On Decision-Making Under Uncertainty

A Brief Study of Decision-Making Under "Complete" Uncertainty

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Abstract

This paper is based on a report at *Nihon Zaimu-Kanri Gakkai*, at Bunkyo Joshi University, June 21, 1997. Contents of the report are just as listed below, however only the last item will be able to be reviewed here due to space limitation. This article will introducing the system of C. R. Barrett and P. K. Pattanaik, and make some interpretations of it within the context of G. L. S. Shackle's view. In section 6 of this paper, an evaluation of the axiomatic system will be given. The system under review is for a personal rational decision making under "complete" uncertainty; no objective or subjective probability can be defined on states of nature of outcomes. All of proofs had to be omitted for want of space.

i. Shackle's view of economic choice, and of probability with respect to Keynes [1921].

ii. A brief review of the notion and axioms of Shackle's *PS* function and expected utilities.

iii. Decision-making under "risk" vs. "complete uncertainty," and an explanation of notation of "uncertainty."

iv. A brief review of a model by Barrett and Pattanaik [1994].

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1 Preferences

X: the set of all conceivable outcomes. $\succeq$: the agent's weak preference relation (*wpr*; *at least as good as*) over X. $\succ$: asymmetric factor of $\succeq$. $\sim$: symmetric factor of $\succeq$. $\succeq$: assumed to be an ordering. Some $x, y, z \in X$, $x \succ y \succ z$: assumed.

There exist at least three distinct "indifference classes" in X, defined in terms of $\succeq$. Z: the class of all conceivable finite sets (elements are possible states of nature); $S, S' \in Z$, $|S| = 2$ and $|S'| = 3$. For all $S \in Z$, $A(S)$: the set of all functions $a[S \to X]$. (The elements of $A(S)$: *actions*.)

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$S = \{s_1, \cdots, s_n\} \in Z$ and $a \in A(S)$. For all i, $1 \le i \le n, a(s_i) = x_i$. For every $S \in Z$, the agent has a *wpr* R over $A(S)$. P: asymmetric factor of R. I: symmetric factor of R.

	s_1	s_2	$\cdots$	s_n
a	x_1	x_1	$\cdots$	x_1
b	y_1	y_1	$\cdots$	y_1

2 Axioms

Axiom 1: *Rationality* For all $S \in Z$, R is an ordering over $A(S)$.

Axiom 2: *Quasi-rationality* For all $S \in Z$, R is *reflexive* and P is *transitive* over $A(S)$.

Definition 3: *Dominance* For all $S \in Z$ and for all $a, b \in A(S)$, a dominates b iff for all $s \in S$, $a(s) \succeq b(s)$ and for *some* $s \in S$, $a(s) \succ b(s)$.

Axiom 4: *Dominance* For all $S \in Z$ and for all $a, b \in A(S)$, if a dominates b, aPb.

Axiom 5: *Weak dominance* For all $S \in Z$ and for all $a, b \in A(S)$, if a dominates b, aRb.

Axiom 6: *Symmetry* Let $S, S' \in Z, (|S| = |S'|)$, and let $a, b \in A(S)$ and $c, d \in A(S')$. In case g is one-to-one function from S to S', $a(s) \sim c(g(s))$ and $b(s) \sim d(g(s))$. Then aRb iff $cR'd$.

Axiom 7: *Independence of the ranking of irrelevant outcomes* Let $S \in Z$, and let $a, b, c, d \in A(S)$ be sure that, for all $s, s' \in S$

$[a(s) \succeq a(s')$ iff $c(s) \succeq c(s')]$;

$[b(s) \succeq b(s')$ iff $d(s) \succeq d(s')]$;

$[a(s) \succeq b(s')$ iff $c(s) \succeq d(s')]$;

$[b(s) \succeq a(s')$ iff $d(s) \succeq c(s')]$.

Then aRb iff cRd.

Notation 8: *Restriction* Let $T \subset S \subset Z$ and $a \in A(S)$. a/T: restriction of a to T.

Notation 9: *Restriction and mapping* Let $S, S' \in Z$; $s_i, s_j \in S$; $s_k \in S'$; $a, b \in A(S)$; $c, d \in A(S')$. $(S; s_i, s_j) \to (S'; s_k)$ iff $S - S' = \{s_k\}$ and $S - S' = \{s_i, s_j\}$.

$(S; s_i, s_j; a, b) \to (S'; s_k : c, d)$, iff $S - S' = \{s_k\}$ and $S - S' = \{s_i, s_j\}$.

$(S; s_i, s_j; a, b) \to (S; s_k : c, d)$

$$\text{iff}[(S; s_i, s_j) \to (S'; s_k) \text{ and } a/S \cap S' = c/S \cap S' \text{ and } b/S \cap S' = d/S \cap S$$
$$\text{and } a(s_i) = a(s_j) = c(s_k) \text{ and } b(s_i) = b(s_j) = d(s_k)].$$

Axiom 10: *Invariance with respect to merger of states* Given $S \in Z$ and $s_i, s_j \in S$, there exist $S' \in X$ and $s_k \in S'$, such that

$$[(S; s_i, s_j) \to (S'; s_k) \text{ and, for all } a, b \in A(S) \text{ and } c, d \in A(S'),$$
$$\text{iff } (S; s_i, s_j; a, b) \to (S'; s_k : c, d), \text{then } (aRb \text{ if } cR'd)].$$

Definition 11: *Relative positions of s under permutations* Let $S \in Z$, $s \in S$ and $a, b \in A(S)$.

$s_1, \cdots, s_{|S|}$: Arbitrary ordering of $s \in S$.
L: The set of all permutations of $\{1, \cdots, |S|\}$.
The relative positions of s under a and b are similar
iff there exist $\ell, \ell' \in L$ and $k, (1 \leq k \leq |S|)$, such that $s = s_{\ell(k)} = s_{\ell'(k)}$ and, for $1 \leq i < j \leq |S|$, $a\left(s_{\ell(j)}\right) \succeq a\left(s_{\ell(i)}\right)$ and $b\left(s_{\ell'(j)}\right) \succeq b\left(s_{\ell'(i)}\right)$.

Axiom 12: *Weak invariance with respect to merger of states* Given $S \in Z$ and $s_i, s_j \in S$, there exist $S' \in Z$ and $s_k \in S'$, such that
$[(S; s_i, s_j) \rightarrow (S'; s_k)$ and, for all $a, b \in S$ and $c, d \in S'$
if $(S; s_i, s_j; a, b) \rightarrow (S'; s_k : c, d)$ and the relative positions of both s_i and s_j under a and b are similar, then $(aRb$ iff $cR'd)]$.

Axiom 13: *Independence of common outcomes* Let $T \subset S \in Z$, and let $a, b, c, d \in A(S)$.
$[a/S - T = c/S - T$ and $b/S - T = d/S - T$ and $a/T = b/T$ and $c/T = d/T]$
then aRb if cRd.

Axiom 14: *Semi-independence of common outcomes* Let $T \subset S \in Z$, and let $a, b, c, d \in A(S)$.
$[a/S - T = c/S - T$ and $b/S - T = d/S - T$ and $a/T = b/T$ and for all $s \in T$, $c(s) \succ d(s)]$
then aPb implies cPd.

Axiom 15: *Ranking of sure outcomes* Let $S \in Z$ and $x, y \in X$, and let $a, b \in A(S)$.
For all $s \in S$, $a(s) = x \succ y = b(s)$.
Then aPb.

3 Axioms for Choice

Axiom 16: *Existence of local pessimism* There exist $S = \{s, s'\} \in Z$, and $a, b \in A(S)$ such that $a(s) \succ b(s) \sim b(s') \succ a(s')$, and bPa.

Axiom 17: *Existence of local optimism* There exist $S = \{s, s'\} \in Z$, and $a, b \in A(S)$ such that $a(s) \succ b(s) \sim b(s') \succ a(s)$, and aPb.

Axiom 18: *Local absence of pessimism and optimism* There exist $S = \{s, s'\} \in Z$, and $a, b \in A(S)$
such that $a(s) \succ b(s) \sim b(s') \succ a(s)$,
and not $(bPa$ or $aPb)$.

Notation 19: *Maximum and minimum of outcomes* Given $S \in Z$, and $a \in A(S)$,
$m(a)$: Least outcome in $a(S)$.
$M(a)$: Greatest outcome in $a(S)$.

Axiom 20: *Existence of local weak pessimism* There exist $S \in Z$, and $a, b \in A(S)$, such that $M(a) \succ M(b)$ and bPa.

Axiom 21: *Existence of local weak optimism* There exist $S \in Z$, and $a, b \in A(S)$, such that $m(a) \succ m(b)$ and bPa.

4 Propositions

Proposition 1 Suppose $S \in Z$ and $a, b \in A(S)$. Assume the agent satisfies
(1) axioms 1, 5, 6, 7, 12, or
(2) axioms 2, 4, 6, 7, 12. Then

Proposition 1-1 Under axiom 16, $m(a) \succ m(b)$ implies aPb.

Proposition 1-2 Under axiom 17, $M(a) \succ M(b)$ implies aPb.

Proposition 1-3 Under axiom 18, $[m(a) \succ m(b)$ and $M(b) \succ M(a)]$ implies not $(aPb$ or $bPa)$.

Proposition 2 Suppose the agent satisfies
(1) axioms 1, 5, 6, 7, 12, or
(2) axioms 2, 4, 6, 7, 12. Then

Proposition 2-1 Axiom 16 holds iff axiom 20 holds.

Proposition 2-2 Axiom 17 holds iff axiom 21 holds.

Proposition 3: *Arrow and Hurwicz [1972]* Assume axioms 1, 5, 6, 10. There exists an ordering $\Re$ over $X \times X$
such that, for all $S \in Z$ and for all $a, b \in A(S)$,
aRb iff $(m(a), M(a)) \Re (m(b), M(b))$.

5 Decision Rules under the Propositions

Definition 1a: *Maximin criterion* For all $S \in Z$ and for all $a, b \in A(S)$, aRb iff $m(a) \succeq m(b)$.

Proposition 2a: *Maximin criterion* Agent follows the maximin criterion iff axioms 1, 5, 6, 7, 10, 14, 16 are satisfied.

Definition 1b: *Maximax criterion* For all $S \in Z$ and for all $a, b \in A(S)$, aRb iff $M(a) \succeq M(b)$.

Proposition 2b: *Maximax criterion* Agent follows the maximax criterion iff axioms 1, 5, 6, 7, 10, 14, 17 are satisfied.

Notation 3: *Restriction of mapping* Let $S \in Z$ and $a, b \in A(S)$, then $a * b$: restriction of a to $\{s \in S \,|\text{not } a(s) \sim b(s)\}$.
$m(a * b)$: least outcomes. $M(a * b)$: greatest outcomes. (in the range of $a * b$.)

Definition 4a: *Min-based protective criterion* For all $S \in Z$ and for all $a, b \in A(S)$, aPb iff $m(a * b) \succ m(b * a)$.

Proposition 5a: *Min-based protective criterion* Agent follows min-based protective criterion
iff axioms 2, 4, 6, 7, 10, 13, 16 are satisfied.

Definition 4b: *Max-based protective criterion* For all $S \in Z$ and for all $a, b \in A(S)$, aPb iff $M(a * b) \succ M(b * a)$.

Proposition 5b: *Max-based protective criterion* Agent follows max-based protective criterion
iff axioms 2, 4, 6, 7, 10, 13, 17 are satisfied.

Notation 6: *Cardinality* Let $S \in Z$,
$x \in X$ and $a \in A(S)$. $n(x, a)$: cardinality of $\{s \in S \,| a(s) \sim x\}$.

Definition 7a: *Leximin criterion* For all $S \in Z$ and for all $a, b \in A(S)$,
[aRb iff not exist $x \in X$ such that $n(x, a) > n(x, b)$ and,
for all $y \in X$, xPy implies $n(y, a) = n(y, b)$].

Proposition 8a: *Leximin criterion* Agent follows leximin criterion
iff axioms 1, 4, 6, 7, 12, 13, 16 are satisfied.

Definition 7b: *Leximax criterion* For all $S \in Z$ and for all $a, b \in A(S)$,
[aRb iff not exist $x \in X$ such that $n(x, a) < n(x, b)$ and,
for all $y \in X$, yPx implies $n(y, a) = n(y, b)$].

Proposition 8b: *Leximax criterion* Agent follows leximax criterion
iff axioms 1, 4, 6, 7, 12, 13, 17 are satisfied.

6 Remarks

Basic axioms are listed in section 2. All of these are for making the point at issue clear from Barrett and Pattanaik [1994]' s place: defining a list of axioms and propositions in compatible with some decision making rules, maximin and minimax, leximin and leximax, and so on.

From the beginning point for making their system as an ordinal over outcomes, the way that to summarize all the available and relevant information about the agent's preferences over outcomes should be crucial for describing rational choice under "complete" uncertainty. The so called Neumann and Morgenstern's utilities need sets of axioms (1) supposing the existence of probabilities or (2) alluding to the existence of probabilities as in Savage's. These basic axioms show preferences. Axioms aim at generalized and diverse rationality of choice under uncertainty, and systems built by these include the ordinal system. Axioms 4 (or 5), 6 and 7 are indispensable for the nature of ordinal approach of Barrett and Pattanaik; however these axioms are quite usual. Axioms 10 (or 12), 13 (or 14), and 15 are essential for the agent's ordering over outcomes and his/her ranking of actions. In Shackle's words, focusing on relevant outcomes corresponds to these axioms. In other words, axiom 10 (and also 12) shows the ranking agent's choice with the restrictive mapping of the outcomes of actions.

Through section 3 to 4, we can see how a rational agent behaves under complete uncertainty in the presence of these axioms. These axioms show certain sets of assumptions impose on the agent a severe type of uniformity concerning agent's possible pessimism or optimism. As in the lexicographic ordering system, and in Shackle's focusing approach †, in the presence of these axioms, the existence of a minimal local amount of pessimism (optimism) is escalated or spread into universal pessimism (optimism).

Barrett and Pattanaik show how well-known usual decision criteria are designated as a result of ordinal descriptions. Their remarks mean, though decision criteria, peculiar to Shackle's model kept intact, cannot consider these decision criteria, Barrett and Pattanaik [1994] does by setting a starting point to make axioms apply at a place where the agent has no probabilities on outcomes the same as Shackle [1952]. Now we should be able to make Shackle's concept of the agent's decision environments remain valid, and could characterize his idea as decisions between of the expected utility hypothesis and of the criteria provided by lexicographic order.

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† Shackle's model is fairly interpreted into a context of expected utility hypothesis under the conditions of further strict definitions on measure of uncertainty of expected and unknowable outcomes (*PS*). See Takayabu [1995]. His *Primary Focus Function* is corresponds to weighting of information of outcomes, or cognitive body of agent in Prospect theory, Kahneman and Tversky [1990].

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