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Symmetric Instability without Symmetry

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Abstract

When a basic flow with linear or axial symmetry has a region of negative potential vorticity (hereafter *PV*), such a disturbance with the same symmetry as the basic flow grows that converts the negative *PV* of the region into non-negative *PV*. This is the so-called symmetric instability.

In this note, the following is shown in a nondissipative adiabatic system. Irrespective of the symmetry, disturbances grow in the region of negative *PV*. That is, irrespective of the symmetry, a flow with negative *PV* is unstable. This statement is based on the conservation law of absolute circulation around a material closed curve on the isentropic surface.

1. Introduction

As is well known, when a basic flow with linear or axial symmetry has a region of negative potential vorticity (*PV* hereafter), such a disturbance with the same symmetry as the basic flow grows that converts the negative *PV* into non-negative *PV* (e.g., Holton 1992).

For mathematical simplicity, a Y-symmetric basic state on an f plane is considered. The basic velocity has only the Y-component $V = V(Z,X)$, which depends only on (Z,X). Here, (X,Y,Z) are the Cartesian coordinates. The basic potential temperature $\theta = \theta(Z,X)$ is also independent of Y. As usual, the vertical wind shear $\partial V/\partial Z$ is set to be positive, and therewith $\partial\theta/\partial X$ is positive, because of the thermal wind relation.

By assumption, the absolute vorticity $f + \partial V/\partial X = (\partial/\partial X)(fX + V) = \partial M/\partial X$ is positive (otherwise the basic flow becomes inertially unstable). Here, $M = fX + V$ is so-called the absolute momentum of the basic state. Further, by assumption $\partial\theta/\partial Z$ is positive (otherwise the basic flow becomes statically unstable).

In this case, the following can be shown. If the Y-component of the vector product of $\nabla\theta$ and ∇M is negative, i.e., the iso-θ surface is steeper than the iso-M surface, the Y-independent disturbance in the (Z,X) plane grows. The negativity of $\boldsymbol{e}_Y \cdot (\nabla\theta \times \nabla M) = \partial(\theta,M)/\partial(Z,X)$ is equivalent to the negativity of the basic *PV*. Here $\boldsymbol{e}_Y$ is the unit vector in the Y-direction (e.g., Holton 1992). The symmetric instability was extended to include moist processes (called moist symmetric instability), in order to explain frontal rainbands (e.g., Benetts and Hoskins 1979; Emanuel 1983).

The above conclusion depends crucially on the symmetry assumption of the basic state. However, instability itself of a basic flow with negative *PV* seems not to depend on the symmetry (although the growth rate etc. are of course dependent on the form of the basic flow). That is, even in the absence of symmetry, disturbances seem to grow in the region of negative *PV* (e.g., Dixon et al. 2002; Mecikalski and Tripoli 2003).

Symmetric instability without exact symmetry was already studied by some authors. For the conventional symmetric instability, a steady symmetric basic flow on an f-plane is assumed. Sun (1994) considered a steady sym-

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metric basic flow on a β-plane. He showed that the growth rate is enhanced or decreased according to the temperature gradient in the horizontal direction normal to the symmetry direction. Clark and Haynes (1996) considered a steady and slowly varying basic flow. The adjective "Slowly varying" means that the basic flow is nearly symmetric. They showed that the small deviation from exact symmetry causes the maximum growth rate at finite vertical wave-number, rather than at infinite wavenumber, as the conventional case.

In this note, the following is shown in a nondissipative adiabatic system on the rotating earth. Disturbances grow in the region of negative *PV*. Whether the basic state is symmetric or not, has nothing to do with the reasoning. The reasoning is based on the conservation law of absolute circulation around a material closed curve on the iso-θ surface. This law is, of course, equivalent to the conservation law of *PV*. First, in section 2, the hydrostatic case on an f-plane is considered, and then in section 3, the general case is considered.

2. Hydrostatic case on an f-plane

2.a Conservation of absolute circulation

When the vertical gradient of the potential temperature θ is everywhere positive, it is possible for θ to be the vertical coordinate. Then the nondissipative adiabatic, and hydrostatic horizontal momentum equation on an f-plane is written as

$$d\boldsymbol{v}/dt = -\nabla_H \Phi - f\boldsymbol{e} \times \boldsymbol{v},$$

$$d/dt = \partial/\partial t + \boldsymbol{v} \cdot \nabla_H. \qquad (2\text{-}1)$$

Here, t is the time, $\boldsymbol{v}$ is the horizontal velocity, ∇_H is the horizontal partial differential operator on the iso-θ surface, Φ is the Montgomery function, f is the Coriolis parameter, and $\boldsymbol{e}$ is the unit vector in the upward-vertical direction. The Montgomery function is the sum of the geopotential and enthalpy.

From (2-1), the following conservation equation of the absolute circulation is obtained (see Appendix).

$$(d/dt) \int_C d\boldsymbol{x} \cdot \{\boldsymbol{v} + (f/2)\boldsymbol{e} \times \boldsymbol{x}\} = 0. \qquad (2\text{-}2)$$

Here, C is a material closed curve on the iso-θ surface. The adjective "material" means that the closed curve C moves with the fluid. Because of the adiabatic assumption, C remains on the same iso-θ surface. The horizontal projection of the position vector is denoted by $\boldsymbol{x}$. The direction of $d\boldsymbol{x}$, which is the horizontal line element vector on C, is so defined that $d\boldsymbol{x} \times \boldsymbol{e}$ points in the direction outward-normal to C (see Fig. 1).

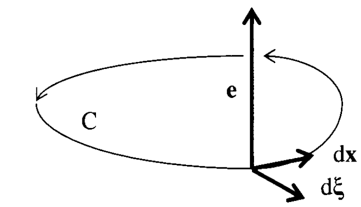

Fig. 1. The direction of integration is so defined that $d\xi = d\boldsymbol{x} \times \boldsymbol{e}$ points in the direction outward-normal to C. Here $\boldsymbol{e}$ and $d\boldsymbol{x}$ are respectively the unit vector in the upward vertical direction and the horizontal projection of line element vector on C.

Together with the mass conservation equation, (2-2) is equivalent to the potential vorticity (*PV* hereafter) conservation. *PV* is the scalar product of the absolute vorticity and $\nabla\theta$ divided by the density. In particular, the sign of the absolute circulation in (2-2) is the same as that of *PV*.

2.b Initial disturbance

Let C' be another material closed curve, which includes C and on the same iso-θ surface as C. We consider such a displacement (i.e., initial disturbance) of the fluid particles from C to C', that the absolute circulation in (2-2) is conserved (see Fig. 2).

$$\int_C d\boldsymbol{x} \cdot \{\boldsymbol{v} + (f/2)\boldsymbol{e} \times \boldsymbol{x}\}$$

$$= \int_{C'} d\boldsymbol{x}' \cdot \{\boldsymbol{v}^P + (f/2)\boldsymbol{e} \times \boldsymbol{x}'\}. \qquad (2\text{-}3)$$

Here, the velocity of the fluid particle on C is denoted by $\boldsymbol{v}$. The velocity $\boldsymbol{v}^P$, which the displaced fluid particle has on C', must satisfy the conservation constraint of absolute circulation. According to the parcel method, the pressure gradient force is assumed not to be altered. Then the force exerted on the fluid particle, which is displaced from C to C', is

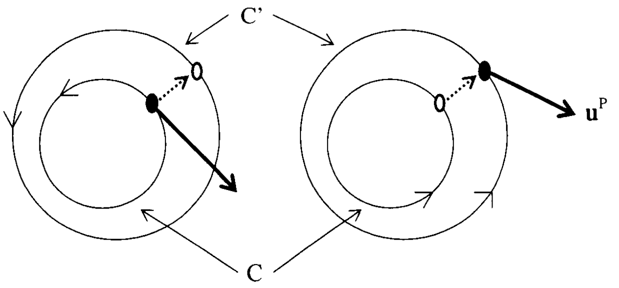

Fig. 2. The solid circles represent the displaced fluid particle from C to C'. The velocity of the fluid particle on C is denoted by $\boldsymbol{u}$. The velocity $\boldsymbol{u}^P$ which the displaced fluid particle has on C' must satisfy the conservation constraint (2-3).

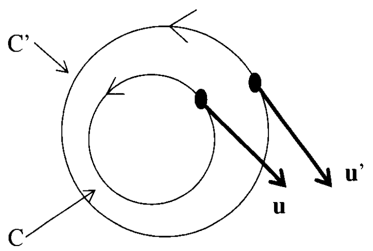

Fig. 3. The velocity $\boldsymbol{u}'$, which the fluid particle has on C' in the absence of displacement, is different from $\boldsymbol{u}^P$.

$$d\boldsymbol{v}^p/dt = -\nabla_H \Phi' - f\boldsymbol{e} \times \boldsymbol{v}^P. \tag{2-4}$$

Here, $\Phi' = \Phi(\boldsymbol{x}', t)$ and $\boldsymbol{x}'$ is on C'. If the displacement (i.e., disturbance) is absent, the fluid particles which were present on C and C' remain on C and C', respectively (see Fig. 3). Then, the force exerted on the fluid particle, which lies originally on C' in the absence of disturbance, is

$$d\boldsymbol{v}'/dt = -\nabla_H \Phi' - f\boldsymbol{e} \times \boldsymbol{v}'. \tag{2-5}$$

Here, $\boldsymbol{v}' = \boldsymbol{v}(\boldsymbol{x}', t)$ is the velocity of fluid particle on C' in the absence of disturbance. Then the anomalous (compared with the disturbance-free case) force $\boldsymbol{F}$, felt by the displaced particle, is the difference between (2-4) and (2-5) (see Fig. 4).

$$\begin{aligned} \boldsymbol{F} &= \{-\nabla_H \Phi' - f\boldsymbol{e} \times \boldsymbol{v}^P\} - \{-\nabla_H \Phi' - f\boldsymbol{e} \times \boldsymbol{v}'\} \\ &= f\boldsymbol{e} \times \boldsymbol{v}' - f\boldsymbol{e} \times \boldsymbol{v}^P. \end{aligned} \tag{2-6}$$

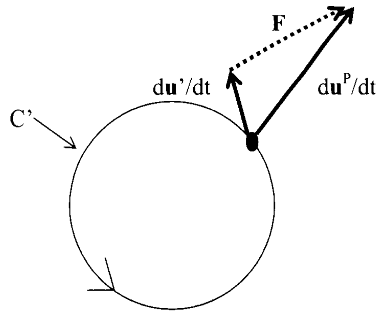

Fig. 4. The anomalous force $\boldsymbol{F}$ (compared with the disturbance free case) is the difference between $d\boldsymbol{u}^P/dt$ and $d\boldsymbol{u}'/dt$.

2.c Instability criterion

Here, such a vector $d\xi$ is defined, that points in the horizontal direction outward-normal to C' (see Fig. 1).

$$d\xi = d\boldsymbol{x}' \times \boldsymbol{e}. \tag{2-7}$$

Then, from Eqs. (2-6) and (2-7),

$$\begin{aligned} \int_{C'} d\xi \cdot \boldsymbol{F} &= f \int_{C'} (d\boldsymbol{x}' \times \boldsymbol{e}) \cdot (\boldsymbol{e} \times \boldsymbol{v}' - \boldsymbol{e} \times \boldsymbol{v}^P) \\ &= f \int_{C'} d\boldsymbol{x}' \cdot (\boldsymbol{v}^P - \boldsymbol{v}'). \end{aligned} \tag{2-8}$$

Here the identity $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$ is used. From Eq. (2-8), using the conservation condition (2-3), the following equation is obtained,

$$\begin{aligned} \int_{C'} d\xi \cdot \boldsymbol{F} &= -f \int_{C'} d\boldsymbol{x}' \cdot \{\boldsymbol{v}' + (f/2)\boldsymbol{e} \times \boldsymbol{x}'\} \\ &\quad + f \int_{C} d\boldsymbol{x} \cdot \{\boldsymbol{v} + (f/2)\boldsymbol{e} \times \boldsymbol{x}\} \\ &= -f \int_{C'-C} d\boldsymbol{x} \cdot \{\boldsymbol{v} + (f/2)\boldsymbol{e} \times \boldsymbol{x}\}. \end{aligned} \tag{2-9}$$

The integral on the right hand side of (2-9) is the absolute circulation around the region between C and C' (see Fig. 5). We consider such a displacement $\delta\boldsymbol{\eta}$ from C to C' that its horizontal projection $\delta\boldsymbol{\eta} - (\boldsymbol{e} \cdot \delta\boldsymbol{\eta})\boldsymbol{e}$ is proportional to $d\xi$. If the PV is negative between C and C', the absolute circulation is negative there, and so the left hand side of (2-9) is positive. The positiveness means that the scalar product of the displacement and force is positive on average, and therefore that the anomalous kinetic energy is produced. That is, the disturbance grows. As a

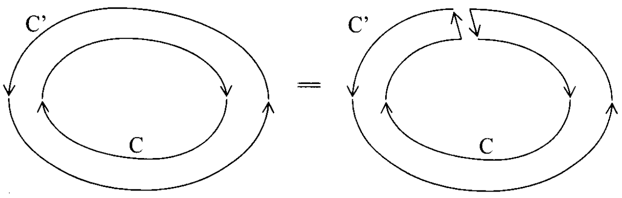

Fig. 5. The difference between the circulations around C' and C is equal to the circulation around the region between C' and C.

result, Eq. (2-9) implies that the region of negative PV is unstable.

3. General case

3.a Conservation of absolute circulation

The nondissipative equation of motion on the rotating earth can be written in the following form in the Cartesian coordinates system.

$$d\boldsymbol{V}/dt = -\theta\nabla\pi - 2\boldsymbol{\Omega}\times\boldsymbol{V} - \nabla\phi. \tag{3-1}$$

Here, $\boldsymbol{V}$ is the wind velocity, θ is the potential temperature, π is the Exner function which is a function of pressure, $\boldsymbol{\Omega}$ is the angular velocity of the earth, which is a constant vector, and ϕ is the gravitational (plus centrifugal) potential. The material temporal differential operator is denoted by d/dt, and the spatial partial differential operator is denoted by ∇.

From (3-1), the following conservation equation of the absolute circulation is obtained (see Appendix).

$$(d/dt)\int_C d\boldsymbol{X}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}) = 0. \tag{3-2}$$

Here, C is a material closed curve on the iso-θ surface, $\boldsymbol{X}$ is the position vector on C, and $d\boldsymbol{X}$ is the line element vector on C. The direction of integration is clockwise viewing in the direction of $\nabla\theta$.

From (3-2), using Stokes' theorem and the mass conservation law, we can derive the usual conservation equation of PV. In particular, the sign of the absolute circulation $\int_C d\boldsymbol{X}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X})$ on C is equal to the sign of PV there.

3.b Initial disturbance

As is defined in section 3.a, C is a material closed curve on an iso-θ surface. Here and hereafter, let C be so small that the direction of $\nabla\theta$ may be regarded to be uniform on and inside of C. In addition to C, another closed curve C' is defined. Let C' be such a material closed curve on the same iso-θ surface that C is included inside of C', at time t. The difference between C and C' is assumed to be infinitesimal.

At this time t, the fluid particles from C to C' are displaced, in such a way that the conservation law (3-2) is satisfied. That is,

$$\int_C d\boldsymbol{X}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}) = \int_{C'} d\boldsymbol{X}'\cdot(\boldsymbol{V}^P+\boldsymbol{\Omega}\times\boldsymbol{X}'). \tag{3-3}$$

Here, $\boldsymbol{X}'$ is the position vector on C', and $d\boldsymbol{X}'$ is the line element vector on C'. The vector $\boldsymbol{V}^P$ is the velocity that the fluid particle, which is displaced from $\boldsymbol{X}$ on C to $\boldsymbol{X}'$ on C', must have in order to satisfy the conservation constraint (3-3). According to the parcel method, the pressure gradient force is assumed not to be altered in this infinitesimal displacement. Then, from (3-1), the force exerted on the fluid particle displaced from $\boldsymbol{X}$ on C to $\boldsymbol{X}'$ on C' is

$$d\boldsymbol{V}^P/dt = -\theta\nabla\pi' - 2\boldsymbol{\Omega}\times\boldsymbol{V}^P - \nabla\phi'. \tag{3-4}$$

Here, $\pi' = \pi(\boldsymbol{X}', t)$ and $\phi' = \phi(\boldsymbol{X}', t)$.

If the displacement (i.e., disturbance) is absent, the fluid particle, which was present on C' remains on C'. The force exerted on this fluid particle, which is originally present on C', is given by (3-1) at $\boldsymbol{X}'$.

$$d\boldsymbol{V}'/dt = -\theta\nabla\pi' - 2\boldsymbol{\Omega}\times\boldsymbol{V}' - \nabla\phi'. \tag{3-5}$$

Here, $\boldsymbol{V}' = \boldsymbol{V}(\boldsymbol{X}', t)$. The anomalous force $\boldsymbol{F}$ felt by the fluid element displaced from $\boldsymbol{X}$ on C to $\boldsymbol{X}'$ on C' is the difference between (3-4) and (3-5).

$$\begin{aligned}\boldsymbol{F} &= d\boldsymbol{V}^P/dt - d\boldsymbol{V}'/dt \\ &= -2\boldsymbol{\Omega}\times\boldsymbol{V}^P + 2\boldsymbol{\Omega}\times\boldsymbol{V}'.\end{aligned} \tag{3-6}$$

The anomalous force $\boldsymbol{F}$ is the extra force which is caused by the infinitesimal displacement, i.e., by the disturbance. If the extra force $\boldsymbol{F}$ amplifies the displacement, then the disturbance grows, and then the flow is unstable.

3.c Instability criterion

As is defined in section 3.b, C' is a material closed curve on an iso-θ surface. On and inside of C', the direction of $\nabla\theta$ may be regarded as uniform. Let $\boldsymbol{n}$ be the unit vector parallel to $\nabla\theta$. Since the line element vector $d\boldsymbol{X}'$ on C' lies on the iso-θ surface, the vectors $\boldsymbol{n}$ and $d\boldsymbol{X}'$ are perpendicular to each other.

Here, a vector element $d\xi$ is defined, which is the vector product of $\boldsymbol{n}$ and $d\boldsymbol{X}'$.

$$d\xi = d\boldsymbol{X}' \times \boldsymbol{n}, \quad \boldsymbol{n} \propto \nabla\theta \quad \text{and} \quad |\boldsymbol{n}| = 1. \tag{3-7}$$

Since $\boldsymbol{n}$ and $d\boldsymbol{X}'$ are perpendicular to each other, the length of $d\xi$ is the line element of C', i.e., $|d\xi| = |d\boldsymbol{X}'|$, and the direction of $d\xi$ is outward-normal to C'.

Let us consider the following line integral of the extra force (3-6) on C'.

$$\int_{C'} d\xi \cdot \boldsymbol{F} = \int_{C'} d\xi \cdot (2\boldsymbol{\Omega} \times \boldsymbol{V}' - 2\boldsymbol{\Omega} \times \boldsymbol{V}^P). \tag{3-8}$$

After substitution of (3-7), and some manipulations, the Eq. (3-8) becomes

$$\int_{C'} d\xi \cdot \boldsymbol{F} = 2(\boldsymbol{\Omega} \cdot \boldsymbol{n}) \int_{C'} d\boldsymbol{X}' \cdot (\boldsymbol{V}^P - \boldsymbol{V}') - 2\int_{C'} (d\boldsymbol{X}' \cdot \boldsymbol{\Omega})\{\boldsymbol{n} \cdot (\boldsymbol{V}^P - \boldsymbol{V}')\}. \tag{3-9}$$

If the basic flow is stationary, i.e., $\partial/\partial t = 0$, the flow velocity is perpendicular to $\nabla\theta$, because of the adiabatic assumption. This means that $\boldsymbol{n} \cdot \boldsymbol{V}'$ in (3-9) vanishes, since $\boldsymbol{n} \propto \nabla\theta$. The velocity of the displaced fluid particle is forced to change from $\boldsymbol{V}$ to $\boldsymbol{V}^P$, in order to satisfy the conservation constraint (3-3). Since the conservation constraint (3-3) says nothing about the vector component parallel to $\nabla\theta$, the component of $\boldsymbol{V}^P$ parallel to $\nabla\theta$ is the same as that of $\boldsymbol{V}$, which is zero because of the adiabatic assumption. Further eliminating $\boldsymbol{V}^P$ from (3-3) and (3-9), we obtain the following equation.

$$\int_{C'} d\xi \cdot \boldsymbol{F} = -2(\boldsymbol{\Omega} \cdot \boldsymbol{n}) \int_{C'-C} d\boldsymbol{X} \cdot (\boldsymbol{V} + 2\boldsymbol{\Omega} \times \boldsymbol{X}). \tag{3-10}$$

The integral in (3-10) is the absolute circulation around the region between C' and C. So, the sign of the integral is equal to the sign of PV there.

We consider such a displacement $\delta\boldsymbol{\eta}$ from C to C' that is proportional to $d\xi$. If the sign of $(\boldsymbol{\Omega} \cdot \nabla\theta)PV$ is negative, the sign of $\int_{C'} d\xi \cdot \boldsymbol{F}$ is positive. The positiveness of $\int_{C'} d\xi \cdot \boldsymbol{F}$ means that the scalar product of the displacement and force is positive on average, and therefore that the kinetic energy is produced compared with the disturbance-free case. That is, the disturbance grows. As a result, $(\boldsymbol{\Omega} \cdot \nabla\theta)PV < 0$ implies instability.

4. Conclusion

As is well known, when a basic flow with linear or axial symmetry has a region of negative potential vorticity (*PV* hereafter), such a disturbance with the same symmetry as the basic flow grows that converts the negative *PV* into non-negative *PV*. This is the so-called symmetric instability.

The proof of conventional symmetric instability is crucially dependent on the symmetry of basic flow. That is, the conventional proof says nothing about the instability of an asymmetric basic flow with negative *PV*. In the real atmosphere, a region of regative *PV* seems to be unstable, seems not to persist. Indeed, *PV* is positive (in the nothern hemisphere) in most parts of the atmosphere.

In this note, the following was shown in a nondissipative adiabatic system. Irrespective of the symmetry, disturbances grow in the region of negative *PV*. That is, irrespective of the symmetry, a flow with negative *PV* is unstable. More precisely speaking, the negativeness of *PV* means that the sign of *PV* is opposite to the sign of $\boldsymbol{\Omega} \cdot \nabla\theta$. Here, $\boldsymbol{\Omega}$ is the angular velocity of the earth, and θ is the potential temperature.

In the hydrostatic case on a f-plane, assuming that the vertical gradient of θ is everywhere positive, and taking θ as the vertical coordinate, we could show that the basic flow (not necessarily steady) with negative *PV* becomes unstable. In the general case (i.e., nonhydrostatic and on the rotating sphere), we could show the instability only of the steady basic flow with negative *PV*.

The proof is based on the conservation law of absolute circulation on the iso-θ surface. Together with the mass conservation, the law is of course equivalent to the conservation law of *PV*.

The instability was shown by employing the parcel method. A particular initial disturbance is assumed by the parcel method. However, the particularity has no problem, because the existence of at least one growing disturbance is sufficient for the instability. The parcel method can say nothing about the form of disturbance, or about its growth rate. These are left for future works.

Appendix

Let C be a material closed curve on an iso-θ surface. Because of the adiabatic assumption, the future C at time $t > 0$ remains to lie on the same iso-θ surface as the initial $C(0)$ at time $t = 0$.

Let us consider the material temporal derivative of the absolute circulation around the region encircled by C.

$$(d/dt)\int_C d\boldsymbol{X}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}). \tag{A-1}$$

Here $\boldsymbol{X}$ is the position vector on C, $d\boldsymbol{X}$ is the line element vector on C, $\boldsymbol{V}$ is the wind velocity, and $\boldsymbol{\Omega}$ is the angular velocity of the earth. The direction of integration is clockwise viewing in the direction of $\nabla\theta$. The integral (A-1) on C at time t can be rewritten as an integral on $C(0)$ at time $t = 0$, and then the differentiation with respect to t can be performed inside of the integral symbol.

$$\begin{aligned}
(d/dt)&\int_C d\boldsymbol{X}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}) \\
&= (d/dt)\int_{C(0)} d\boldsymbol{X}^{(0)}\cdot\{\nabla^{(0)}\boldsymbol{X}\}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}) \\
&= \int_{C(0)} d\boldsymbol{X}^{(0)}\cdot\{\nabla^{(0)}\boldsymbol{V}\}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}) \\
&\quad + \int_{C(0)} d\boldsymbol{X}^{(0)}\cdot\{\nabla^{(0)}\boldsymbol{X}\}\cdot(d\boldsymbol{V}/dt+\boldsymbol{\Omega}\times\boldsymbol{V}) \\
&= \int_C d\boldsymbol{V}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}) \\
&\quad + \int_C d\boldsymbol{X}\cdot(d\boldsymbol{V}/dt+\boldsymbol{\Omega}\times\boldsymbol{V}) \\
&= \int_C d\boldsymbol{V}\cdot\boldsymbol{\Omega}\times\boldsymbol{X} \\
&\quad + \int_C d\boldsymbol{X}\cdot(d\boldsymbol{V}/dt+\boldsymbol{\Omega}\times\boldsymbol{V}).
\end{aligned} \tag{A-2}$$

Here $\boldsymbol{X}^{(0)}$ is the initial position vector on $C(0)$ corresponding to $\boldsymbol{X}$ on C, $d\boldsymbol{X}^{(0)}$ is the initial line element vector on $C(0)$ corresponding to $d\boldsymbol{X}$ on C, and $\nabla^{(0)} = \partial/\partial\boldsymbol{X}^{(0)}$ is the partial differential operator with respect to the initial position. Further, partially integrating, substituting (3-1) into (A-2), and noticing that θ is constant on C, we obtain the following conservation equation.

$$(d/dt)\int_C d\boldsymbol{X}\cdot(\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{X}) = 0. \tag{A-3}$$

In the hydrostatic system, almost the same result is obtained, except that $d\boldsymbol{X}$, $\boldsymbol{V}$ and $\boldsymbol{X}$ in (A-3) are replaced with their horizontal projections, and that $\boldsymbol{\Omega}$ is replaced with $(f/2)\boldsymbol{e}$.

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