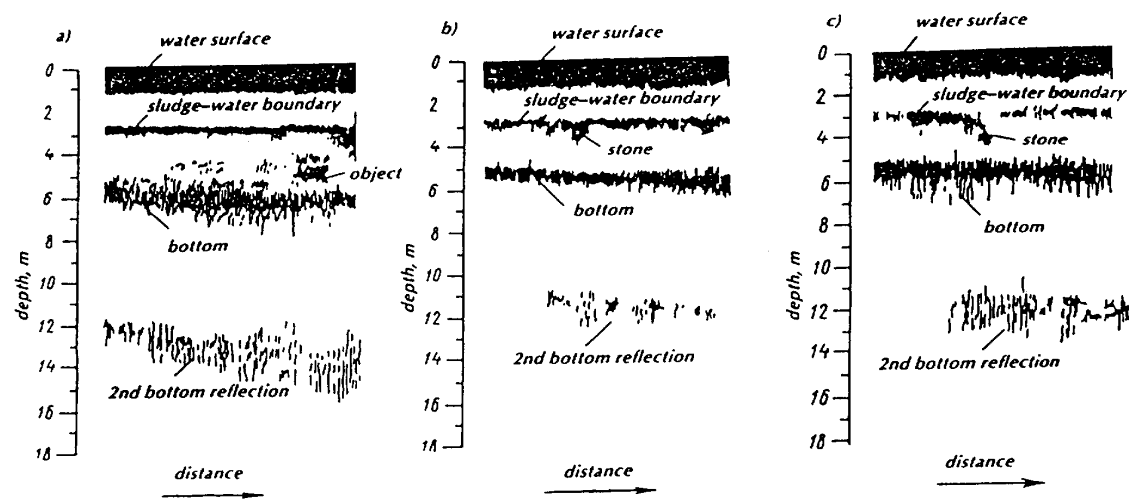


Fig. 8 Parametric array sonar used in search of buried treasure (Novikov *et al.*[23]).

collinear. Numerous papers, both theoretical and experimental, have been lined up on the two sides of this question. It would appear that the resolution of the controversy lies in the boundary conditions that we have applied to the propagating waves, and whether or not the medium is dispersive.

A surprise application of the Tartini tones has been found in small radios, whose loud speakers were too small in diameter to allow effective radiation of the bass notes in musical compositions. The ear recognizes the second and third harmonics of the bass tone and therefore perceives the difference frequency which, is of course the missing fundamental tone. Between this and the underwater sound apparatus discussed above, we thus have applications of *engineering acoustics*.[25)]

The nonlinearities of ultrasonic propagation are of two kinds. One is due to the nonlinearity of the wave equation itself, which takes the form

$$\frac{\partial^2 \xi}{\partial t^2}=\frac{c_0^2}{\left(1+\frac{\partial \xi}{\partial a}\right)^{\gamma+1}}\frac{\partial^2 \xi}{\partial a^2}. \qquad (2)$$

in gases or

$$\frac{\partial^2 \xi}{\partial t^2}=\frac{c_0^2}{\left(1+\frac{\partial \xi}{\partial a}\right)^{2+B/A}}\frac{\partial^2 \xi}{\partial a^2} \qquad (3)$$

in liquids. But the nonlinearity may also stem from a nonlinearity in the relationship between the excess pressure and the density changes, which can be written in the form

$$p=p_0+As+\frac{B}{2!}s^2 \qquad (4)$$

where

$$A=\rho_0\left(\frac{\partial p}{\partial \rho}\right)_{S,\rho=\rho_0}=\rho_0 c_0^2$$

$$B=\rho_0^2\left(\frac{\partial^2 p}{\partial \rho^2}\right)_{S,\rho=\rho_0} \qquad s=\frac{\rho-\rho_0}{\rho_0}$$

The pursuit of the ratio of B/A has formed an important part of my own research career, but it has attracted the attention of many others. In 1940, the Indian physicist M. R. Rao[26)] set up a relationship between the sound velocity and the specific volume of a liquid,

$$c^{1/3}v=R\,,$$

where R is a constant for a given liquid, called the Rao number. About 10 years later, the Japanese physicist Y. Wada introduced a variant of this relationship, called Wada's rule,[27)] in which the adiabatic compressibility κ_S and the density were related in the form

$$B=(M/\rho)\kappa_S^{-1/7}$$

where B is a pressure-independent constant for a given liquid. These two empirical formulas are closely related to one another, but what is of interest from our point of view is that they yield an expression for the ratio $B/A=6$. Thus, they bear an intimate relationship with nonlinear acoustics, albeit a very approximate one. The ratio B/A for liquids has been found to vary from about 2 up to 11, although there is a cluster of liquids for which the value is close to 6. There are still many unanswered questions involving the ratio of B/A and the structure of liquids that might give a bearing on these values, and, as it is said at the end of every contract report, more research needs to be done.

Another well-studied feature of nonlinearity is radiation pressure, the small but measurable one-directional force produced by a traveling wave. This phenomenon has resulted in some levitation (Fig. 9[28)]) and in a control system (Fig. 10[29)]) that can lead to containerless processing of materials.

Another result of the nonlinearity of wave propagation is the production of acoustic streaming, a phenomenon first observed by Michael Faraday in the 1830's (Fig. 11).[30)] My colleague, Peter Westervelt would point out that streaming is the result of the self-interaction of a finite amplitude wave, which produces both the sum frequency—the second har-

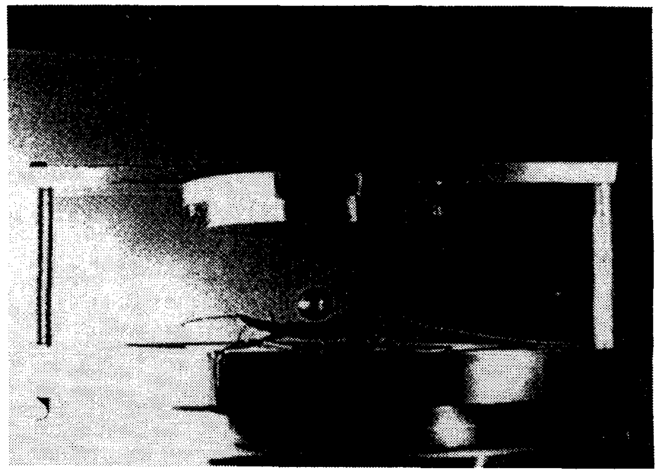


Fig. 9 Leviation of a steel ball above a siren. The diameter of the ball is 1.9 cm (Barmatz[28)]).

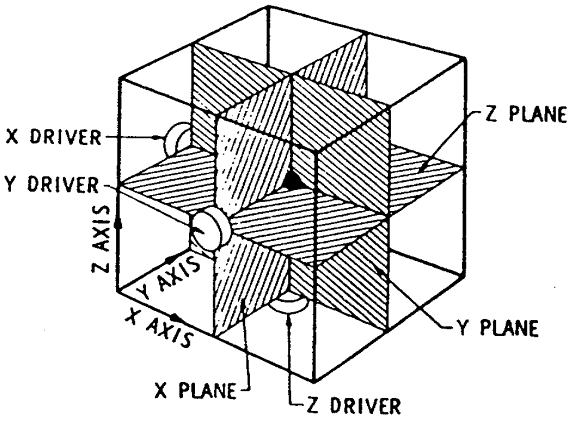


Fig. 10 Triple-axis levitator for holding a sample at the center by beams of three orthogonal sound waves (Wang *et al.*[29]).

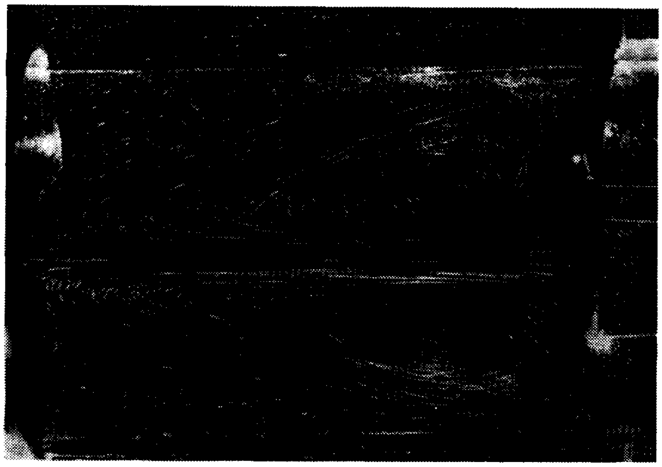


Fig. 11 Acoustic streaming due to an ultrasonic source at the left (L. Liebermann[30]).

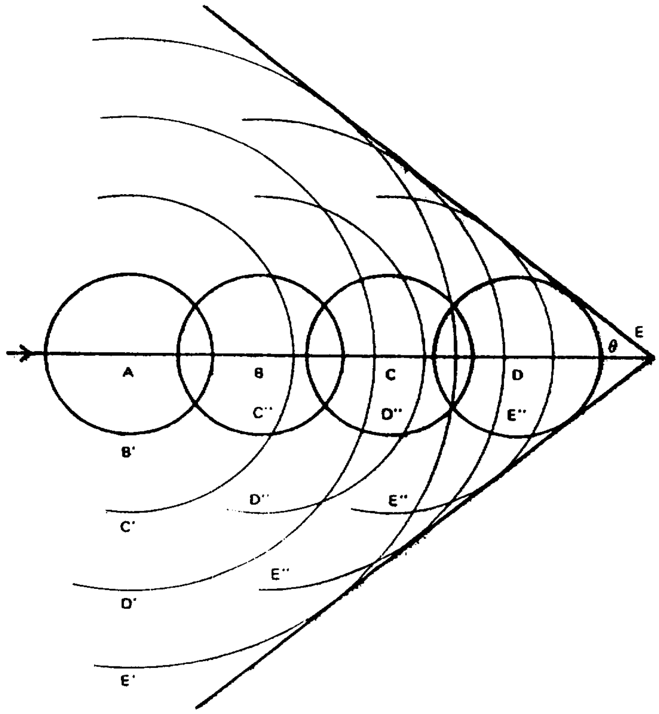


Fig. 12 Generation of a sonar boom (Beyer[31]).

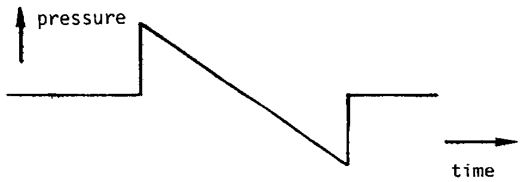


Fig. 13 N wave (Beyer[32]).

monic—and the difference frequency—the zeroth harmonic or acoustic streaming.

The field of shock waves is still one more branch of nonlinear acoustics. Here, the acoustical phenomenon has led to acute noise problems on the one hand—through the appearance of the sonic boom generated by high-speed aircraft (Fig. 12)[31] or projectiles (N waves, Fig. 13[32])—and to the medical field of lithotripsy, or the focusing of shock waves to break up kidney and other stones embedded in body organs (Fig. 14[33]).

Solitons.* One of the basic characteristics of linear acoustics is the principle of superposition, according to which the different solutions of a single equation can exist independently of one another. But, in recent years, it has been found that the solutions of certain nonlinear differential equations, known as evolution equations, have, in the words of one observer, G. L. Lamb, Jr., "a special type of elementary solution. These special solutions take the form of localized disturbances or pulses that retain their shape even after interaction among themselves, and thus act somewhat like particles."[34] In 1965, Zabusky and Kruskal remarked that these "interacting localized pulses do not scatter irreversibly, and gave them the name of solitons—*i.e.*, solitary waves."[35]

It has been a lot easier to produce solitons in hydrodynamic waves than in acoustical ones, and the acoustical research has mainly been in the former area. Some samples of solitons or solitary waves, going back to Russell in the 1840's are shown in Fig.

* Those who attended the talk in Honolulu will recognize that about here the author became ill and had to leave the platform. He did, however, survive.

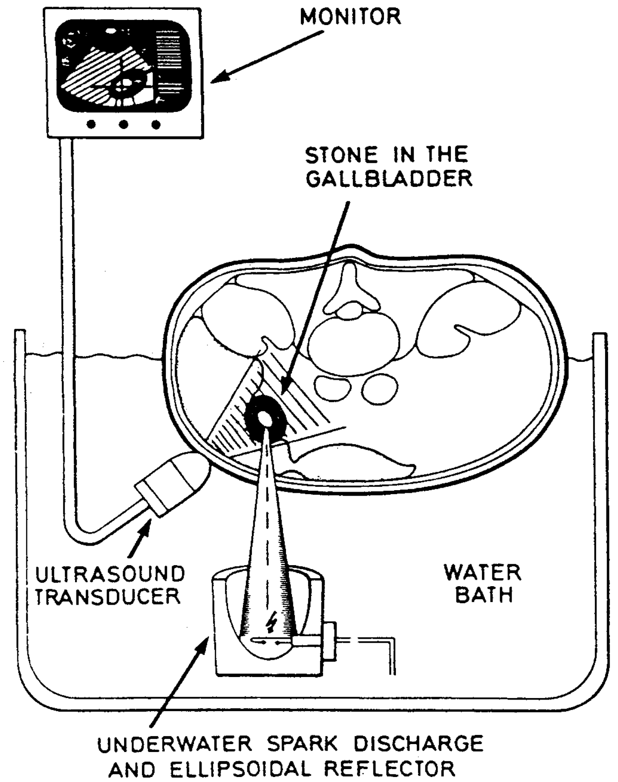


Fig. 14 Shock-wave lithotripter for gallstones (Sauerbruch *et al.*[33]).

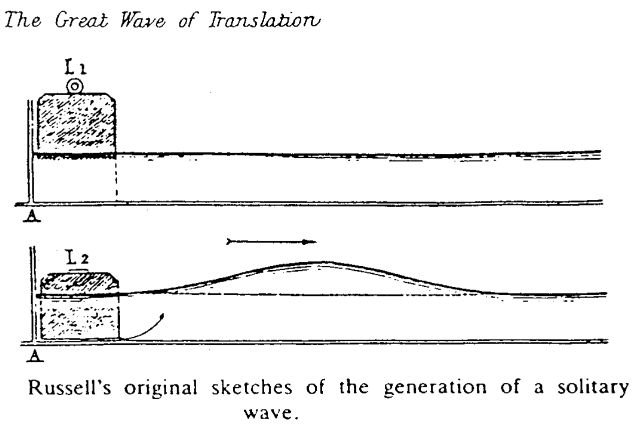


Fig. 15 Generation of a solitary wave (Russell[36]).

15,[36] and Figs. 16[37] and 17.[38]

Sonoluminescence. The phenomenon of sonoluminescence was first observed by Frenzel, Hinsberg and Schulte back in 1929—the appearance of luminescence when air or oxygen had been dissolved in water and a sufficiently strong ultrasonic signal irradiated the liquid.[39] After a long quiet period, the study of sonoluminesence was once again been taken up, vigorously, by Larry Crum and his associates.[40] They have colored slides, whereas this paper is in back and white, but the results of both single bubble light scattering and multiple

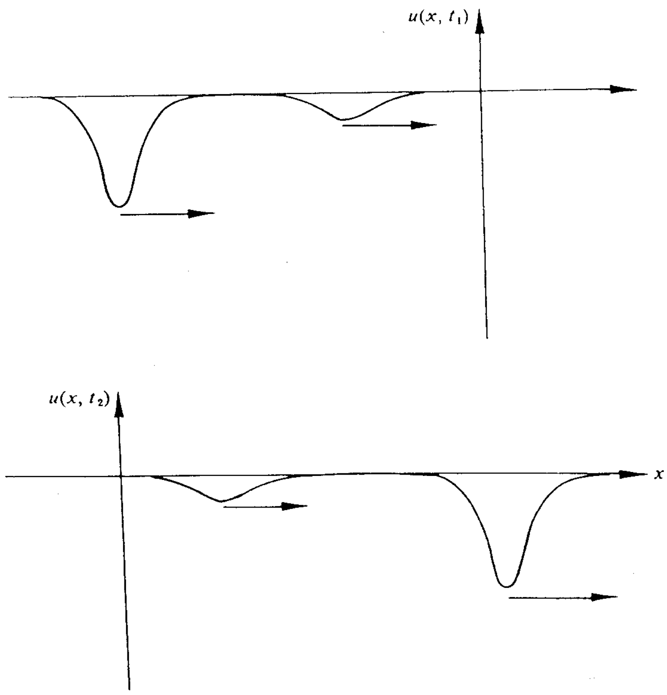


Fig. 16 Interaction of two solitons (Drazin and Johnson[37]).

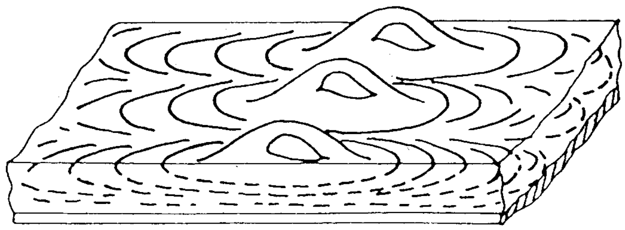


Fig. 17 Array of solitons: $l_x = 54$ mm, $l_y = 18$ cm, exciting frequency 10 Hz for (0,6) mode (Wei *et al.*[38]).

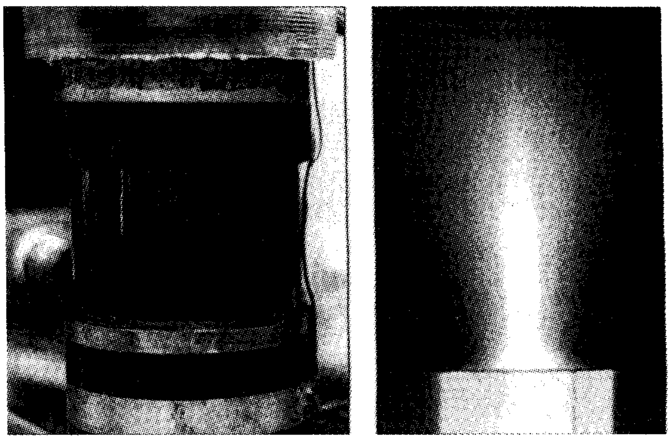


Fig. 18 Single bubble and multiple bubble sonoluminescence (Crum and Roy[40]).

bubble scattering have been spectacular (Fig. 18). These studies have led to a new branch of sonochemistry, but they have also interconnected with cavitation—the appearance of holes or bubbles in a liquid upon intense ultrasonic irradiation, still another branch of nonlinear phenomena in acoustics. The study of the collapsing cavitation bubble goes back to Lord Rayleigh. The mathematics have been formidable and checkable results in the area of sonoluminescence have not yet appeared.

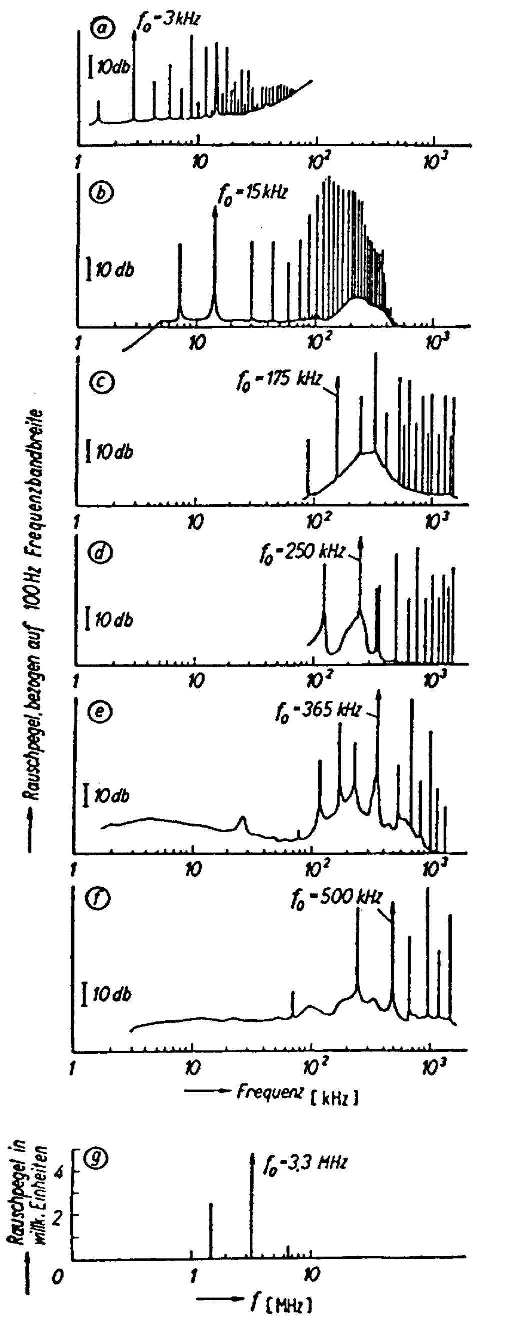


Fig. 19 The spectra of acoustic cavitation noise. Nota (e), where a second period doubling occurs (Esche[42]).

The study has led to an intriguing development —the creation of very high temperatures during the bubble collapse. Estimates of these temperatures by Seth Putterman and others have been as high as 100 million degrees, which approximates the level necessary for nuclear fusion.[41] It may be that the collapsing bubbles of cavitation have an great undeveloped future before them.

The phenomenon of cavitation has lead to still another field prominent in modern physics, that of chaos. In addition to the luminescence produced by cavitation that we have just discussed, cavitation was also known to produce a hissing noise. In 1952, Esche plotted the spectra of such noise for sound sources of different frequencies (Fig. 19[42]). As can be plainly seen from the figure, the noise contains the fundamental of the exciting frequency and its higher harmonics. But what is also present is a collection of half frequencies. In modern terms this is period doubling. And period doubling leads to chaos. More recent work by Lauterborn is shown in Fig. 20.[43]

I have now escorted you on a tour of nonlinear acoustics, from the harmonies of Tartini's music, to chaos, which latter is possibly a symbol of our modern times. I have dropped the names of more than half of our technical committees in dealing with nonlinear processes, and I hope that I have therefore made the point that the thread of nonlinearity runs through much of this magical coat of acoustics.

REFERENCES

1) A. Nakamura, J. Acoust. Soc. Jpn. (J) **29**, 316–319 (1973) (in Japanese).
2) L. E. Kinsler and A. R. Frey, *Fundamentals of Acoustics* (John Wiley, New York, 1950); A. T. Jones, *Sound* (Van Nostrand, New York, 1937).
3) Oil painting by an unknown artist (Conservatorio Musicale G. B. Martini, ex "Liceo Musicale, Bologna, Italy).
4) H. v. Helmholtz, *On Sensations of Tone*, English Translation by Alexander J. Ellis (Reprinted by Dover Publ., New York, 1954), p. 156.
5) S. D. Poisson, J. l'Ecole Polytech. **7**, 319 (1808). A partial English translation appears in *Nonlinear Underwater Sound*, R. T. Beyer, Ed. (Van Nostrand Reinhold Co., New York, 1984), pp. 23–28.
6) B. Riemann, Ges. Wiss. Göttingern Abh. **8**, (1860); A partial English translation appears in *Nonlinear Underwater Sound*, R. T. Beyer, Ed. (Van Nostrand

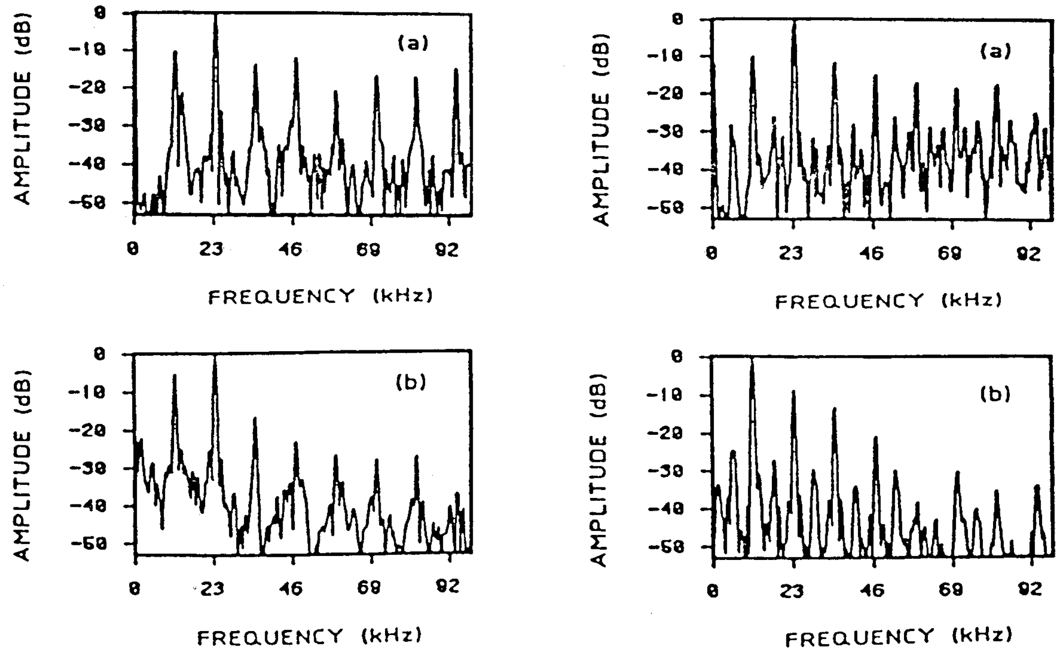


Fig. 20 Power spectra of the noise (a) and transmitted light (b) obtained from a cavitation bubble driven at 23.1 kHz. The figures on the left represent first period doubling, while those on the right are second period doubling (Lauterborn[43]).

Reinhold Co., New York, 1984), pp. 42–61.

7) S. Earnshaw, Philos. Trans. R. Soc. London **150**, 133 (1860).

8) A. Rücker and E. Edser, Philos. Mag. Ser. 5, **39**, 341–357 (1895).

9) E. Fubini, Alta Freq. 4, 532–581 (1925). A partial English translation appears in *Nonlinear Underwater Sound*, R. T. Beyer, Ed. (Van Nostrand Reinhold Co., New York, 1984), pp. 118–177.

10) A. L. Thuras, R. T. Jenkins, and H. T. O'Neil, J. Acoust. Soc. Am. **6**, 173–180 (1935).

11) D. T. Blackstock, J. Acoust. Soc. Am. **34**, 9–30 (1962).

12) R. Fay, J. Acoust. Soc. Am. **3**, 222–241 (1931).

13) J. S. Mendousse, J. Acoust. Soc. Am. **25**, 51–54 (1953).

14) S. I. Soluyan and R. V. Khokhlov, Vestn. Moskov. Univ. **3**, 52–61 (1967). An English translation appears in *Nonlinear Underwater Sound*, R. T. Beyer, Ed. (Van Nostrand Reinhold Co., New York, 1984), pp. 193–206.

15) E. G. Wever and M. Lawrence, *Physiological Acoustics* (Princeton University Press, Princeton, 1954).

16) B. K. Novikov, O. V. Rudenko, and V. I. Timoshenko, *Nonlinear Underwater Acoustics*, Translated from the Russian by R. T. Beyer (Acoustical Society of America, New York, 1987).

17) E. A. Zabolotskaya and R. V. Khokhlov, Akust. Zh. **15**, 40–47 (1969) [Sov. Phys. Acoust. **15**, 35–40 (1969)].

18) V. P. Kuznetsov, Akust. Zh. **16**, 548 (1970) [Sov. Phys. Acoust. **16**, 467 (1970)].

19) J. M. M. Lighthill, Proc. R. Soc. London **A211**, 564–587 (1952).

20) G. M. Walsh, Electron. Progr. **13**, No. 17 (1971).

21) B. K. Novikov, O. V. Rudenko, and V. I. Timoshenko, *Nonlinear Underwater Acoustics*, Translated from the Russian by R. T. Beyer (Acoustical Society of America, New York, 1987), p. 163.

22) B. K. Novikov, O. V. Rudenko, and V. I. Timoshenko, *Nonlinear Underwater Acoustics*, Translated from the Russian by R. T. Beyer (Acoustical Society of America, New York, 1987), p. 231.

23) B. K. Novikov, O. V. Rudenko, and V. I. Timoshenko, *Nonlinear Underwater Acoustics*, Translated from the Russian by R. T. Beyer (Acoustical Society of America, New York, 1987), p. 236.

24) P. J. Westervelt, J. Acoust. Soc. Am. **29**, 199–203, 934–935 (1957).

25) Since this talk, it has been pointed out to the author by Adrian Houtsma that this explanation has been shown to be at least partially incorrect by J. F. Schouten in Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam **41**, 1086–1093 (1938); **43**, 356–465 (1940). See also J. R. Licklider, in *Psychology: A Study of a Science*, S. Koch, Ed. (McGraw-Hill, New York, NY, 1959), Vol. 1, pp. 141–144. Schouten showed that a "residue pitch" endured even after all corrections had been made for the presence of Tartini tones.

26) M. R. Rao, Nature **139**, 885 (1937); Proc. Indian Acad. Sci. Sect. A **7**, 163 (1938).

27) Y. Wada, J. Phys. Soc. Jpn. **4**, 280 (1949).

28) M. Barmatz, in *Materials Processing in the Reduced*

Gravity Environment of Space, G. E, Rindone, Ed. (Elsevier North Holland, New York, 1982).
29) T. G. Wang, M. M. Saffren, and D. D. Elleman, AIAA Paper No. 74-155 (1974).
30) L. Liebermann, Phys. Rev. **75**, 1415-1422 (1949).
31) R. T. Beyer, *Nonlinear Acoustics* (Naval Ship Systems Command, Washington, D. C., 1974), p. 190.
32) R. T. Beyer, *Nonlinear Acoustics* (Naval Ship Systems Command, Washington, D. C., 1974), p. 197.
33) T. Sauerbruch, M. Delius, G. Paumgartner, J. Holl, D. Wess, W. Weber, W. Hepp, and W. Brendel, New Engl. J. Med. **314**, 818 (1986).
34) G. L. Lamb, Jr., *Elements of Soliton Theory* (John Wiley & Sons, New York, 1980), p. 1.
35) N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240-245 (1965).
36) J. S. Russell, Report, 14th Meeting, Brit. Ass'n Adv. Sci. **64**, 167 (1844).
37) P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, U. K., 1989), pp. 1-22; D. J. Acheson, *Elementary Fluid Dynamics* (Clarendon Press, Oxford, 1990), p. 110.
38) R.-J. Wei, B.-R. Wang, Y. Mao, X.-Y. Zheng, and G.-O. Miao, J. Acoust. Soc. Am. **88**, 469-472 (1990).
39) H. Frenzel, Hinsberg, and H. Schulte, Z. Gesamte Exp. Med. **89**, 246 (1929); H. Frenzel and H. Schulte, Z. Phys. Chem. **27B**, 421 (1934).
40) L. Crum and R. Roy, Science **266**, 233-234 (1994); Phys. Today, Sept. 1994, pp. 22-29.
41) B. P. Barber and S. Putterman, Phys. Rev. Lett. **69**, 3839 (1992); S. Putterman, Sci. Am., Feb. 1995, p. 46.
42) R. Esche, Acustica **2**, AB208 (1952).
43) W. Lauterborn, Acustica **82**, S46-S55 (1996).