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On the Heaving Motion of a Circular Disk in Shallow Water

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Summary

The hydrodynamic forces on a circular disk as a shallow draft ship for the forced heaving oscillation in shallow water are investigated. The boundary value problem is formulated by the use of the concept of the surface distributed sources so that integral equations for the source densities are obtained. In the case of long waves, the problem is solved analitically. The numerical solution of the integral equations is found and the added mass and damping factor are calculated. The exciting forces on a fixed circular disk in an incident wave are calculated according to Haskind-Newman's relation. The influence of the shallow water effect on the forces is illustrated graphically and discussed.

1. Introduction

When large size of floating type offshore structures such as airport, atomic power plant and so on are considered, the draft of the structures and the water depth of seas where the structures will be operated may be considered relatively as shallow draft the limiting case of which is that of a zero draft structure, and as shallow water, respectively. C. H. Kim¹⁾ investigated the shallow water effect on two dimensional problem.

The shallow draft problems on a circular disk in deep water were investigated by MacCamy²) and W. D. Kim³). This paper extends some of these ideas to the shallow water problems of a circular disk. The numerical procedure is based on source distribution method.

The wave exciting forces are obtained according to the Haskind-Newman's relation in three dimensional shallow water.

This paper concerns only the theoretical analysis on above problem.

2. Formulation of the Problem

A Cartesian coordinate system is defined with its origin on a mean free surface of a fluid as shown in Fig. 1. Let $\phi e^{-i\omega t}$ be the velocity potential which refers to a fluid motion due to heave oscillation, of unit velocity amplitude, of a circular disk *C* floating on the free surface of the fluid. The corresponding boundary value problem may be written as follows

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$
 in fluid
 $\phi_z - K\phi = 0$ on the free surface (z=0)



Fig. 1 Coordinates

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 $\phi_z = 1$ on C $\phi_z = 0$ on the bottom of the finid (z = -h) ϕ represents a diverging wave as $\sqrt{x^2 + y^2} \rightarrow \infty$

where deep water wave number $K = \omega^2/g$, and g is the acceleration of gravity. Let G(x, y, z, a, b, c) be the velocity potential of the point source or the so-called Green function with its singularity located at a point (a, b, c). G is obtained as [John⁴), Thorne⁵, Wehausen⁶]

$$G_{s} = G_{c} + iG_{s}$$

$$G_{c} = \frac{1}{r} + \frac{1}{r_{2}} + \int_{0}^{\infty} \frac{2(k+K)e^{-kh}\cosh k(c+h)\cosh k(z+h)}{k\sinh kh - K\cosh kh} J_{0}(kR)dk$$

$$G_{s} = 2\pi \frac{(m_{0}+K)e^{-m_{0}h}\sinh m_{0}h}{Kh+\sinh^{2}m_{0}h}\cosh m_{0}(c+h)\cosh m_{0}(z+h)J_{0}(m_{0}R)$$
(2.2)

where \oint refers to the principal value of the integral,

$$\left. \begin{array}{l} r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \\ r_2 = \sqrt{(x-a)^2 + (y-b)^2 + (z+c+2h)^2} \\ R = \sqrt{(x-a)^2 + (y-b)^2} \end{array} \right\}$$

$$(2.3)$$

and shallow water wave number m_0

$$m_0 \tanh m_0 h = K \quad (m_0 > 0).$$

The Green function G defined above leads to the expression of the velocity potential ϕ by means of source distribution, that is

$$\phi(P) = \frac{1}{4\pi} \iint_{\sigma} \sigma(Q) G(P, Q) dS_Q, \qquad (2.4)$$

where P and Q refer to points (x, y, z) in fluid and (a, b, c) on C respectively, and σ is the density of the distribution. From the boundary condition on C, the following integral equation is obtained:

$$1 = \sigma(P) + \frac{1}{4\pi} \iint_{C} \sigma(Q) \frac{\partial G(P, Q)}{\partial z} dS_{Q} \qquad P \in C.$$
(2.5)

Substitution of the free surface condition:

$$\frac{\partial G(P, Q)}{\partial z} - KG(P, Q) = 0 \quad \text{on } z = 0$$

leads to the following version of the integral equation (2.5) [Kim⁸)]

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$$1 = \sigma(P) + \frac{K}{4\pi} \iint_{\sigma} \sigma(Q) G(P, Q) dS_Q \qquad P \in C.$$
(2.6)

The existence of the solution of the integral equation (2.6) was proved by W. D. Kim [Kim³), page 9. (25)]. Using the solution σ of (2.6), the velocity potential ϕ on C can be written as

$$\phi(P) = \frac{1 - \sigma(P)}{K} \qquad P \in C.$$
(2.7)

The added mass \bar{M} and the damping \bar{N} are now given as

$$\bar{M} = \rho \iint_{c} \phi_{c} dS = \rho \iint_{c} \frac{1 - \sigma_{c}}{K} dS
\bar{N} = \rho \omega \iint_{c} \phi_{s} dS = -\rho \omega \iint_{c} \frac{\sigma_{s}}{K} dS ,$$
(2.8)

where ρ is the density of the fluid, and subscripts c and s refer to the real and imaginary parts of the corresponding quantity.

3. Numerical Procedure to Solve the Integral Equation

Using the cylindrical coordinates (ρ_p, θ_p, z) and (ρ_q, θ_q, c) as shown in Fig. 2, $R = \sqrt{(x-a)^2 + (y-b)^{2z}}$ can be written as

(2.1)

$$R = \sqrt{\rho_P^2 + \rho_Q^2 - 2\rho_P \rho_Q \cos\left(\theta_P - \theta_Q\right)}.$$
 (3.1)

 $J_0(m_0R)$ is then decomposed into a Fourier series by the addition theorem of the Bessel function [Watson¹⁴)]:

$$J_{0}(m_{0}R) = J_{0}(m_{0}\sqrt{\rho_{P}^{2} + \rho_{Q}^{2} - 2\rho_{P}\rho_{Q}\cos\left(\theta_{P} - \theta_{Q}\right)})$$

$$= J_{0}(m_{0}\rho_{P})J_{0}(m_{0}\rho_{Q})$$

$$+ 2\sum_{n=1}^{\infty}J_{n}(m_{0}\rho_{P})J_{n}(m_{0}\rho_{Q})\cos\left(\theta_{P} - \theta_{Q}\right). \quad (3.2)$$

Since, in the case of heave oscillation of a circular disk, $\sigma(Q) = \sigma_c(Q) + i\sigma_s(Q) \ Q \in C$ is a function of ρ_Q alone,

 $\iint_{\sigma} \sigma(\rho_Q) G_{s}(P, Q) dS_Q$

may be reduced to the simpler form:

$$\frac{K}{4\pi} \iint_{\mathcal{C}} \sigma(\rho_Q) G_{\mathfrak{s}}(P, Q) dS_Q \qquad \text{Fig. 2 Cylindrical coordinates}$$
$$= \frac{KA(z, 0)}{4\pi} \iint_{\mathcal{C}} \sigma(\rho_Q) J_0(m_0 R) dS_Q = \left[\frac{KA(z, 0)}{4\pi} \iint_{\mathcal{C}} \sigma(\rho_Q) J_0(m_0 \rho_Q) dS_Q\right] J_0(m_0 \rho_P), \quad (3.3)$$

where

$$A(z, c) = 2\pi \frac{(m_0 + K)e^{-m_0h}\sinh m_0h}{Kh + \sinh^2 m_0h} \cosh m_0(c+h) \cosh m_0(z+h) .$$
(3.4)

The integral equation (2.6) now can be written as a set of simultaneous integral equation, that is

$$1 = \sigma_{c}(\rho_{P}) + \frac{K}{4\pi} \iint_{c} \sigma_{c}(\rho_{Q})G_{c}(P, Q)dS_{Q} - P_{s} \cdot J_{0}(m_{0}\rho_{P}) \qquad P \in C$$

$$0 = \sigma_{s}(\rho_{P}) + \frac{K}{4\pi} \iint_{c} \sigma_{s}(\rho_{Q})G_{c}(P, Q)dS_{Q} + P_{o} \cdot J_{0}(m_{0}\rho_{P}) \qquad P \in C$$

$$P_{o} = \frac{KA(0, 0)}{4\pi} \iint_{c} \sigma_{c}(\rho_{Q})J_{0}(m_{0}\rho_{Q})dS_{Q}$$

$$P_{s} = \frac{KA(0, 0)}{4\pi} \iint_{c} \sigma_{s}(\rho_{Q})J_{0}(m_{0}\rho_{Q})dS_{Q}$$

$$(3.5)$$

Let $\gamma^1(\rho_Q)$ and $\gamma^d(\rho_Q)$, $Q \in C$, be the solutions of the following integral equations respectively, that is

$$-J_0(m_0\rho_P) = \gamma^1(\rho_P) + \frac{K}{4\pi} \iint_C \gamma^1(\rho_Q) G_c(P, Q) dS_Q \qquad P \in C$$
(3.6)

and

$$1 = \gamma^{d}(\rho_{P}) + \frac{K}{4\pi} \iint_{\mathcal{O}} \gamma^{d}(\rho_{Q}) G_{c}(P, Q) dS_{Q} \qquad P \in C.$$
(3.7)

The above integral equations for unknowns γ^1 and γ^d are Fredholm equations of the second kind. Then σ_o and σ_s may be written as

$$\sigma_{c} = \gamma^{d} - P_{s} \cdot \gamma^{1}$$

$$\sigma_{s} = P_{c} \cdot \gamma^{1}$$

$$(3.8)$$

where P_{σ} and P_{s} are yet solved. Substitution of (3.8) into the equations in (3.5), which define P_{σ} and P_{s} , leads to the solution:

$$P_c = P_d / (1 + P_1^2)$$
, $P_s = P_1 \cdot P_c$, (3.9)

where

$$P_{1} P_{d} = \frac{KA(0, 0)}{4\pi} \iint_{C} \left\{ \gamma^{1}(\rho_{Q}) \atop \gamma^{d}(\rho_{Q}) \right\} J_{0}(m_{0}\rho_{Q}) dS_{Q} .$$

$$(3.10)$$

In the case of a shallow water of the constant finite depth, the real part G_c of the Green function G can be expanded into countable infinite number of component waves as [John⁴), Wehausen et al⁶]

$$G_{c} = -2\pi \frac{m_{0}^{2} - K^{2}}{hm_{0}^{2} - hK^{2} + K} \cosh m_{0}(c+h) \cosh m_{0}(z+h) Y_{0}(m_{0}R)$$





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$$+4\sum_{n=1}^{\infty}\frac{m_n^2+K^2}{hm_n^2+hK^2-K}\cos m_n(c+h)\cos m_n(z+h)K_0(m_nR).$$
(3.11)

 $K_0(m_n R)$ and $Y_0(m_0 R)$ in (3.11) can, furthermore, be expanded by the addition theorem of Bessel functions [Watson¹⁴] as follows, where $m_n \tan m_n h = -K$

a) in the case of $\rho_P > \rho_Q$,

$$K_{0}(m_{n}R) = K_{0}(m_{n}\rho_{P})I_{0}(m_{n}\rho_{Q}) + 2\sum_{p=1}^{\infty} K_{p}(m_{n}\rho_{P})I_{p}(m_{n}\rho_{Q})\cos p(\theta_{P} - \theta_{Q})$$

$$Y_{0}(m_{0}R) = Y_{0}(m_{0}\rho_{P})J_{0}(m_{0}\rho_{Q}) + 2\sum_{n=1}^{\infty} Y_{n}(m_{0}\rho_{P})J_{n}(m_{0}\rho_{Q})\cos n(\theta_{P} - \theta_{Q})$$
(3.12 a)

b) in the case of $\rho_P < \rho_Q$,

$$K_{0}(m_{n}R) = I_{0}(m_{n}\rho_{P})K_{0}(m_{n}\rho_{Q}) + 2\sum_{p=1}^{\infty} I_{p}(m_{n}\rho_{P})K_{p}(m_{n}\rho_{Q})\cos p(\theta_{P} - \theta_{Q})$$

$$Y_{0}(m_{0}R) = J_{0}(m_{0}\rho_{P})Y_{0}(m_{0}\rho_{Q}) + 2\sum_{n=1}^{\infty} J_{n}(m_{0}\rho_{P})Y_{n}(m_{0}\rho_{Q})\cos n(\theta_{P} - \theta_{Q})$$
(3.12 b)

(3.11) and (3.12) lead to the following expression of $\int_0^{2\pi} G_o d\theta_Q$:

a) in the case of $\rho_P > \rho_Q$,

$$\int_{0}^{2\pi} G_{c} d\theta_{Q} = \int_{0}^{2\pi} G_{c}(\rho_{P}, \theta_{P}, z, \rho_{Q}, \theta_{Q}, c) d\theta_{Q}$$

= $A_{0} J_{0}(m_{0}\rho_{Q}) + \sum_{n=1}^{\infty} A_{n} I_{0}(m_{n}\rho_{Q}) ,$ (3.13 a)

b) in the case of $\rho_P < \rho_Q$,

$$\int_{0}^{2\pi} G_{c} d\theta_{Q} = B_{0} Y_{0}(m_{0}\rho_{Q}) + \sum_{n=1}^{\infty} B_{n} K_{0}(m_{n}\rho_{Q}), \qquad (3.13 b)$$

where

$$\begin{array}{l}
\left. A_{0}(\rho_{P}, z, c) \\
B_{0}(\rho_{P}, z, c) \\
B_{n}(\rho_{P}, z, c) \\
\end{array} \right\} = -4\pi^{2} \frac{m_{0}^{2} - K^{2}}{hm_{0}^{2} - hK^{2} + K} \cosh m_{0}(c+h) \cosh m_{0}(z+h) \begin{cases} Y_{0}(m_{0}\rho_{P}) \\
J_{0}(m_{0}\rho_{P}) \\
\end{array} \\
\left. A_{n}(\rho_{P}, z, c) \\
B_{n}(\rho_{P}, z, c) \\
\end{array} \right\} = 8\pi \frac{m_{n}^{2} + K^{2}}{hm_{n}^{2} + hK^{2} - K} \cos m_{n}(c+h) \cos m_{n}(z+h) \begin{cases} K_{0}(m_{n}\rho_{P}) \\
I_{0}(m_{n}\rho_{P}) \\
\end{array} \\
\left. \left. A_{n}(p_{P}, z, c) \\
\end{array} \right\} = 8\pi \frac{m_{n}^{2} + K^{2}}{hm_{n}^{2} + hK^{2} - K} \cos m_{n}(c+h) \cos m_{n}(z+h) \begin{cases} K_{0}(m_{n}\rho_{P}) \\
K_{0}(m_{n}\rho_{P}) \\
\end{array} \\
\left. A_{n}(p_{P}, z, c) \\
\end{array} \right\}$$

$$(3.14)$$

Let $\chi(\rho_P, \rho_Q)$ be such that

$$\chi(\rho_{P}, \rho_{Q}) = \int_{0}^{2\pi} G_{c}(\rho_{P}, \theta_{P}, 0, \rho_{Q}, \theta_{Q}, 0) d\theta_{Q}$$

$$= \begin{cases} A_{0}(\rho_{P}, 0, 0) J_{0}(m_{0}\rho_{Q}) + \sum_{n=1}^{\infty} A_{n}(\rho_{P}, 0, 0) I_{0}(m_{n}\rho_{Q}) & \text{for } \rho_{P} > \rho_{Q} \\ B_{0}(\rho_{P}, 0, 0) Y_{0}(m_{0}\rho_{Q}) + \sum_{n=1}^{\infty} B_{n}(\rho_{P}, 0, 0) K_{0}(m_{n}\rho_{Q}) & \text{for } \rho_{P} < \rho_{Q} \end{cases}$$
(3.15)

The integral equations (3.6) and (3.7) may then be reduced to the upper and lower ones of the following integral equations respectively:

$$-J_{0}(m_{0}\rho_{P}) = \begin{cases} \gamma^{1}(\rho_{P}) \\ \gamma^{a}(\rho_{P}) \end{cases} + \frac{K}{4\pi} \int_{0}^{\overline{a}} \begin{cases} \gamma^{1}(\rho_{Q}) \\ \gamma^{a}(\rho_{Q}) \end{cases} \chi(\rho_{P}, \rho_{Q})\rho_{Q}d\rho_{Q} \qquad 0 < \rho_{P} < \overline{a} ,$$
(3.16)

where \bar{a} is the radius of the circular disk.

The numerical solutions of the integral equations (3.16) must now be obtained by applying a proper numerical quadrature. Let the interval $[0, \bar{a}]$ consist of N small segments, that is

$$[0, \bar{a}] = \bigcup_{\nu=1}^{N} [a_{\nu-1}, a_{\nu}], \qquad (3.17)$$

where

$$0 = a_0 < a_1 < a_2 < \dots < a_{\nu-1} < a_\nu < \dots < a_N = \bar{a} .$$
(3.18)

The integral $\int_{0}^{\bar{a}} \gamma(\rho_Q) \chi(\rho_P, \rho_Q) \rho_Q d\rho_Q$ in (3.16) may then be approximated as follows

$$\int_{0}^{\overline{a}} \gamma(\rho_Q) \chi(\rho_P, \rho_Q) \rho_Q d\rho_Q = \sum_{\nu=1}^{N} \int_{a_{\nu-1}}^{a_{\nu}} \gamma(\rho_Q) \chi(\rho_P, \rho_Q) \rho_Q d\rho_Q$$
$$\approx \sum_{\nu=1}^{N} \gamma\left(\frac{a_{\nu-1}+a_{\nu}}{2}\right) \int_{a_{\nu-1}}^{a_{\nu}} \chi(\rho_P, \rho_Q) \rho_Q d\rho_Q . \tag{3.19}$$

Substitution of (3.15) into the integral $\int_{a_{\nu-1}}^{a_{\nu}} \chi(\rho_P, \rho_Q) \rho_Q d\rho_Q$ in (3.19) yields the following expression of this integral:

a) when
$$\rho_P < a_{\nu-1}$$
,

$$\int_{a_{\nu-1}}^{a_{\nu}} \chi(\rho_{P}, \rho_{Q}) \rho_{Q} d\rho_{Q}$$

$$= -4\pi^{2} \frac{m_{0}^{2}}{hm_{0}^{2} - hK^{2} + K} J_{0}(m_{0}\rho_{P}) \left[\frac{-\rho_{Q}}{m_{0}} Y_{1}(m_{0}\rho_{Q}) \right]_{a_{\nu-1}}^{a_{\nu}}$$

$$+ 8\pi \sum_{n=1}^{\infty} \frac{m_{n}^{2}}{hm_{n}^{2} + hK^{2} - K} I_{0}(m_{n}\rho_{P}) \left[\frac{-\rho_{Q}}{m_{n}} K_{1}(m_{n}\rho_{Q}) \right]_{a_{\nu-1}}^{a_{\nu}}, \qquad (3.20 a)$$

b) when $a_{\nu-1} < \rho_P < a_{\nu}$,

$$\int_{a_{\nu-1}}^{a_{\nu}} \chi(\rho_{P}, \rho_{Q}) \rho_{Q} d\rho_{Q} = \left\{ \int_{a_{\nu-1}}^{\rho_{P}} + \int_{\rho_{P}}^{a_{\nu}} \right\} \chi(\rho_{P}, \rho_{Q}) \rho_{Q} d\rho_{Q}$$

$$= \left[4\pi^{2} \frac{m_{0}a_{\nu-1}}{hm_{0}^{2} - hK^{2} + K} Y_{0}(m_{0}\rho_{P}) J_{1}(m_{0}a_{\nu-1}) - 8\pi \sum_{n=1}^{\infty} \frac{m_{n}a_{\nu-1}}{hm_{n}^{2} + hK^{2} - K} K_{0}(m_{n}\rho_{P}) I_{1}(m_{n}a_{\nu-1}) \right]$$

$$- \left[4\pi^{2} \frac{m_{0}a_{\nu}}{hm_{0}^{2} - hK^{2} + K} J_{0}(m_{0}\rho_{P}) Y_{1}(m_{0}a_{\nu}) + 8\pi \sum_{n=1}^{\infty} \frac{m_{n}a_{\nu}}{hm_{n}^{2} + hK^{2} - K} I_{0}(m_{n}\rho_{P}) K_{1}(m_{n}a_{\nu}) \right]$$

$$- \frac{4\pi}{K}, \qquad (3.20 \text{ b})$$

c) when $a_{\nu} < \rho_P$,

$$\int_{a_{\nu-1}}^{a_{\nu}} \chi(\rho_{P}, \rho_{Q})\rho_{Q}d\rho_{Q}$$

$$= -4\pi^{2} \frac{m_{0}^{2}}{hm_{0}^{2} - hK^{2} + K} Y_{0}(m_{0}\rho_{P}) \left[\frac{\rho_{Q}}{m_{0}} J_{1}(m_{0}\rho_{Q})\right]_{a_{\nu-1}}^{a_{\nu}}$$

$$+8\pi \sum_{n=1}^{\infty} \frac{m_{n}^{2}}{hm_{n}^{2} + hK^{2} - K} K_{0}(m_{n}\rho_{P}) \left[\frac{\rho_{Q}}{m_{n}} I_{1}(m_{n}\rho_{Q})\right]_{a_{\nu-1}}^{a_{\nu}}, \qquad (3.20 \text{ c})$$

where the following relations are used, that is

$$\begin{aligned} \cosh^{2} m_{0}h &= \frac{m_{0}^{2}}{m_{0}^{2} - K^{2}} \quad \cos^{2} m_{n}h = \frac{m_{n}^{2}}{m_{n}^{2} + K^{2}} \\ \int z \left\{ J_{0}(\alpha z) \atop Y_{0}(\alpha z) \right\} dz &= \frac{z}{\alpha} \left\{ J_{1}(\alpha z) \qquad (\alpha = \text{const.}) \right. \\ \int z \left\{ I_{0}(\alpha z) \atop K_{0}(\alpha z) \right\} dz &= \frac{z}{\alpha} \left\{ I_{1}(\alpha z) \qquad (\alpha = \text{const.}) \right. \\ J_{0}(z) Y_{1}(z) - J_{1}(z) Y_{0}(z) &= -\frac{2}{\pi z} \\ I_{0}(z) K_{1}(z) + I_{1}(z) K_{0}(z) &= \frac{1}{z} \\ \frac{1}{hm_{0}^{2} - hK^{2} + K} - \sum_{n=1}^{\infty} \frac{1}{hm_{n}^{2} + hK^{2} - K} \\ &= \frac{1}{2K} \left[\frac{\int_{-h}^{0} (1) \cosh m_{0}(z+h) dz}{\int_{-h}^{0} \cosh^{2} m_{0}(z+h) dz} \cosh m_{0}(z+h) \right] \end{aligned}$$

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$$+\sum_{n=1}^{\infty} \frac{\int_{-h}^{0} (1) \cos m_n(z+h) dz}{\int_{-h}^{0} \cos^2 m_n(z+h) dz} \cos m_n(z+h) \Big]_{z=0} = \frac{1}{2K}.$$
 (3.21)

The integral equations (3.16) can now be approximated by the following simultaneous linear equations for unknowns

 $\gamma_{\mu^{1}}$ or $\gamma_{\mu^{d}}$ ($\mu = 1, 2, \dots, N$):

$$-J_{0}\left(m_{0}\frac{a_{\mu-1}+a_{\mu}}{2}\right)_{1} = \begin{cases} \gamma_{\mu}^{1} \\ \gamma_{\mu}^{a} \end{cases} + \frac{K}{4\pi} \sum_{\nu=1}^{N} A_{\mu\nu} \begin{cases} \gamma_{\nu}^{1} \\ \gamma_{\nu}^{a} \end{cases} \qquad \mu = 1, 2, \dots, N, \qquad (3.22)$$

where

$$\begin{cases} \gamma_{\mu}^{1} \\ \gamma_{\mu}^{d} \end{cases} = \begin{cases} \gamma^{1} \left(\frac{a_{\mu-1} + a_{\mu}}{2} \right) \\ \gamma^{d} \left(\frac{a_{\mu-1} + a_{\mu}}{2} \right) \\ A_{\mu\nu} = \left[\int_{a_{\nu-1}}^{a_{\nu}} \chi(\rho_{P}, \rho_{Q}) \rho_{Q} d\rho_{Q} \right]_{\rho_{P}} = \frac{a_{\mu-1} + a_{\mu}}{2} \end{cases}$$

$$(3.23)$$

A must be evaluated numerically according to (3.20).

4. Some Important Relations Used in the Numerical Calculations [Bessho¹¹), Newman¹²), Isshiki et al.¹³]

4.1 A Relation Between the Damping and the Amplitude of Diverging wave

Let \overline{A} be the amplitude of the diverging wave due to the forced heave oscillation, with unit velocity amplitude, of the circular disk that is

$$\phi \sim \frac{g}{i\omega} \frac{\bar{A}}{\sqrt{m_0 \rho_P}} \frac{\cosh m_0(z+h)}{\cosh m_0 h} e^{im_0 \rho_P}$$

as $\rho_P = \sqrt{x^2 + y^2} \rightarrow \infty$. (4.1)

The damping \overline{N} defined by (2.8) may also be calculated by [Isshiki et al.¹³]

$$\bar{N} = \frac{2\pi\rho g^2}{\omega \cosh^2 m_0 h} \left(\frac{h}{2} + \frac{1}{2m_0} \cosh m_0 h \sinh m_0 h \right) |\bar{A}|^2 .$$
(4.2)

Since the assymptotic expansion of the Green function G is given by

$$G \sim B(z, c) H_0^{(1)}(m_0 R)$$
as $\rho_P \rightarrow \infty$, (4.3)

where

$$B(z, c) = 2\pi i \frac{(m_0 + K)e^{-m_0h}\sinh m_0h}{Kh + \sinh^2 m_0h}\cosh m_0(c+h)\cosh m_0(z+h), \qquad (4.4)$$

that of the velocity potential ϕ becomes as follows

$$\phi(P) = \frac{1}{4\pi} \iint_{\sigma} \sigma(Q) G(P, Q) dS_Q$$

$$\sim \frac{B(z, 0)}{4\pi} \int_{0}^{\bar{a}} \int_{0}^{2\pi} \sigma(\rho_Q) H_0^{(1)}(m_0 R) \rho_Q d\rho_Q d\theta_Q$$
(4.5)

According to the addition theorem of the Bessel function [Watson¹⁴], $H_0^{(1)}(m_0R)$ can be written as

$$H_0^{(1)}(m_0 R) = \sum_{n=-\infty}^{\infty} H_n^{(1)}(m_0 \rho_P) J_n(m_0 \rho_Q) \exp \{in(\theta_Q - \theta_P)\}$$

when $\rho_P > \rho_Q$. (4.6)

Substitution of (4.6) into (4.5) yields that

$$\phi(P) \sim \frac{B(z, 0)}{2} H_0^{(1)}(m_0 \rho_P) \int_0^{\bar{a}} \sigma(\rho_Q) J_0(m_0 \rho_Q) \rho_Q d\rho_Q$$

as $\rho_P \to \infty$. (4.7)

The integral in (4.7) can be written as $(\rightarrow (3.5))$

 $\int_{0}^{\bar{a}} \sigma(\rho_Q) f_0(m_0 \rho_Q) \rho_Q d\rho_Q = \frac{2}{KA(0, 0)} \left(P_c + i P_s \right).$ (4.8)

It then follows that

$$\phi(P) \sim \frac{B(z, 0)}{KA(0, 0)} (P_{c} + iP_{s}) H_{0}^{(1)}(m_{0}\rho_{P})$$

$$\sim \frac{i}{K} e^{-(\pi/4)i} \sqrt{\frac{2}{\pi}} (P_{c} + iP_{s}) \frac{1}{\sqrt{m_{0}\rho_{P}}} \frac{\cosh m_{0}(z+h)}{\cosh m_{0}h} e^{im_{0}\rho_{P}}$$
as $\rho_{P} \to \infty$. (4.9)

From (4.1) and (4.9), the amplitude \bar{A} of the diverging wave can now be expressed by P_c+iP_s as follows

$$\bar{A} = -\frac{1}{\omega} e^{-(\pi/4)i} \sqrt{\frac{2}{\pi}} (P_c + iP_s) .$$
(4.10)

The damping \bar{N} may then be written as

$$\bar{N} = \frac{4\rho g^2}{\omega^3} \frac{1}{\cosh^2 m_0 h} \left(\frac{h}{2} + \frac{1}{2m_0} \cosh m_0 h \sinh m_0 h\right) (P_o^2 + P_s^2)$$
(4.11)

The difference of the damping \overline{N} calculated from (2.8) and that from (4.11) may be considered as a measure of the accuracy of the numerical calculations.

4.2 The Haskind-Newman Relation

According to the Haskind-Newman Relation, the heave component $E e^{-i\omega t}$ of the wave excitation force due to a plane incident wave with the amplitude ζ_0 can be written as [Isshiki et al.¹³]

$$\frac{E}{\zeta_0} = -\frac{2\sqrt{2\pi}\rho g^2 e^{i\pi/4}}{\omega\cosh^2 m_0 h} \left(\frac{h}{2} + \frac{1}{2m_0}\cosh m_0 h\sinh m_0 h\right) \cdot \bar{A} , \qquad (4.12)$$

where \overline{A} is defined by (4.1) or (4.10). Substitution of (4.10) into (4.12) yields that

$$\frac{E}{\zeta_0} = \frac{4\rho g^2}{\omega^2} \frac{1}{\cosh^2 m_0 h} \left(\frac{h}{2} + \frac{1}{2m_0} \cosh m_0 h \sinh m_0 h\right) (P_c + iP_s) .$$
(4.13)

5. Long Wave Approximations

When the depth h of water is sufficiently small compared with the radius \bar{a} of the circular disk, the so-called long wave approximations [John⁷), Stoker⁸), Tuck⁹), Chen et al.¹⁰, Bessho¹⁵)] may be applicable. The wave number m_0 of the shallow water is now given by

$$m_0^2 h = K \qquad \left(K = \frac{\omega^2}{g}\right) \,. \tag{5.1}$$

Neglecting the variation of the velocity potential ϕ along z-axis, ϕ may be expressed as

x =

$$\phi = \begin{cases} \phi_e(r, \theta) & \text{for } r > \bar{a} \\ \phi_i(r, \theta) & \text{for } \bar{a} > r \ge 0 \end{cases}$$
(5.2)

where (r, θ) is a polar coordinate system such that

$$= r \cos \theta \qquad y = r \sin \theta . \tag{5.3}$$

 ϕ_e and ϕ_i must then satisfy the following equations:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi_e + m_0^2\phi_e = 0 \quad \text{for} \quad r > \bar{a}$$
(5.4 a)

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi_i = -\frac{1}{h} \qquad \text{for} \quad 0 \le r < \bar{a} , \qquad (5.4 \text{ b})$$

and on $r = \bar{a}$

$$\phi_e = \phi_i \tag{5.5 a}$$

$$\frac{\partial \phi_e}{\partial r} = \frac{\partial \phi_i}{\partial r} . \tag{5.5 b}$$

The axis-symmetric solution of (5.4 a) with an outgoing progressive wave is given as

$$\phi_e = \alpha H_0^{(1)}(m_0 r) \qquad (\alpha: \text{ an integral const.}), \tag{5.6 a}$$

and that of (5.4 b) regular for $0 \le r < \bar{a}$ is given as

$$\phi_i = \beta - \frac{1}{4h} r^2$$
 (β : an integral const.). (5.6 b)

Substitutions of (5.6) into (5.4) lead to the solution:

$$\phi_{e} = \frac{\bar{a}}{2h} \frac{H_{0}^{(1)}(m_{0}r)}{m_{0}H_{1}^{(1)}(m_{0}\bar{a})} \qquad \text{for } r \ge \bar{a} \\
\phi_{i} = \frac{\bar{a}}{2h} \frac{H_{0}^{(1)}(m_{0}\bar{a})}{m_{0}H_{1}^{(1)}(m_{0}a)} + \frac{1}{4h}(\bar{a}^{2} - r^{2}) \qquad \text{for } 0 \le r \le \bar{a}$$
(5.7)

The added mass $ar{M}$ and the damping $ar{N}$ may then be derived as

$$\frac{\overline{M}}{\rho \bar{a}^{3}} = \frac{1}{\bar{a}^{3}} \begin{cases} R_{e} \\ I_{m} \end{cases} \left[\int_{0}^{\bar{a}} \int_{0}^{2\pi} \phi_{i} \cdot r \, dr \, d\theta \right] \\
= \begin{cases} R_{e} \\ I_{m} \end{cases} \left[\frac{\pi}{2} \left\{ \frac{1}{m_{0}h} \frac{H_{0}^{(1)}(m_{0}\bar{a})}{H_{1}^{(1)}(m_{0}\bar{a})} + \frac{\bar{a}}{4h} \right\} \right] \\
= \begin{cases} \frac{\pi}{2} \left[\frac{1}{m_{0}\bar{a}} \frac{\{J_{0}(m_{0}\bar{a})J_{1}(m_{0}\bar{a}) + Y_{0}(m_{0}\bar{a})Y_{1}(m_{0}\bar{a})\}^{\frac{3}{2}}}{\{J_{1}(m_{0}\bar{a})\}^{2} + \{Y_{1}(m_{0}\bar{a})\}^{2}} + \frac{1}{4} \right] \left(\frac{\bar{a}}{h} \right) \\ \left[\frac{1}{(m_{0}\bar{a})^{2}} \frac{1}{\{J_{1}(m_{0}\bar{a})\}^{2} + \{Y_{1}(m_{0}\bar{a})\}^{2}} \right] \left(\frac{\bar{a}}{h} \right), \qquad (5.8)$$

where ρ is the density of the fluid, and R_e and I_m mean to take the real and imaginary parts of the corresponding quantity. (5.8) indicates that $\frac{\bar{M}}{\rho \bar{a}^3} \left(\frac{h}{\bar{a}}\right)$ and $\frac{\bar{N}}{\rho \bar{a}^3 \omega} \left(\frac{h}{\bar{a}}\right)$ are functions of $m_0 \bar{a}$ alone.

6. Results of Calculations

The computation was performed for a circular disk with the ratio of the radius of the disk \bar{a} to the depth of the shallow water, h, $\bar{a}/h=0.2$, 0.8, 1.0, 2.0, 5.0, 10.0, 20.0 and long wave $(\bar{a}/h\to\infty)$ as the limiting case. The nondimensional frequency $K\bar{a}=(\omega^2/g)\cdot\bar{a}$ assume the values 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0 for the case of $\bar{a}/h=0.2$, 0.8, 1.0, 2.0 and the values $K\bar{a}=0.5$, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0 for the case of $\bar{a}/h=5.0$, 10.0, 20.0.

The accuracy of the computation may be checked by evaluating the second equation of (2.8) and the combined equations of (4.2) and (4.11) by the Haskind-Newman relation. The equations for checking the accuracy for the case of shallow water, deep water and long wave are written as follows, for shallow water

$$\frac{|e|}{\pi \bar{a}^2} = \frac{2}{\pi^2} \cdot \frac{1}{m_0 \bar{a}} \cdot N \cdot \left\{ \frac{m_0 h + \cosh m_0 h \sinh m_0 h}{(\cosh m_0 h)^2} \right\}$$

for deep water

 $\frac{|e|}{\pi\bar{a}^2} = \frac{2}{\pi^2} \cdot \frac{1}{K\bar{a}} \cdot N$

for long wave

$$\frac{|e|}{\pi \bar{a}^2} = \frac{4}{\pi^2} \cdot N \cdot \left(\frac{h}{\bar{a}}\right)$$

where |e| is the normalized wave exciting force as $|e| = |E|/\rho g\zeta_0$ and N is the nondimensionalized damping factor as $N = \bar{N}/\rho \omega \bar{a}^3$. The accuracy also depends on the number of segments of which the interval [0, \bar{a}] of the circular disk consist, that is to say, the accuracy depends on the ratio of the

Kā	m₀ā	$n_0=10$				$n_0=20$			
		M	Ν	e	error (%)	M	N	e	error (%)
0.5	2.255	5.425	6.521	0.505	0.320	5.411	6.538	0.506	0.081
1.0	3.216	4.770	4.698	0.421	0.649	4.753	4.722	0.423	0.163
2.0	4.627	4.425	3.299	0.340	1.347	4.402	3.334	0.343	0.332
4.0	6.778	4.266	2.240	0.259	2.914	4.230	2.292	0.265	0.704
6.0	8.611	4.237	1.739	0.211	4.748	4.183	1.805	0.219	1.125
8.0	10.324	4.246	1.423	0.176	6.910	4.168	1.504	0.186	1.601
10.0	11.997	4.275	1.196	0.149	9.476	4.168	1.290	0.160	2.144

On the Heaving Motion of a Circular Disk in Shallow Water

Table 1	$(\bar{a}/h=10.0)$
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ength of each segments to wave length by the forced oscillation. According to results of calculation making use of the equation (6.1), the relative error is limited less than 1% when $m_0\bar{a}$ is less than $0.4n_0$, where n_0 , is the number of segments. The relative error is less than 5% when $m_0\bar{a}<0.9n_0$. In the case of 10 segments and 1% error, length of segments has to be under one fifteenth of the wave length by forced oscillation. In Table 1, the influence of the effect of the number of segments n_0 ($\bar{a}/h=10.0$) is listed.

In Fig. 3, the relation between the nondimensional shallow water wave number $m_0\bar{a} = \frac{2\pi}{\lambda}\bar{a}$ (where λ is the wave length by forced oscillation) and nondimensional frequency $K\bar{a} = (\omega^2/g)\bar{a}$ which is based on the last equation of (2.3) is plotted for several values of the parameter \bar{a}/h . The limiting value for long wave approximations is derived from the equation (5.1) and tends to $m_0\bar{a} = \sqrt{K\bar{a}\cdot(\bar{a}/h)}$.

In Figs. 4 and 5 we plot the added mass and damp-



Fig. 3 Shallow water wave number $m_0 \bar{a}$ as functions of frequency $K \bar{a}$

ing factors as functions of the nondimensional frequency $K\bar{a}$ for several values of the parameter \bar{a}/h : together with comparison curve for the circular disk in the limiting value as infinite water depth. $\bar{a}/h=0.0$ taken from MacCamy²). The added mass and damping factor are nondimensionalized as $M=\bar{M}/\rho\bar{a}^3$ and $N=\bar{N}/\rho\bar{a}^3\omega$ respectively. As the parameter for shallow water \bar{a}/h decreases, the curve tends to approach that for deep water $\bar{a}/h=0$. The curve for $\bar{a}/h=0.2$ is almost the same as that for deep water $\bar{a}/h=0$. And the curve for $\bar{a}/h=1.0$ is still also close to that for deep water.

In Figs. 6 and 7, the added mass and damping factor are plotted as functions of nondimensional shallow water wave number $m_0\bar{a}$ together with comparison curve for long wave approximations $(\bar{a}/h \rightarrow \infty)$, given by the equations (5.8). We see from them that as the shallow water parameter \bar{a}/h increases, the curve tends to approach that for long wave approximations. The limiting values for long: wave approximations are derived from the equations (5.8) as follows,



Fig. 4 Added mass $M = \overline{M} / \rho \overline{a}^3$ as functions of frequency $K \overline{a}$





at $m_0\bar{a}\rightarrow 0$



 $M\left(\frac{h}{\bar{a}}\right) \sim \frac{\pi}{8} + \frac{\pi}{4} \frac{1}{(m_0\bar{a})^2}$

 $N\left(\frac{h}{\bar{a}}\right) \sim \frac{\pi}{2} \cdot \frac{1}{m_0 \bar{a}}$

at $m_0 \bar{a} \rightarrow \infty$



Fig. 5 Damping factor $N = \bar{N} / \rho \bar{a}^3 \omega$ as functions of frequency $K \bar{a}$



Fig. 7 Damping factor $N \cdot (h/\bar{a})$ as functions of shallow water wave number $m_0 \bar{a}$



Fig. 9 In phase and out of phase components of pressure for disk $(\bar{a}/h=1.0)$

The curves for the parameter \bar{a}/h more than 5.0 may be replaced by that for long wave approximations. In Fig. 8 we plot the amplitude of the normalized pressure in case of $\bar{a}/h=1.0$ as functions of ρ/\bar{a} together with comparison curve for deep water $(\bar{a}/h\rightarrow 0)$ taken from MacCamy²). Curves for $\bar{a}/h=1.0$ are close to those for deep water. In phase and out of phase components of the radiation pressure

of the circular disk $\bar{a}/h=1.0$ are plotted in Fig. 9 as functions of $K\bar{a}$ for several values of the parameter ρ/\bar{a} . Pressure p and its in phase component p_c , out of phase component p_s are obtained from the equation (2.7) as follows,

$$p = p_c + ip_s$$

$$p_c = \rho g \bar{Z}_0 (1 - \sigma_c)$$

$$p_s = -\rho g \bar{Z}_0 \sigma_s$$

where \overline{Z}_0 is the amplitude of the forced heaving oscillation. Both components of the radiation pressure increase almost in proportion to the nondimensional frequency $K\overline{a}$.

Finally in Fig. 10, we plot the amplitude of the exciting forces and their phase lag as functions of nondimensional shallow water wave number $m_0\bar{a}$ for several values of the parameter \bar{a}/h together with



Fig. 10 Heave exciting force coefficient for disk and its phase lag as functions of shallow water wave number $m_0 \bar{a}$

comparison curves for both of the limiting cases of infinite water depth and long wave approximation. As the shallow water parameter \bar{a}/h increases at any shallow water wave number, then the exciting force increases. But the phase lag is almost independent of the shallow water parameter \bar{a}/h .

Contour lines of nondimensional frequency are also plotted in Fig. 10. From the point of view of nondimensional frequency, the exciting forces for several values of a parameter \bar{a}/h are rather close to each other, while their phase lag are different from **e**ach other.

7. Discussion and Conclusions

The dynamic qualities evaluated in the present paper converge to both of the limiting case of deep water and long wave approximation. Therefore, the results of the calculation are believed to yield good approximations for shallow draft ships

in shallow water performing forced heaving oscillation. The accuracy of the results were checked by Haskind-Newman relation and fairly good agreement was observed.

It is interesting to note that the hydrodynamic forces for shallow water parameter \bar{a}/h less than or equal to 1.0 may be approximated by those for deep water and that hydrodynamic forces for \bar{a}/h more than or equal to 10.0 are very close to those for long wave. In addition, from the point of view of nondimensional shallow water wave number, the exciting forces in heaving mode for a circular disk increase monotonously according to the shallow water parameter \bar{a}/h , while their phase lag are almost independent of the parameter \bar{a}/h .

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