J. For. Res. 3: 67–74 (1998)

Representation of Tree Stem Taper Curves and Their Dynamic, Using a Linear Model and the Centroaffine Transformation[1]

Dieter Gaffrey,*, [2] Branislav Sloboda,* and Naoto Matsumura**

*University of Göttingen, Faculty of Forest Sciences and Forest Ecology, Institute of Forest Biometry and Informatics, Büsgenweg 4, 37077 Göttingen, Germany.
**Shikoku Research Center, Forestry and Forest Products Research Institute, Kochi 780–8064, Japan.

One main task of forestry is a reliable estimation of the stem form and its development applied in calculating total and log volume. As long as process-oriented models are not available for this practical use, empirical models must serve instead. Taper curve data of trees within stands normally show a rank maintenance, *i.e.*, a tree which has a greater diameter than another one at a certain height, is most probably bigger at any other height, too. This property also applies to the analysed tree species, sugi (*Cryptomeria japonica*) and hinoki (*Chamaecyparis obtusa*), and will be well-represented by a linear model formulation. As model parameter sets of single stands have a very limited time validity, two approaches for future stem form prediction are discussed. The one, the centroaffine transformation of a linear taper curve function, is not suitable for representing the time-depending change of the stem form. However, this can be done by a linear unit taper curve model, the parameters of which are based on sample trees of stands of several age classes. Temporary unit parameter sets are derived for sugi and hinoki and the estimated stand volumes are compared to the real ones to evaluate the model performance, which turned out to be very good.

Key words: centroaffine transformation, hinoki, linear taper curve model, sugi, taper curve development

One of the most important aspects of forestry is the description of tree stem taper curves, as the utilization of logs depends essentially on their form. The development of the taper curve over time is influenced by a multitude of events (*e.g.*, Larson, 1963) and of especial interest are those which can be controlled by man. However, the comprehension of tree growth processes is the basic prerequisite for treating a tree or a stand in a way that will lead to a desired result.

The roots of research on stem taper formation date back to the last century (Pressler, 1865; Hartig, 1871). Apart from mechanically oriented theories (Tirén, 1928; Ylinen, 1952), Pressler and Hartig introduced the aspect of photosynthesis production and its allocation, which is of central importance in modern theories such as the 'pipe model' (Shinozaki *et al.*, 1964a, b) and derived stem taper models (Oohata and Shinozaki, 1979; Chiba, 1990; Chiba and Shinozaki, 1994). Detailed empirical analyses on pruning (Fujimori, 1970; Takeuchi and Hatiya, 1977) demonstrate how to achieve a desired stem form, using the knowledge of the close relation between crown and stem taper development.

Unfortunately, at present, physiologically-related models which predict the stem taper curve and its dynamic are not available for practical use (Deleuze and Houllier, 1995). Therefore, this demand must yet be satisfied by models with a mainly descriptive character.

When regarding all tree stems within a stand, a linear model, which estimates the taper curve by the diameter at breast height (d.b.h.), performs very well (Sloboda, 1984, 1985). Its independency of tree species and stand age has been proved by a great variety of applications (Gaffrey, 1988 (*Picea abies*), 1996 (*Pseudotsuga menziesii*); Böckmann, 1990 (*Tilia cordata*); Röös, 1990 (*Prunus avium*); Lee, 1993 (*Pinus densiflora*) and further tree species such as *Populus spec.*, *Fagus sylvatica*, *Pinus sylvestris*, *Sorbus aucuparia*; for more details see Gaffrey, 1996).

In this contribution, a detailed presentation of the linear model approach is given first. This model, which was originally conceived for a specific stand of a defined age, is extended to a timeless as well as to a unit model. Alternatively, the estimation of future stem taper curves using a centroaffine transformation (Sloboda, 1976, 1977) is proved.

Materials

Data of the Japanese tree species sugi (*Cryptomeria japonica*) and hinoki (*Chamaecyparis obtusa*) serve for model parametrization and verification. They consist of stem analysis data recording in five-year intervals, which have been published by the University Forest in Chichibu & Laboratory of Forest Management in Tokyo (1987).

The data used for taper curve models contain three collectives for sugi (compartments 1a13, 1a15, 6a1) and hinoki (1a13, 3a11, 6a1), respectively. Stems with an obvious abnormal shape, caused by crown break, were sorted out. All trees are relatively young, the age most often ranges from 30 to 40, but seldom over 50 years.

Methods

1 The linear taper curve model and its applications

The object of interest of a stem taper curve model can be a single tree or a whole collective of trees. In reference to the latter, a more or less great variation in stem form can be observed within a stand. Of course, a taper curve model should be able to handle this variety, moreover the internal structural conception should coincide with the data structures.

Until now, all investigated tree collectives showed a typical

[1] We would like to express our gratitude to the Japan Science and Technology Corporation for the financial promotion, which enabled this research work.

[2] Corresponding author.

stochastic tendency of maintaining rank in regard to the stem taper form. This means that a tree which has a greater diameter than another one at a fixed, absolute height (*e. g.*, 1.3 m) is most probably bigger at any other height, too (Fig. 1(a)).

The comparison of trees with different top heights requires a normalization of the length. Let H_i denote the top height of the ith tree and X an absolute height value in the interval $[0, H_i]$. Then the transformation $x := X/H_i$, $x \in [0, 1]$ results in trees with a unit height, which also show the tendency of maintaining rank (Fig. 1(b)).

Regression analyses of all n normalized stems of any collective show, for any relative point x_k, that there are very straight relationships between the diameters at breast height D^i_{13} iand the diameters $D_i(x_k)$ at point x_k (Fig. 1(c)). This justifies the linear formulation

$$\hat{D}(x_k, D_{13}) = a_1^k \cdot D_{13} + a_2^k, \qquad x_k \in [0, 1]$$

to estimate the diameter $\hat{D}$ at the height x_k. When regarding $k = 1, \ldots, m$ relative positions x_k, there will be m pairs of regression coefficients a_1^k and a_2^k. If there are continuous, interpolating functions $\alpha_1(x)$ and $\alpha_2(x)$ over x, $x \in [0, 1]$, the function

$$\hat{D}(x, D_{13}) = \alpha_1(x) \cdot D_{13} + \alpha_2(x), \qquad x \in [0, 1] \quad (1.1a)$$

estimates the relative stem taper and

$$\hat{D}(X, H, D_{13}) = \alpha_1\left(\frac{X}{H}\right) \cdot D_{13} + \alpha_2\left(\frac{X}{H}\right), \quad X \in [0, H] \qquad (1.1b)$$

the absolute one. If additional information, for instance the diameter at a height of seven meters (D_7), exists, it should be included in (1.1) to reduce the residual variance (Sloboda, 1984, 1985).

$$\hat{D}(x, D_{13}, D_7) = \alpha_1(x) \cdot D_{13} + \alpha_2(x) \cdot D_7 + \alpha_3(x),$$
$$x \in [0, 1] \qquad (1.2)$$

In this case, however, model (1.2) will not be further developed, because in most cases only the d.b.h. is available.

The functions $\alpha_1(x)$ and $\alpha_2(x)$ must be now specified. Initially, methodical reasons require the transformation $\mathrm{x} := 1 - x$ of the stem curves, resulting in the trees' tops being positioned into the origin. The m gradients α_1^k and intercepts α_2^k, respectively, can be interpolated over x, $\mathrm{x} \in [0, 1]$ by polynomials of nth degree without constants (Fig. 1(d)).

$$\alpha_i(\mathrm{x}) := \alpha_i(\mathrm{x}, \vec{a}_i) = \sum_{j=1}^{n} a_{ij} \cdot \mathrm{x}^j, \quad \mathrm{x} \in [0, 1]$$
$$\vec{a}_i = [a_{i1}, \ldots, a_{in}], \quad i = 1, 2 \qquad (1.3)$$

A concrete determination of a polynomial parameter set for a stand demands more focus on practical details. Questions about the necessary number and the selection method of the tree probands, the choice of the relative positions for the linear regressions and of the polynomial degree must be answered.

A stable parameter estimation can be achieved by choosing (10 to) 20 trees, if their diameters are equally spread over the stand diameter interval. There is no improvement using random sampling, but the required sample size must be increased considerably to obtain the same accuracy (Dehn *et al.*, 1985).

The selection of the relative positions x_k shall gaurantee a satisfying representation of the stem taper. Normally, 20 to 40 equidistantly spread points are sufficient. The first point should not be the foot of the tree because exactly measured values are rare or are, in the case of a thick buttress, not useful. Therefore, a position which lies a little bit above the ground should be chosen (Gaffrey, 1996).

For both polynomials, a degree of six is enough in most cases. Sometimes a higher one is necessary, but a degree of more than eight is not advisable in order to avoid the danger of overshooting. Fortunately, slight deviations from an exact interpolation are not so important (Gaffrey, 1996).

The most important application of stem taper curve models is the prediction of a single tree's assortment(s) or of the assortment distribution of a stand or a subset of it. Calculating the log volume $V_{[h_1, h_2]}$ by integrating from a lower value h_1 (*e. g.*, the tree stump's height) to an upper value h_2 can be done by

$$V_{[h_1, h_2]} = \frac{\pi}{4} \cdot \int_{h_1}^{h_2} \left[\alpha_1\left(\frac{\mathrm{X}}{H}, \vec{a}_1\right) \cdot D_{13} + \alpha_2\left(\frac{\mathrm{X}}{H}, \vec{a}_2\right)\right]^2 d\mathrm{X},$$
$$X := H - \mathrm{X}, \quad X \in [0, H], \quad 0 \le h_1 < h_2 \le H. \qquad (1.4)$$

On the other hand, taper curve models are often used for the construction of volume tables. The integral volume $V_{[0, H]}$ of the total stem from foot to top can be easily calculated using a vector of form factors $\vec{F} = (F_1, F_2, F_3)$ (Sloboda, 1984; Gaffrey, 1988):

$$V_{[0, H]} = \frac{\pi}{4} \cdot \left(F_1 \cdot H \cdot D_{13}^2 + F_2 \cdot H \cdot D_{13} + F_3 \cdot H\right),$$
$$F_1 = \int_0^1 \left[\alpha_1(\mathrm{x}, \vec{a}_1)\right]^2 d\mathrm{x},$$
$$F_2 = 2 \cdot \int_0^1 \alpha_1(\mathrm{x}, \vec{a}_1) \cdot \alpha_2(\mathrm{x}, \vec{a}_2) d\mathrm{x}, \qquad (1.5)$$
$$F_3 = \int_0^1 \left[\alpha_2(\mathrm{x}, \vec{a}_2)\right]^2 d\mathrm{x}.$$

The component F_1 represents the known form factor. The use of this single reduction factor would be justified if all observed intercepts a_2^k would be equal to zero, but none of the analysed stands showed this property. Therefore, the application of the complete vector is recommended.

Lastly let us take a closer look at the object of the application of the linear taper curve model, the 'tree collective.' It

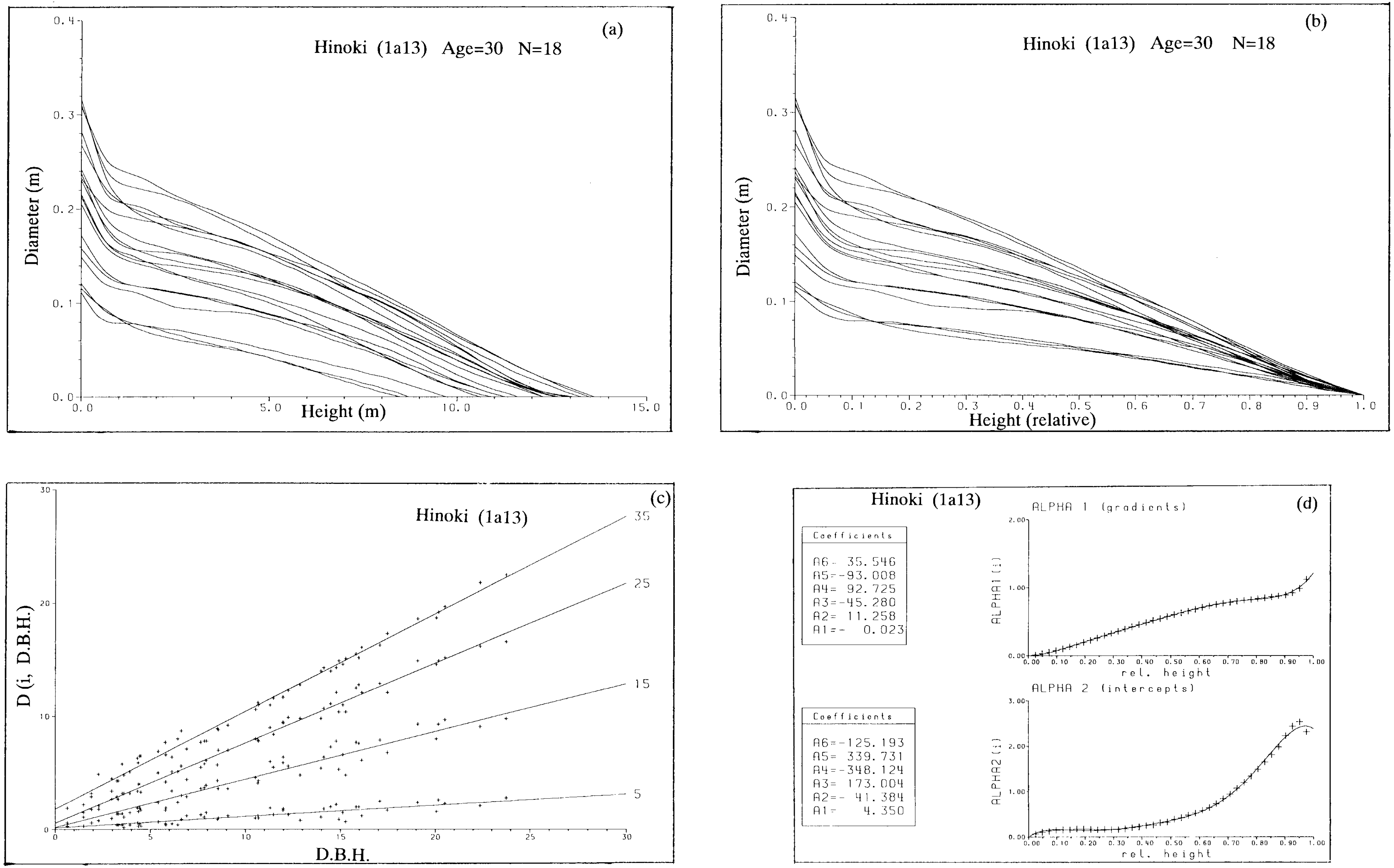


Fig. 1 (a) Rank maintenance of stem taper curves with absolute height values. (b) Rank maintenance of stem taper curves with normalized height values. (c) Linear relationships between d.b.h. and the diameter at the relative heights $x_k = 1 - k/41$, $k = 5, 15, 25, 35$. (d) Polynomial interpolation of the linear regression coefficients, the gradients and the intercepts.

shall first be defined by a 'single stand collective' because within a stand, rank maintenance has been developed due to inter-tree competition. As analyses show that stem taper formation is similar in many stands, the sample trees of these stands can be united in a 'unit stand collective' and its stem form structure can be represented by a unit model parameter set. (The problems of how to define similarity and to realize clustering are not to be discussed here and now (see Gaffrey, 1994, 1996).) Lastly, there is a very special application by using stem analysis taper curves of only one tree as a 'single tree collective.' This can be useful to estimate missing taper curves, if data are only measured at only a few points in time.

A further important aspect of taper curve models is its validity over time. The model function (1.1) is a static one because it does not include time as an argument. Thus, the applicability of a single stand parameter set is limited, for the future relative stem form structure in a stand will differ due to the initial stand age and the length of the considered growing period. However, by using unit parameter sets, there is a way to guarantee a time-independent validity. If a unit stand consists of stands of different age (age classes), the derived parameters contain an inherent time dynamic. They can then be applied to all stands (classifiable to a certain unit stand) with an age that lies within the age range of the unit stand.

The question arises whether long-term prognoses for stand taper curves are possible, if there is no time intrinsic unit stand model but only a static single stand model. Assuming a taper curve function exists for a certain stand age, an adjustment for a future taper curve can probably be done by an equiform or a centroaffine transformation — if the trees' stems really follow such a growth pattern.

2 Centroaffine transformation of taper curves

One way to describe the time-dependent change of a stem taper curve is its adjustment by an equiform or a centroaffine transformation. The theory was presented by Sloboda (1976, 1977) and has been proved for spruce (*Picea abies*). In this study, the general applicability of this method in combination with the linear stem taper curve equation is to be tested for sugi and hinoki. At first, the mathematical derivation of the transformed taper curve and volume functions will be given in detail.

It is assumed that a continuous diameter taper curve function φ_t of a tree exists at a time t with top height H_t and basal diameter D_0^t.

$$\varphi_t(X), \quad X \in [0, H_t] \tag{2.1}$$

If equiform stem growth will occur during the time interval $[t, \tau]$, $\tau = t + \Delta t$, then the equation

$$\lambda(t, \Delta t) = \frac{H_\tau}{H_t} = \frac{D_0^\tau}{D_0^t} \tag{2.2}$$

must hold. For simplification, the definition $\lambda := \lambda(t, \Delta t)$ is chosen. Analyses of tree growth, for any given species, will normally show an inequality of the two ratios. That means

$$\lambda_H = \frac{H_\tau}{H_t} \neq \frac{D_0^\tau}{D_0^t} = \lambda_{D_0}. \tag{2.3}$$

Stretching a taper curve, given by a function φ_t, with the factors λ_H and λ_{D_0} in x- and in y-direction, respectively, represents a centroaffine transformation with its fixed point in the origin. The time adjusted function is then

$$\varphi_t(X) = \lambda_{D_0} \cdot \varphi_t(X / \lambda_H), \quad X \in [0, H_\tau]. \tag{2.4}$$

The use of the total stem volume V_t at time t, calculated by

$$V_t = \frac{\pi}{4} \cdot \int_0^{H_t} \varphi_t^2(X) dX, \tag{2.5}$$

allows a very easy computation of the volume V_τ at time τ:

$$V_\tau = \lambda_{D_0}^2 \cdot \lambda_H \cdot V_t. \tag{2.6}$$

With the aid of the equations (2.4) and (2.6), stem analysis data can be tested as to whether stem form development is centroaffine, or not.

Referring to the linear stem form model given above, the required function φ_t shall be specified by the linear function (1.1). As the basal diameter D_0 is usually not measured, but rather the d.b.h., there is a need of defining a stretching factor $\lambda_{D_{13}}$ for the diameter at breast height.

$$\lambda_{D_{13}} = \frac{D_{13}^\tau}{D_{13}^t} \tag{2.7a}$$

If differences are slight, $\lambda_{D_{13}}$ may substitute λ_{D_0}.

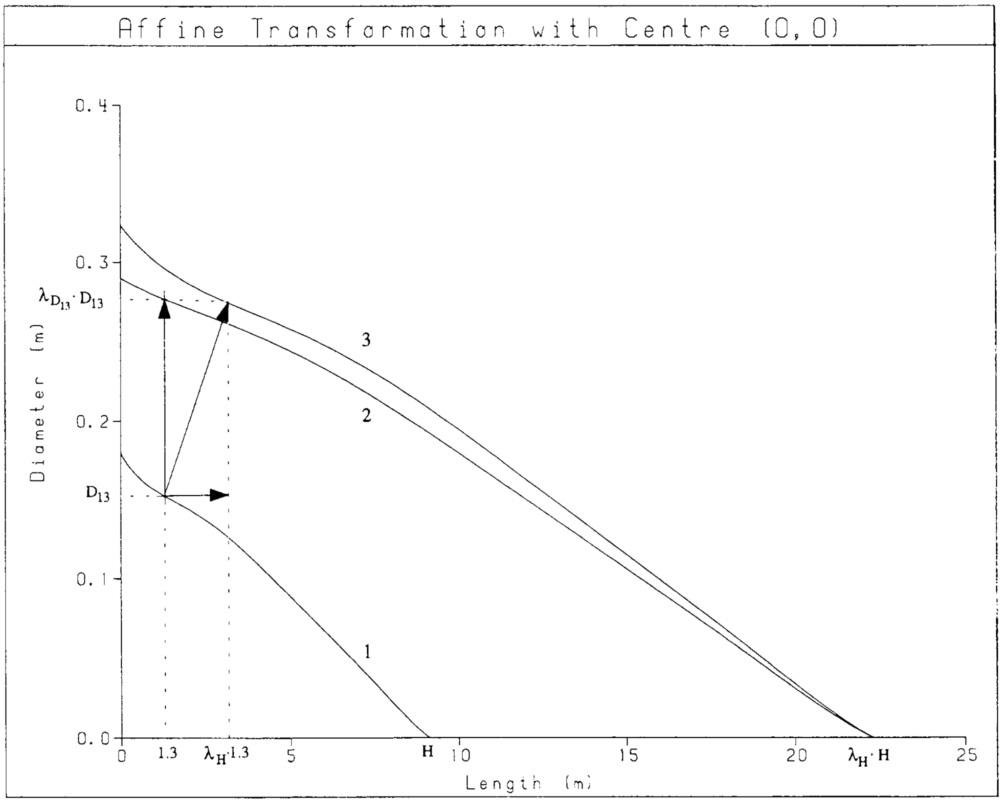
However, using $\lambda_{D_{13}}$ and the d.b.h.-dependent taper curve function (1.1) in (2.4) causes a problem, as you can see in Fig. 2(a). Line 1 describes a taper curve at time t and lines 2 and 3 taper curves of the same tree at time τ. Assuming the d.b.h. at time t (cross on line 1) is known as well as at time τ (cross on line 2), the centroaffine transformation with $\lambda_{D_{13}}$ and λ_H results in a taper curve (line 3) which overestimates considerably, compared to the expected curve 2. In explanation, the transformation of a single point, the d.b.h. coordinate (1.3, D_{13}^t) is indicated. Its movement has not only a vector component in y-direction, but also in x-direction, which has to be avoided for this point.

Translating the taper curve prior to transformation solves the problem. After a transformation parallel to the x-axis by 1.3 m to the left ($X' = X - 1.3$), the abscissa of the d.b.h. point is zero (Fig. 2(b)) and will not be affected by the height stretching factor. It should be mentioned that this operation causes a minor defect in representing the butt swell (curve 2 *vs.* curve 3 in Fig. 2(a)). It is not the foot, but a point somewhere above it, which is transformed into the future foot. Normally, this error can be neglected as the butt swell is of less importance for the estimation of usable log volume.

The parallel translation requires a correction of the height stretching factor. Instead of λ_H, now λ'_H is given by

$$\lambda'_H = \frac{H_\tau - 1.3}{H_t - 1.3}. \tag{2.7b}$$

(a)



(b)

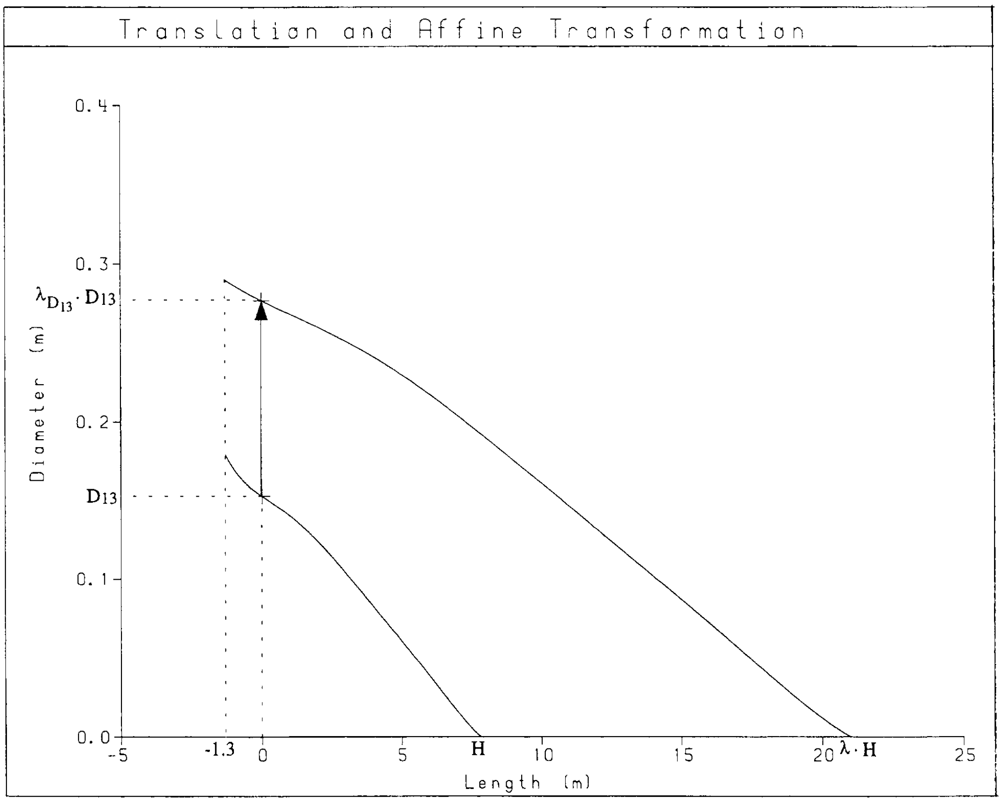


Fig. 2 (a) Centroaffine transformation of the linear taper curve model (1). The use of $\lambda_{D_{13}}$ instead of λ_{D_0} leads to an overestimation of the future taper curve (3), whereas curve (2) is to be expected. (b) A translation of the taper curve of 1.3 m to the left, which preceeds the centroaffine transformation, solves the problem. Only a minor defect in representing the butt swell still remains.

By applying the modifications to (2.4), we obtain

$$\varphi(X')=\lambda_{D_{13}}\cdot\varphi_t(X'/\lambda'_H+1.3),\quad X'\in[-1.3, H_\tau-1.3] \tag{2.8}$$

and involving the normalized linear taper curve model gives

$$\begin{aligned}\varphi(X')&=\lambda_{D_{13}}\cdot\varphi_t\left(\frac{X'/\lambda'_H+1.3}{H_t}\right)\\&=\lambda_{D_{13}}\cdot\left[\alpha'_1\left(\frac{X'/\lambda'_H+1.3}{H_t}\right)\cdot D'_{13}\right.\\&\left.+\alpha'_2\left(\frac{X'/\lambda'_H+1.3}{H_t}\right)\right],\\&X'\in[-1.3, H_\tau-1.3].\end{aligned} \tag{2.9}$$

Unfortunately, the volume V_τ cannot be derived from the volume V_t in such an easy manner as in (2.6), for in the following integral, the lower boundary x_1 is not equal to zero:

$$\begin{aligned}V_\tau&=\frac{\pi}{4}\cdot\int_{-1.3}^{H_\tau-1.3}\varphi_\tau^2(X')dX'\\&=\frac{\pi}{4}\cdot\lambda_{D_{13}}^2\cdot\lambda'_H\cdot H_t\cdot\int_{x_1}^{x_2}\varphi_t^2(x)dx,\\x&:=\frac{X'/\lambda'_H+1.3}{H_t},\ x_1=\frac{-1.3/\lambda'_H+1.3}{H_t},\ x_2=1.\end{aligned} \tag{2.10a}$$

Estimating V_τ similarly to (2.6) requires a reduction, as λ'_H is greater than λ_H.

$$V_\tau=\lambda_{D_{13}}^2\cdot\lambda'_H\cdot\left[V_t-\frac{\pi}{4}\cdot H_t\cdot\int_0^{x_1}\varphi_t^2(x)dx\right] \tag{2.10b}$$

Data analyses have to first prove whether the centroaffine transformation can principally predict stem taper curve development in a satisfying way. In that case, the next important task will be to find a practicable approach for estimating well-performing stretching factors $\lambda_{D_{13}}$ and λ'_H.

Results and Discussion

There is no necessity to use all available data for testing the assumption of equiform or centroaffine tree growth. If a centroaffine growth model shall be applicable in general, then analyses may concentrate on some few typically developed stems.

Sugi and hinoki do not at all show an equiform growth pattern. This observation is assumed to be universally valid for other tree species, too.

When now regarding the centroaffine model, the demanded prerequisite of equality between λ_{D_0} and $\lambda_{D_{13}}$ is probably not met. Decisions are difficult when arguing on the mere figures of the factors (*e. g.*, the stretching factors for the taper curve development from age 20 to 50 of the sugi tree no.8, stand 1a15 are: $\lambda_{D_0}(20, 50)=1.731$, $\lambda_{D_{13}}=1.801$.) Actually, there is no need to answer, as there is no real alternative to the use of $\lambda_{D_{13}}$. The recommended way of comparing the transformed stem taper curve to the real one is the visual evaluation of the plotted graphs (Fig. 3). When referring to the chosen

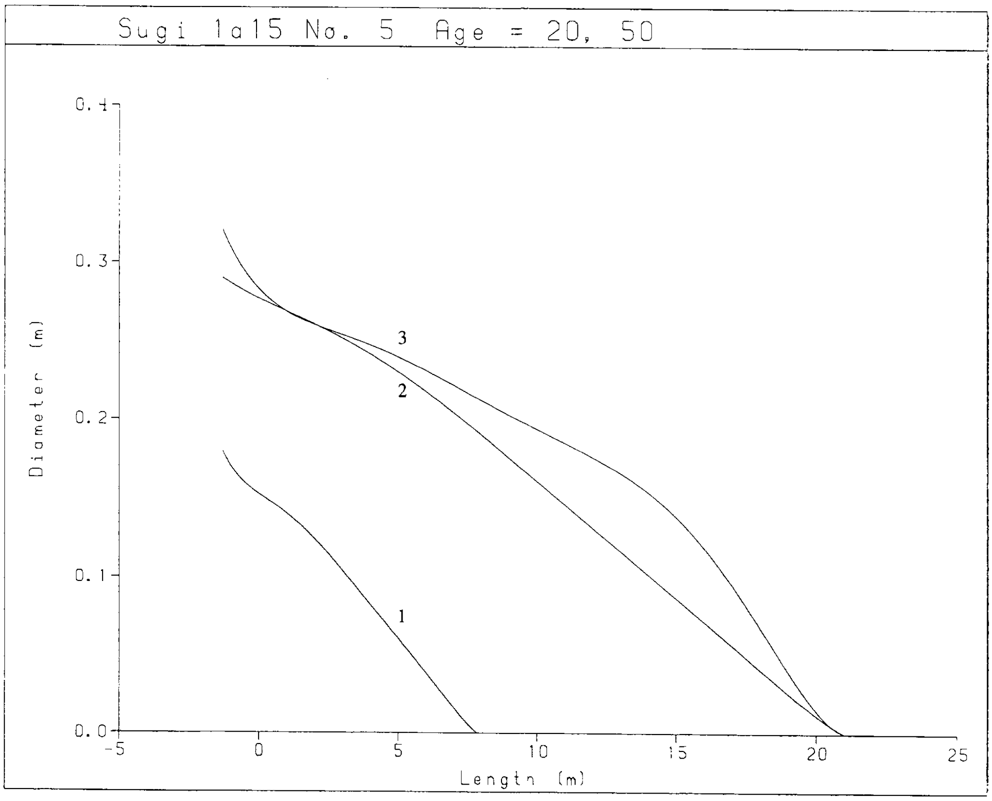


Fig. 3 Sugi and hinoki trees do not show a centroaffine development of their stem taper curves. (1) and (2) are a tree's taper curves at age 20 and 50, respectively, whereas curve (2) is the prognosticated one.

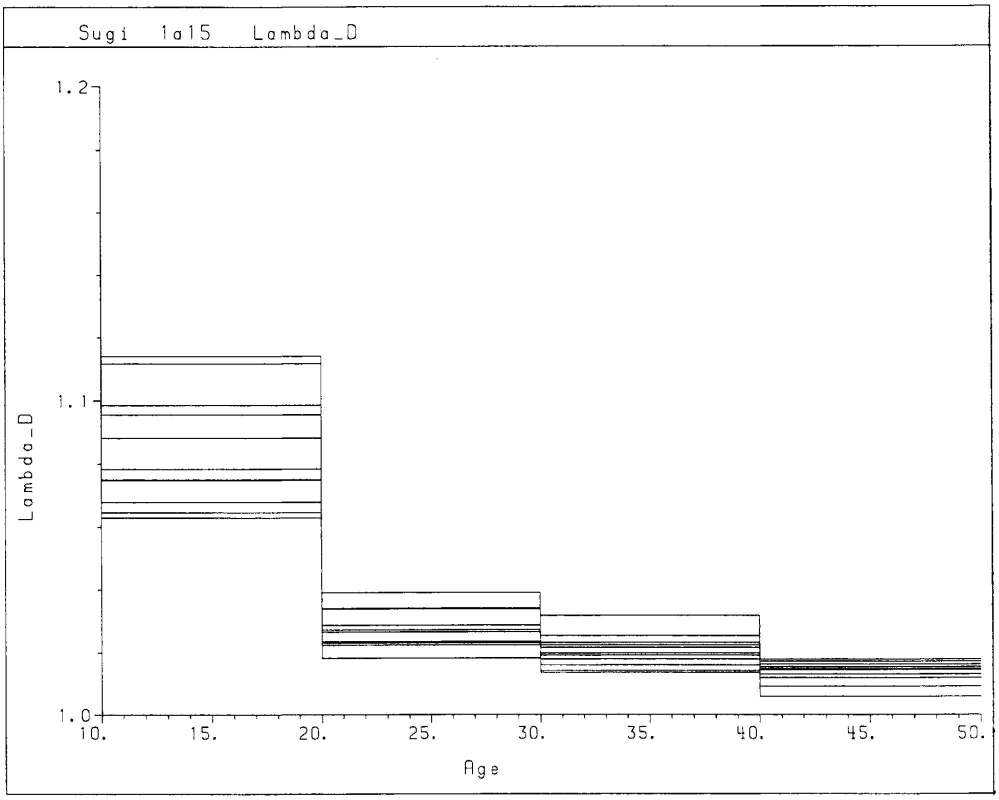


Fig. 4 The stretching factors (λ_{D_0} = Lambda_D) depend on age, the length of the time prognosis and are tree-individual.

tree, the taper curve development is typical of the other sample trees, too. It must be stated that the centroaffine transformation fails to represent the decrease in taper, which is almost always observed in growing closed stands. In addition to the optical comparison, the tree's stem volume is calculated. There is an underestimation by more than 17% for a 30-year prediction period (age 50) (−6%, age 30; −12%, age 40). In contrast, the application of the tree-specific linear taper curve model results in deviations of only +2% (age 30), +1% (age 40) and −2% (age 50).

Thus, it must be concluded that stem form dynamic cannot be represented by a centroaffine transformation. This saves the difficult additional work of estimating well-performing stretching factors, for as you see in Fig. 4, they are tree-specific, as well as age-dependent.

The remaining way to predict future stem form exists in the application of a linear unit taper curve model, which is valid within a wide range of time. For its parametrization, the stem analysis data of the selected stands were used and from each tree the taper curve data of each ten years were chosen. Subsequently all data of sugi and hinoki, respectively, were merged and processed to obtain the unit stand parameter sets for these tree species.

The polynomial coefficients and the form factor vectors are given in Table 1(a) and 1(b). The presented parameter sets are temporary because of the very restricted data base. The sugi model is valid for trees within a d.b.h. range of 5 to 35 cm and for hinoki from 5 to 30 cm. Therefore, a prediction of taper curves of larger trees is not covered by the models' data bases.

Table 1

(a) Parameter set of the sugi unit taper curve model.

Polynomial coefficients				Form factor vector	
a_{11}	1.23587	a_{21}	−4.39138	F_1	0.431258
a_{12}	7.67383	a_{22}	−12.41204	F_2	0.009319
a_{13}	−38.99138	a_{23}	132.6382	F_3	0.000114
a_{14}	77.57639	a_{24}	−288.62230		
a_{15}	−71.93629	a_{25}	264.09990		
a_{16}	25.52121	a_{26}	−88.91119		

(b) Parameter set of the hinoki unit taper curve model.

Polynomial coefficients				Form factor vector	
a_{11}	0.47869	a_{21}	1.25226	F_1	0.413828
a_{12}	10.69494	a_{22}	−40.44562	F_2	0.009922
a_{13}	−46.84522	a_{23}	203.65450	F_3	0.000092
a_{14}	94.20950	a_{24}	−410.19420		
a_{15}	−91.10876	a_{25}	382.00300		
a_{16}	33.75589	a_{26}	−134.16870		

The evaluation of the unit models' performance is done by comparing the exact stand volumes (computed by stem-individual spline functions) to the estimated ones (Table 2(a), 2(b)). The maximum error for sugi and hinoki does not exceed 5% and the average is less than 2%. This superior result has to be relativized because of the low number of tested stands. As experiences with other tree species show (Gaffrey, 1996), normally an average error of 3−5% and a maximum error of 8−10% can be expected. By the way, it should be mentioned that single or unit stand taper curve models, regardless of the mathematical function used, are not suitable in predicting a single tree's volume with the same precision. Here the calculated average error is about 6−

Table 2

(a) Accuracy of stand volume estimations using the sugi unit taper curve model (volumes in m^3).

Collective	Number of Obs.	Spline volume	Model volume	Difference (%)
Sugi 1a13	48	6.3477	6.3239	−0.4
Sugi 1a15	67	13.4966	12.8329	−4.9
Sugi 6a1	36	16.4453	16.8046	2.2

(b) Accuracy of stand volume estimations using the hinoki unit taper curve model (volumes in m^3)

Collective	Number of Obs.	Spline volume	Model volume	Difference (%)
Hinoki 1a13	54	3.1229	3.1315	0.3
Hinoki 3a11	60	5.2882	5.2177	−1.3
Hinoki 6a1	36	3.6754	3.6410	−0.9

8%, but the maximum can be 20% or even more. An error reduction is only achievable by using specific single-tree models.

Data of both examined Japanese tree species, sugi and hinoki, show the same linear structure of maintaining rank as any other species analysed, so far. Thus, it is advised to choose a corresponding, linear model type. Other model approaches are possible in principle, but they generally perform less well (Lee, 1993). Despite the advantages of the linear (unit) taper curve model, it should be kept in mind that its main weakness is the lack of causal explanation, for the observed linearity itself does not contribute to a deeper insight of stem form formation.

Literature cited

Böckmann, T. (1990) Wachstum und Ertrag der Winterlinde (*Tilia cordata* Mill.) in Niedersachsen und Nordhessen. Dissertation, 143 pp, Univ. Göttingen, Forstwiss. Fachbereich.

Chiba, Y. (1990) Plant form analysis based on the pipe model theory. I. A statical model within the crown. Ecol. Res. 5: 207−220.

Chiba, T. and Shinozaki, K. (1994) A simple mathematical model of growth pattern in tree stems. Ann. Bot. 73: 91−98.

Dehn, R., Taube, D., and Sloboda, B. (1985) Schaftvermessung an stehenden Bäumen mit einem eindimensionalen Paβpunktsystem. Allg. Forstztg. 15: 350−353.

Deleuze, C. and Houllier, F. (1995) Prediction of stem profile of *Picea abies* using a process-based tree growth model. Tree Physiol. 15: 113−120.

Fujimori, T. (1970) Fundamental studies on pruning I. Discussion about pruning on the basis of an ecological research. Bull. Gov. For. Exp. Stat. 228: 1−38. (in Japanese with English summary)

Gaffrey, D. (1988) Forstamts-und bestandesindividuelles Sortimentierungsprogramm als Mittel zur Planung, Aushaltung und Simulation. Thesis, 86 pp, Univ. Göttingen, Forstwiss. Fachbereich.

Gaffrey, D. (1994) Regionales, baumartenspezifisches Sortenmodell zur bestandesindividuellen Sortenprognose. *In* Proceedings of Deutscher Verband Forstlicher Forschungsanstalten, Sektion Forstl. Biometrie u. Informatik in Martin/Zvolen. Sloboda, B. and Šmelko, Š. (eds.), 182−204.

Gaffrey, D. (1996) Sortenorientiertes Bestandeswachstums-Simulationmodell auf der Basis intraspezifischen, konkurrenzbedingten Einzelbaumwachstums—insbesondere hinsichtlich des Durchmessers—am Beispiel der Douglasie. Berichte des Forschungszentrums Waldökosysteme, Reihe A 133, 413 pp.

Hartig, R. (1871) Ueber das Dickenwachsthum der Waldbäume. Zeitschrift für Forst- u. Jagdwesen 3: 66−104.

Larson, P. R. (1963) Stem form development of forest trees. For. Sci. Monogr. 5: 1−41.

Lee, W.-K. (1993) Wachstums-und Ertragsmodelle für *Pinus densiflora* in der Kangwon-Provinz, Korea. Dissertation, 178 pp, Univ. Göttingen, Forstwiss. Fachbereich.

Oohata, S. and Shinozaki, K. (1979) A statistical model of plant form—further analysis of the pipe model theory. Jpn. J. Ecol. 29: 323−335.

Pressler, M. R. (1865) Das Gesetz der Stammbildung und dessen forstwirtschaftliche Bedeutung für den Waldbau höchsten Reinertrags. 153 pp, Arnold, Leipzig.

Röös, M. (1990) Zum Wachstum der Vogelkirsche (*Prunus avium* L.) in Nordrhein-Westfalen und angrenzenden Gebieten. Dissertation, 162 pp, Univ. Göttingen, Forstwiss. Fachbereich.

Shinozaki, K., Yoda, K., Hozumi, K., and Kira, T. (1964a) A quantitative analysis of plant form — the pipe-model theory. I. Basic analysis. Jpn. J. Ecol. 14: 97−105.

Shinozaki, K., Yoda, K., Hozumi, K., and Kira, T. (1964b) A quantitative analysis of plant form — the pipe-model theory. II. Further evidence of the theory and its applications in forest ecology. Jpn. J. Ecol. 14: 133−139.

Sloboda, B. (1976) Mathematische und stochastische Modelle zur Beschreibung der Statik und Dynamik von Bäumen und Beständen, insbesondere das bestandesspezifische Wachstum als stochastischer Prozeβ. Habilitation thesis, 265 pp, Freiburg.

Sloboda, B. (1977) Die Beschreibung der Dynamik der Schaftformfortpflanzung mit Hilfe der Ähnlichkeitsdifferentialgleichung und der Affinität. Mitteilungen d. forstl. Bundesversuchsanstalt Wien 120: 53−60.

Sloboda, B. (1984) Bestandesindividuelles biometrisches Schaftformmodell zur Darstellung und zum Vergleich von Formigkeit und Sortimentausbeute sowie Inventur. *In* Proceedings of Deutscher Verband Forstlicher Forschungsanstalten, Sektion Ertragskunde in Neustadt a. d. Weinstraβe, 4/1−4/36.

Sloboda, B. (1985) Bestandesindividuelles biometrisches Schaftformmodell zur Darstellung und zum Vergleich von Formigkeit und Sortimentausbeute sowie Inventur. *In* Proceedings of the IUFRO Conference Zürich, Eidgenössische Anstalt für das Forstliche Versuchswesen, Birmensdorf, 345−353.

Takeuchi, I. and Hatiya, K. (1977) Effect of pruning on growth (I). A pruning experiment on model stands of *Cryptomeria japonica*. J. Jpn. For. Soc. 59: 313−320. (in Japanese with English summary)

Tirén, L. (1928) Einige Untersuchungen über die Schaftform. Meddelanden från statens skogsförsöksanstalt (Reports of the Forest Research Institute of Sweden) 24: 81−152.

University Forest in Chichibu & Laboratory of Forest Management, Department of Forestry; Faculty of Agriculture, Univ. of Tokyo (1987) Records of the Stem Analyses in the Tokyo University Forest in Chichibu. Misc. Inf. the Tokyo Univ. For. 25: 161−204. (in Japanese)

Ylinen, A. (1952) Über die mechanische Schaftformtheorie der Bäume. Techn. Universität in Helsinki. Wissenschaftl. Forschungen 87: 1−57.

(Accepted February 18, 1998)