The Society of Naval Architects of Japan

A Calculation of the Statistical Distribution of the Maxima of Nonlinear Responses in Irregular Waves

(2nd Report)

by Motohiro Hineno*, *Member*

Summary

An approximate method is proposed to calculate the statistical distributions of the maxima and minima of a weakly nonlinear response in irregular waves. Different from the previous method which dealt with the same problem, the present formulation includes the effect of the band width of the response spectrum. Therefore, the present approximation is even applicable to the response whose spectrum is not narrow banded but arbitrarily wide banded, as may be frequently experienced in most seakeeping problems.

First, the formulation for the quadratic nonlinear response is derived in the form expressed by the Hermite polynomial.

Second, calculations are carried out on the probability density and the $1/n$ th highest expected amplitude of wave elevation. The comparisons of the calculated results with simulated time domain results as well as experimental ones suggest that the present method may be useful from an engineering point of view.

1. Introduction

In the previous paper[1], it was shown that the probability densities of maxima and minima of a quadratic nonlinear response can be calculated by the Rayleigh distribution multiplied by the higher order moments of the response based on the Vinje method[2]. Recently Kato and Ando[3] also derived an asymptotic solution. These formulations were derived with the assumption that the spectrum of the response may be narrow banded. Most of the problems we dealt with, however, involves rather arbitrary wide band spectra, which means the number of maxima is not identical with the number of zero up-crossings. It is necessary to derive a more accurate analytical method applicable to those responses, including the effect of band width from the point of safety and economical operation of marine structures. Even in this case, the formulation can be obtained by the same approach. The joint probability density function from which the probability density of maxima or minima is obtained can be represented by the joint moments of the responses, as for the narrow band case. In the arbitrary wide band case, however, the joint probability density function of the response and its two derivatives must be considered, while only one derivative is needed for narrow banded.

Dalzell[4] applied the foregoing approach to a cubic nonlinear response including the band width, and derived the joint probability density expressed by the two dimensional Hermite polynomial. Through extensive study of the comparisons between the analytical results and simulated ones, he showed that this method may be useful in a practical sense, though it needs a cumbersome manipulation.

This paper deals with the quadratic case which will be a useful model for some nonlinear problems, and of course makes the manipulation easier and the computation time shorter compared with the cubic case.

First, the formulation of the statistical distribution of maxima is derived in the form expressed by the Hermite polynomial without using the two dimensional Hermite polynomial by means of variable change.

Second, in order to examine the effectiveness of the present method, the calculations are carried out for the surface elevation of irregular waves and compared with time domain results as well as experimental ones.

* Akishima Laboratory, Mitsui Engineering and Shipbuilding Co., Ltd.

NII-Electronic Library Service

The Society of Naval Architects of Japan

2. Formulation

2.1 General expression of the response

It is assumed in the following procedure that the input, such as irregular waves, is a stationary Gaussian stochastic process with zero mean. The response to the input, such as wave induced forces or motions of a floating structure, is assumed to be expressed by a functional polynomial of degree two in a time domain as follows:

$$z(t)=\int g_1(\tau)x(t-\tau)d\tau +\alpha\iint g_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \quad (1)$$

(Limits on integrals, $-\infty$ to ∞ are omitted here in this paper)

where $x(t)$: input at time t
$z(t)$: response to input
$g_1(\tau)$: linear kernel function
$g_2(\tau_1, \tau_2)$: quadratic kernel function
α : small quantity parameter.

The first term of the right hand side of Eq. (1) is for linear response, and the second for quadratic response which is assumed to be smaller than the linear term.

The kernel functions are related to the FRF (frequency response function) through the Fourier transform as follows:

$$\left.\begin{aligned} g_1(\tau)&=\frac{1}{2\pi}\int G_1(\omega)e^{i\omega\tau}d\omega \\ g_2(\tau_1,\tau_2)&=\frac{1}{(2\pi)^2}\iint G_2(\omega_1,\omega_2)\times e^{i(\omega_1\tau_1+\omega_2\tau_2)}d\omega_1 d\omega_2 \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} G_1(\omega)&=\int g_1(\tau)e^{-i\omega\tau}d\tau \\ G_2(\omega_1,\omega_2)&=\iint g_2(\tau_1,\tau_2)e^{-i(\omega_1\tau_1+\omega_2\tau_2)}d\tau_1 d\tau_2 \end{aligned}\right\} \quad (3)$$

where $G_1(\omega)$: linear FRF
$G_2(\omega_1, \omega_2)$: quadratic FRF
ω : circular frequency.

Then the first and the second time derivatives are derived from Eq. (1).

$$\dot{z}(t)=\int g_1(\tau)\dot{x}(t-\tau)d\tau +2\alpha\iint g_2(\tau_1,\tau_2)x(t-\tau_1)\dot{x}(t-\tau_2)d\tau_1 d\tau_2 \quad (4)$$

$$\ddot{z}(t)=\int g_1(\tau)\ddot{x}(t-\tau)d\tau +2\alpha\iint g_2(\tau_1,\tau_2)[\dot{x}(t-\tau_1)\dot{x}(t-\tau_2) +x(t-\tau_1)\ddot{x}(t-\tau_2)]d\tau_1 d\tau_2 \quad (5)$$

where $\dot{}$ signifies differentiation with respect to time, t. It should be noted that the quadratic kernel function is symmetrical in its arguments.

2.2 Joint cumulant function

The necessary fundamental relations used here will be explained briefly prior to the derivation of the probability density function.

Let ϕ be the moment generating function. We have

$$\phi=\iiint \exp[i(\theta_1 z_1+\theta_2 z_2+\theta_3 z_3)] \times f(z_1,z_2,z_3)dz_1 dz_2 dz_3 \quad (6)$$

where $f(z_1, z_2, z_3)$: joint probability density function of z_1, z_2 and z_3
$\theta_1, \theta_2, \theta_3$: arbitrary dummy real variables
$z_1\equiv z,\ z_2\equiv\dot{z},\ z_3\equiv\ddot{z}$.

Applying the Taylor series expansion to Eq. (6), it follows that

$$\phi=1+\sum_l\sum_m\sum_n\frac{\mu_{lmn}}{l!m!n!}(i\theta_1)^l(i\theta_2)^m(i\theta_3)^n \quad (7)$$

where l, m and n are positive integers whose sum is greater than zero. μ_{lmn} is a joint moment of the density and represented as follows:

$$\mu_{lmn}=\iiint z_1^l z_2^m z_3^n f(z_1,z_2,z_3)dz_1 dz_2 dz_3. \quad (8)$$

In order to make the calculation easier in the following procedure, the joint cumulant function is defined as follows:

$$K=\log\phi. \quad (9)$$

Then substituting Eq. (7) into Eq. (9), and again applying the Taylor series expansion, we have

$$K=\sum_l\sum_m\sum_n\frac{K_{lmn}}{l!m!n!}(i\theta_1)^l(i\theta_2)^m(i\theta_3)^n \quad (10)$$

where K_{lmn} is a joint cumulant of the response and its two derivatives.

2.3 Joint probability density function

The joint probability density function can be represented by the joint cumulants from Eq. (6) by means of the threefold Fourier transform and sbstituting Eqs. (9) and (10),

$$\begin{aligned} f(z_1,z_2,z_3) &=\frac{1}{(2\pi)^3}\iiint e^{-i(\theta_1 z_1+\theta_2 z_2+\theta_3 z_3)}e^K d\theta_1 d\theta_2 d\theta_3 \\ &=\frac{1}{(2\pi)^3}\iiint e^{-i(\theta_1 z_1+\theta_2 z_2+\theta_3 z_3)} \\ &\quad\times\exp\left[\sum_l\sum_m\sum_n\frac{K_{lmn}}{l!m!n!}(i\theta_1)^l(i\theta_2)^m(i\theta_3)^n\right] \\ &\quad\times d\theta_1 d\theta_2 d\theta_3. \end{aligned} \quad (11)$$

Thus it is evident that the joint probability density function will be obtained if we know all of the cumulants. In this paper, however, we assumed that the quadratic term of the response is in the order of $\alpha, O(\alpha)$, and is smaller than the linear term. Therefore Eq. (11) can be expanded in a Taylor series and truncated at $O(\alpha)$.

Before applying the Taylor series expansion to Eq. (11), the relationships between the joint cumu-

NII-Electronic Library Service

The Society of Naval Architects of Japan

lants and the FRF's will be described. From Eqs. (7), (9) and (10), the joint cumulants are represented by the joint moments as follows:

$$\left.\begin{aligned}
&K_{100}=\mu_{100}\\
&K_{010}=\mu_{010}\\
&K_{001}=\mu_{001}\\
&K_{200}=\mu_{200}-\mu_{100}^2 \text{ (variance of the response)}\\
&K_{020}=\mu_{020}-\mu_{010}^2\\
&K_{002}=\mu_{002}-\mu_{001}^2\\
&K_{101}=\mu_{101}-\mu_{100}\mu_{001}\\
&K_{110}=\mu_{110}-\mu_{100}\mu_{010}\\
&K_{011}=\mu_{011}-\mu_{010}\mu_{001}\\
&K_{300}=\mu_{300}-3\mu_{100}\mu_{200}+2\mu_{100}^3\\
&K_{003}=\mu_{003}-3\mu_{001}\mu_{002}+2\mu_{001}^3\\
&K_{201}=\mu_{201}-\mu_{001}\mu_{200}-2\mu_{100}\mu_{101}+2\mu_{001}\mu_{100}^2\\
&K_{120}=\mu_{120}-\mu_{100}\mu_{020}-2\mu_{010}\mu_{110}+2\mu_{100}\mu_{010}^2\\
&K_{102}=\mu_{102}-\mu_{100}\mu_{002}-2\mu_{001}\mu_{101}+2\mu_{100}\mu_{001}^2\\
&K_{021}=\mu_{021}-\mu_{001}\mu_{020}-2\mu_{010}\mu_{011}+2\mu_{001}\mu_{010}^2.
\end{aligned}\right\} \tag{12}$$

The other terms of degree three such as K_{030}, K_{210}, K_{012} and K_{111} will not be necessary for the calculation because these terms are related to z_2 which will be set at zero in the calculation later. And the other terms greater than degree three are also not necessary because their orders are higher than $O(\alpha)$.

On the other hand, the joint moments can be represented by the input spectrum and the FRF's from Eqs. (1), (3), (4) and (5) in the same way as in Ref. 1. The present manipulation, however, is more cumbersome than the previous work but not different in principle. After some manipulation we have the following relationships considering the nonlinear term of $O(\alpha)$.

The first degree moments are

$$\left.\begin{aligned}
&\mu_{100}=\alpha\int S(\omega)G_2(\omega,-\omega)d\omega\\
&\mu_{010}=0\\
&\mu_{001}=0
\end{aligned}\right\} \tag{13}$$

where $S(\omega)$: input spectrum (double sided spectrum).

The second degree moments are

$$\left.\begin{aligned}
&\mu_{200}=\int S(\omega)|G_1(\omega)|^2d\omega\\
&\mu_{020}=\int \omega^2S(\omega)|G_1(\omega)|^2d\omega\\
&\mu_{002}=\int \omega^4S(\omega)|G_1(\omega)|^2d\omega\\
&\mu_{101}=-\mu_{020}\\
&\mu_{110}=0\\
&\mu_{011}=0.
\end{aligned}\right\} \tag{14}$$

The third degree moments are

$$\left.\begin{aligned}
\mu_{300}=&3\mu_{100}\mu_{200}+6\alpha\iint S(\omega_1)S(\omega_2)\\
&\times G_1(-\omega_1)G_1(-\omega_2)G_2(\omega_1,\omega_2)d\omega_1d\omega_2\\
\mu_{003}=&-6\alpha\iint \omega_1^2\omega_2^2(\omega_1+\omega_2)^2S(\omega_1)S(\omega_2)\\
&\times G_1(-\omega_1)G_1(-\omega_2)G_2(\omega_1,\omega_2)d\omega_1d\omega_2\\
\mu_{201}=&2\mu_{100}\mu_{101}\\
&-4\alpha\iint(\omega_1^2+\omega_1\omega_2+\omega_2^2)S(\omega_1)S(\omega_2)\\
&\times G_1(-\omega_1)G_1(-\omega_2)G_2(\omega_1,\omega_2)d\omega_1d\omega_2\\
\mu_{120}=&\mu_{100}\mu_{020}\\
&+2\alpha\iint(\omega_1^2+\omega_1\omega_2+\omega_2^2)S(\omega_1)S(\omega_2)\\
&\times G_1(-\omega_1)G_1(-\omega_2)G_2(\omega_1,\omega_2)d\omega_1d\omega_2\\
\mu_{102}=&\mu_{100}\mu_{002}\\
&+2\alpha\iint[(\omega_1^2+\omega_2^2)(\omega_1+\omega_2)^2+\omega_1^2\omega_2^2]\\
&\times S(\omega_1)S(\omega_2)G_1(-\omega_1)G_1(-\omega_2)\\
&\times G_2(\omega_1,\omega_2)d\omega_1d\omega_2\\
\mu_{021}=&0.
\end{aligned}\right\} \tag{15}$$

Then applying the Taylor series expansion about $\alpha=0$ to Eq. (11) and truncating at $O(\alpha)$, we have

$$\begin{aligned}
f(z_1,z_2,z_3)=&\frac{1}{(2\pi)^3}\iiint\exp\Big(-\frac{1}{2}K_{200}\theta_1^2\\
&-\frac{1}{2}K_{020}\theta_2^2-\frac{1}{2}K_{002}\theta_3^2-K_{101}\theta_1\theta_3-i\theta_1z_1\\
&-i\theta_2z_2-i\theta_3z_3\Big)\Big[1+iK_{100}\theta_1\\
&+\sum_l\sum_m\sum_n\frac{K_{lmn}}{l!m!n!}(i\theta_1)^l(i\theta_2)^m(i\theta_3)^n\Big]d\theta_1d\theta_2d\theta_3
\end{aligned} \tag{16}$$

where l, m and n are positive integers whose sum is equal to 3. It should be noted that the small order components higher than $O(\alpha)$ must be neglected for each cumulant in Eq. (16).

In order to represent Eq. (16) by the Hermite polynomial, the following variables are introduced.

$$\left.\begin{aligned}
&x_1\equiv\theta_1\cos\varphi+\theta_3\sin\varphi\\
&x_3\equiv-\theta_1\sin\varphi+\theta_3\cos\varphi\\
&\tan 2\varphi\equiv\frac{2K_{101}}{K_{200}-K_{002}}
\end{aligned}\right\} \tag{17}$$

Thus with this notation we have

$$\begin{aligned}
f(z_1,z_2,z_3)=&\frac{(2\pi)^{-3/2}}{\sqrt{\Delta K_{020}}}\exp\Big[\frac{-1}{2\Delta}(K_{002}z_1^2+K_{200}z_3^2\\
&-2K_{101}z_1z_3)-\frac{z_2^2}{2K_{020}}\Big]\Big[1+C_1H_3\Big(\frac{C}{\sqrt{A}}\Big)\\
&+C_2H_2\Big(\frac{C}{\sqrt{A}}\Big)H_1\Big(\frac{D}{\sqrt{B}}\Big)+C_3H_1\Big(\frac{C}{\sqrt{A}}\Big)H_2\Big(\frac{D}{\sqrt{B}}\Big)\\
&+C_4H_3\Big(\frac{D}{\sqrt{B}}\Big)+C_5H_1\Big(\frac{C}{\sqrt{A}}\Big)+C_6H_1\Big(\frac{D}{\sqrt{B}}\Big)\\
&+C_7H_1\Big(\frac{C}{\sqrt{A}}\Big)H_2\Big(\frac{z_2}{\sqrt{K_{020}}}\Big)+C_8H_1\Big(\frac{D}{\sqrt{B}}\Big)\\
&\times H_2\Big(\frac{z_2}{\sqrt{K_{020}}}\Big)+C_9H_2\Big(\frac{C}{\sqrt{A}}\Big)H_1\Big(\frac{z_2}{\sqrt{K_{020}}}\Big)\\
&+C_{10}H_2\Big(\frac{D}{\sqrt{B}}\Big)H_1\Big(\frac{z_2}{\sqrt{K_{020}}}\Big)+C_{11}H_1\Big(\frac{C}{\sqrt{A}}\Big)\\
&\times H_1\Big(\frac{D}{\sqrt{B}}\Big)H_1\Big(\frac{z_2}{\sqrt{K_{020}}}\Big)+C_{12}H_3\Big(\frac{z_2}{\sqrt{K_{020}}}\Big)\Big]
\end{aligned} \tag{18}$$

NII-Electronic Library Service

The Society of Naval Architects of Japan

where

$$\Delta=K_{200}K_{002}-K_{101}^2$$
$$A=\frac{1}{2}K_{200}(1+\cos 2\varphi)+\frac{1}{2}K_{002}(1-\cos 2\varphi)+K_{101}\sin 2\varphi$$
$$B=\frac{1}{2}K_{200}(1-\cos 2\varphi)+\frac{1}{2}K_{002}(1+\cos 2\varphi)-K_{101}\sin 2\varphi$$
$$C=z_1\cos\varphi+z_3\sin\varphi$$
$$D=-z_1\sin\varphi+z_3\cos\varphi$$
$$H_n(x)=\frac{1}{\sqrt{\pi}}\int e^{-t^2}(x+i\sqrt{2}t)^n dt$$
(Hermite polynomial)
$$C_1=\frac{1}{24A\sqrt{A}}[3(K_{300}+K_{102})\cos\varphi+3(K_{003}+K_{201})\sin\varphi+(K_{300}-3K_{102})\cos 3\varphi-(K_{003}-3K_{201})\sin 3\varphi]$$
$$C_2=\frac{1}{8A\sqrt{B}}[-(K_{300}+K_{102})\sin\varphi+(K_{003}+K_{201})\cos\varphi-(K_{300}-3K_{102})\sin 3\varphi-(K_{003}-3K_{201})\cos 3\varphi]$$
$$C_3=\frac{1}{8B\sqrt{A}}[(K_{300}+K_{102})\cos\varphi+(K_{003}+K_{201})\sin\varphi-(K_{300}-3K_{102})\cos 3\varphi+(K_{003}-3K_{201})\sin 3\varphi]$$
$$C_4=\frac{1}{24B\sqrt{B}}[-3(K_{300}+K_{102})\sin\varphi+3(K_{003}+K_{201})\cos\varphi+(K_{300}-3K_{102})\sin 3\varphi+(K_{003}-3K_{201})\cos 3\varphi]$$
$$C_5=\frac{1}{\sqrt{A}}K_{100}\cos\varphi$$
$$C_6=-\frac{1}{\sqrt{B}}K_{100}\sin\varphi$$
$$C_7=\frac{1}{2K_{020}\sqrt{A}}(K_{120}\cos\varphi+K_{021}\sin\varphi)$$
$$C_8=\frac{1}{2K_{020}\sqrt{B}}(-K_{120}\sin\varphi+K_{021}\cos\varphi)$$
$$C_9=\frac{1}{4A\sqrt{K_{020}}}[(K_{210}-K_{012})\cos 2\varphi+2K_{111}\sin 2\varphi+K_{210}+K_{012}]$$
$$C_{10}=\frac{-1}{4B\sqrt{K_{020}}}[(K_{210}-K_{012})\cos 2\varphi+2K_{111}\sin 2\varphi-K_{210}-K_{012}]$$
$$C_{11}=\frac{-1}{2\sqrt{ABK_{020}}}[(K_{210}-K_{012})\sin 2\varphi-2K_{111}\cos 2\varphi]$$
$$C_{12}=\frac{K_{030}}{6K_{020}\sqrt{K_{020}}}.$$

It can be seen that Eq. (18) is the same result that Cartwright and Longuet-Higgins[5] derived, provided that all the quadratic terms are neglected.

2.4 Probability density function

The probability density function of maxima can be obtained by using the joint probability density. Because a maximum occurs when the 1st derivative is zero and the 2nd is negative, the expected number of maxima per unit time lying in the range (z_1, z_1+dz_1) is[6]

$$E[N(z_1)]=-dz_1\int_{-\infty}^{0} z_3 f(z_1, 0, z_3)dz_3. \quad (19)$$

On the other hand, the expected total number of maxima per unit time regardless of their magnitudes is

$$E[N(-\infty)]=-\int_{-\infty}^{\infty}\int_{-\infty}^{0} z_3 f(z_1, 0, z_3)dz_3 dz_1. \quad (20)$$

Carrying out the straightforward integration of Eq. (20), it is found that all the terms of $O(\alpha)$ become null which is the same result with that Dalzell[4] derived for a cubic nonlinear system. This means the expected total number of maxima is exactly the same as in that for a linear system; that is, the average period between successive maxima does not change from the linear value even though the nonlinearity of quadrature is considered. Then we have

$$E[N(-\infty)]=\frac{1}{2\pi}\sqrt{\frac{K_{002}}{K_{020}}}. \quad (21)$$

Thus the probability density function of the maxima of response to lie in the range (z_1, z_1+dz_1) is obtained from the ratio of Eq. (19) and Eq. (21).

$$F(z_1)dz_1=\frac{E[N(z_1)]}{E[N(-\infty)]}$$

After some manipulation we have

$$F(z_1)=\frac{1}{\sqrt{2\pi K_{200}}}\Big[\varepsilon\Big\{1+\Big(\varepsilon^2\sqrt{1-\varepsilon^2}K_{002}{}^{3/2}P_1+2\varepsilon^2\frac{K_{002}}{\sqrt{K_{200}}}P_2+\frac{P_3}{\sqrt{K_{200}}}\Big)\eta+\frac{P_4}{K_{200}{}^{3/2}}\eta^3\Big\}e^{-\eta^2/2\varepsilon^2}+\Big\{\sqrt{1-\varepsilon^2}\frac{P_4}{K_{200}{}^{3/2}}\eta^4+\Big(\varepsilon^3\sqrt{1-\varepsilon^2}\frac{K_{002}}{\sqrt{K_{200}}}P_2+\sqrt{1-\varepsilon^2}\frac{P_3}{\sqrt{K_{200}}}-\varepsilon^2\frac{\sqrt{K_{002}}}{K_{200}}P_5\Big)\eta^2+\sqrt{1-\varepsilon^2}\eta-3\varepsilon^4K_{002}{}^{3/2}P_1-\varepsilon^2\sqrt{K_{002}}P_6\Big\}e^{-\eta^2/2}\int_{-\infty}^{\eta/\delta}e^{-x^2/2}dx\Big] \quad (22)$$

where

$$P_1=C_1E_1^3+C_4E_3^3+E_1E_3(C_2E_1+C_3E_3)$$
$$P_2=3(C_1E_1^2E_2-C_4E_3^2E_4)+C_2E_1(2E_2E_3-E_1E_4)-C_3E_3(2E_1E_4-E_2E_3)$$
$$P_3=(C_5-C_7)E_2-(C_6-C_8)E_4-3C_1E_2+3C_4E_4+C_2E_4-C_3E_2$$
$$P_4=C_1E_2^3-C_4E_4^3-C_2E_2^2E_4+C_3E_2E_4^2$$
$$P_5=3C_1E_1E_2^2+3C_4E_3E_4^2+C_2E_2(E_2E_3-2E_1E_4)+C_3E_4(E_1E_4-2E_2E_3)$$
$$P_6=(C_5-C_7)E_1+(C_6-C_8)E_3-3(C_1E_1+C_4E_3)-C_2E_3-C_3E_1$$
$$E_1=\frac{\sin\varphi}{\sqrt{A}} \qquad E_2=\frac{1}{\sqrt{A}}(K_{200}\cos\varphi+K_{101}\sin\varphi)$$
$$E_3=\frac{\cos\varphi}{\sqrt{B}} \qquad E_4=\frac{1}{\sqrt{B}}(K_{200}\sin\varphi-K_{101}\cos\varphi)$$
$$\eta=\frac{z_1}{\sqrt{K_{200}}} \qquad \delta=\frac{\sqrt{\Delta}}{-K_{101}}=\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}$$

NII-Electronic Library Service

The Society of Naval Architects of Japan

$$\varepsilon^2=1-\frac{K_{020}{}^2}{K_{200}K_{002}}.\ (\varepsilon : \text{band width parameter})$$

It is obvious that Eq. (22) becomes the well known result that Cartwright and Longuet-Higgins[5] derived for the linear case if all the quadratic terms, P_1 through P_6, tend to zero.

Then the probability distribution function can be obtained from Eq. (22).

$$P(z_1)=\int_{z_1}^{\infty} F(z_1)dz_1 \tag{23}$$

and the $1/n$ th highest expected value of maxima is

$$\bar{z}_{1/n}=\int_{z_n}^{\infty} z_1F(z_1)dz_1/P(z_n) \tag{24}$$

where z_n : $1/n$ th highest maximum.

The formulations for minima also can be derived in the same way as for maxima.

3. Comparison of the calculated results with the experimental ones

With the FRF's and the input spectrum, the probability density, distribution functions and $1/n$ th highest expected value of response can be calculated through the above formulations. In order to provide the examples of a quadratic response with which the present analytical results can be compared, the surface elevation of irregular waves in infinitely deep water will be considered. The reason why the wave elevation was adopted for the comparison is that the exact FRF's can be obtained analytically and the nonlinearity of the wave elevation has long been well known. The FRF's of the wave elevation are analytically derived as follows[1] :

$$\left.\begin{array}{l} G_1(\omega)=1 \\ G_2(\omega_1,\omega_2)=\dfrac{1}{2g}(\omega_1{}^2+\omega_2{}^2) \\ \quad \text{for sum frequency component} \\ G_2(\omega_1,-\omega_2)=-\dfrac{1}{2g}|\omega_1{}^2-\omega_2{}^2| \\ \quad \text{for difference frequency component} \end{array}\right\} \tag{25}$$

where g is gravitational acceleration.

The instantaneous wave elevation at time t is also represented by a different form of Eq. (1) as follows :

$$\begin{aligned} z(t)=&Re\sum_m a_mG_1(\omega_m)e^{i(\omega_m t+\varepsilon_m)} \\ &+\frac{1}{2}Re\sum_m\sum_n a_ma_n[G_2(\omega_m,\omega_n)e^{i\{(\omega_m+\omega_n)t+\varepsilon_m+\varepsilon_n\}} \\ &+G_2(\omega_m,-\omega_n)e^{i\{(\omega_m-\omega_n)t+\varepsilon_m-\varepsilon_n\}}] \end{aligned} \tag{26}$$

where

$a_m=\sqrt{2S(\omega_m)\Delta\omega_m}$

$S(\omega_m)$: linear wave spectrum (one sided spectrum)

$\Delta\omega_m$: interval of discrete circular frequencies

ε_m : random number equally distributed between 0 to 2π.

The amplitudes and the circular frequencies of the component waves are calculated from the 2-parameter Pierson-Moskowitz wave spectrum divided into 50 sections between 0.1777 and 0.9036 rad/sec in such a way that each area of the sections is identical. It is necessary to consider 50 component waves at least, in order to obtain the statistically stable data[7]. It should be noted that the statistical characteristics of wave elevation depend on the range of circular frequencies of the wave spectrum[8]. Therefore, the lowest and the highest frequencies of the component waves were decided in exactly the same values for the analytical calculation. The simulations were conducted for four irregular waves which have average period of 16.1 sec and significant wave heights of 5.8 m through 34.8 m. The simulation time is 10000 sec in which each number of maxima and the minima is about 800 respectively. The calculated results are shown with the normalized maxima or minima by using the positive square root of the variance, namely, $K_{200}{}^{1/2}$, which is called the standard deviation of the wave elevation. Figs. 1 through 4 show the probability density functions obtained from the simulations compared with the results by the present method and also by linear theory in accordance with ascending values of significant wave height. It should be noted that the ordinate is normalized, and the minima are shown with reversed signs in these figures. From these figures, it can be seen that the probability density functions for maxima and minima have some value at negative amplitudes, which is a well known characteristic of a broad wide spectrum. Furthermore, those functions are asymmetrical regarding maxima and minima ; that is, the probability for the maxima is greater than for the minima in the larger amplitudes and vice versa in the smaller amplitudes. This tendency becomes more recognizable as the wave height becomes larger, which is consistent with the fact we experienced. The present analytical results show fairly good agreement with simulated ones, although the present ones have some negative value at larger amplitudes in the highest one as seen in Fig. 4. $1/n$ th highest expected amplitudes for the maxima becomes larger than for the minima as shown in Figs. 5 through 8. The agreement between the present approximation and the simulated results is good except for the highest one. For the highest waves which could not be occured in actual, the present method does not explain well the statistical characteristic of the simulated results in relatively large value of n. It is, however, confirmed that the present method is applicable to the surface elevation of irregular waves even for considerably high waves.

The other comparison between the present method and the experimental results is conducted to examine the applicability of the present method

NII-Electronic Library Service

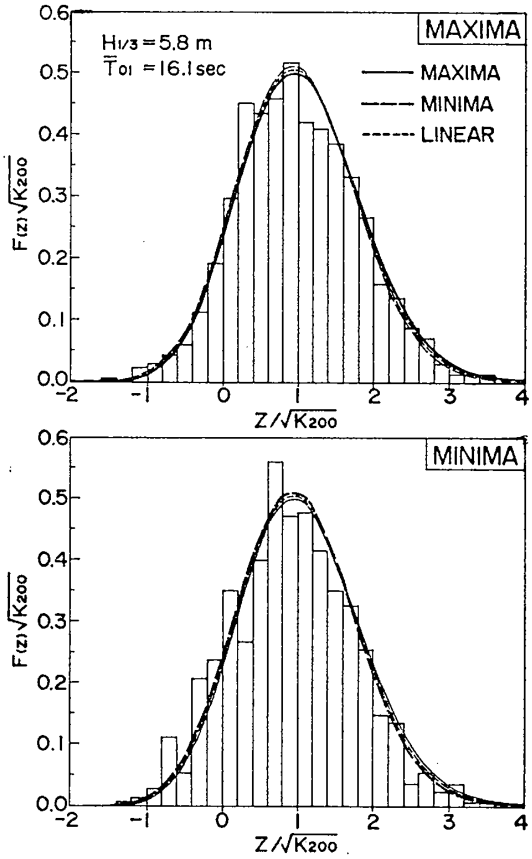

Fig. 1 Probability density function of wave amplitude

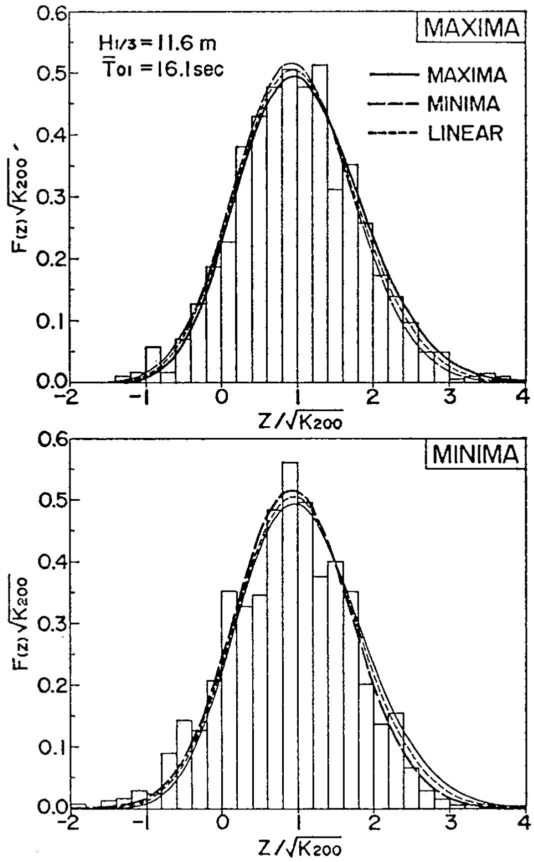

Fig. 2 Probability density function of wave amplitude

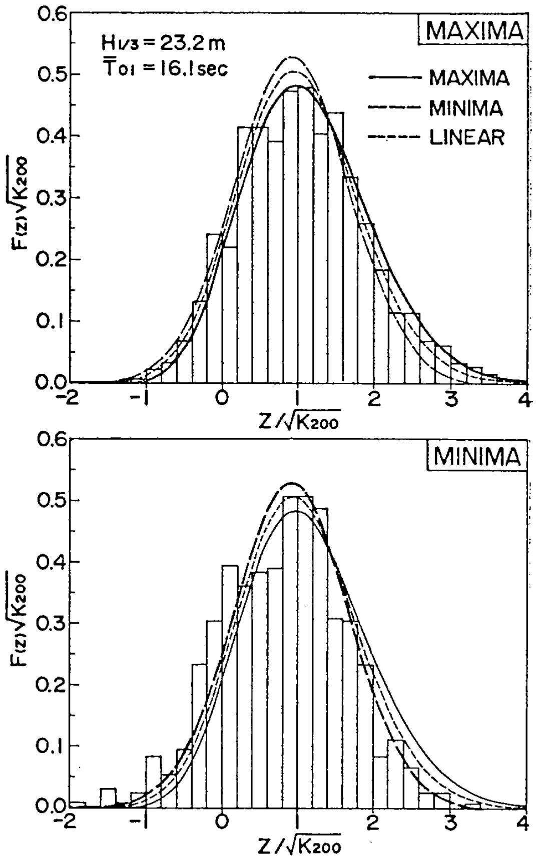

Fig. 3 Probability density function of wave amplitude

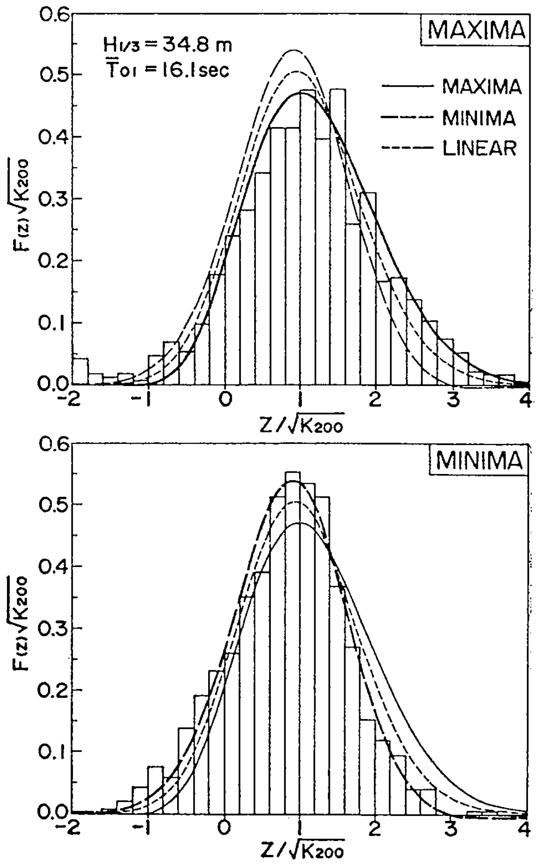

Fig. 4 Probability density function of wave amplitude

The Society of Naval Architects of Japan

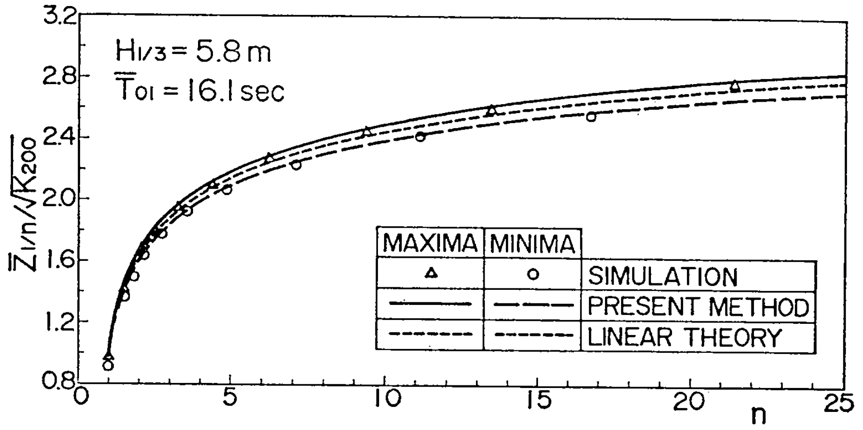

Fig. 5 $1/n$ th highest expected wave amplitude

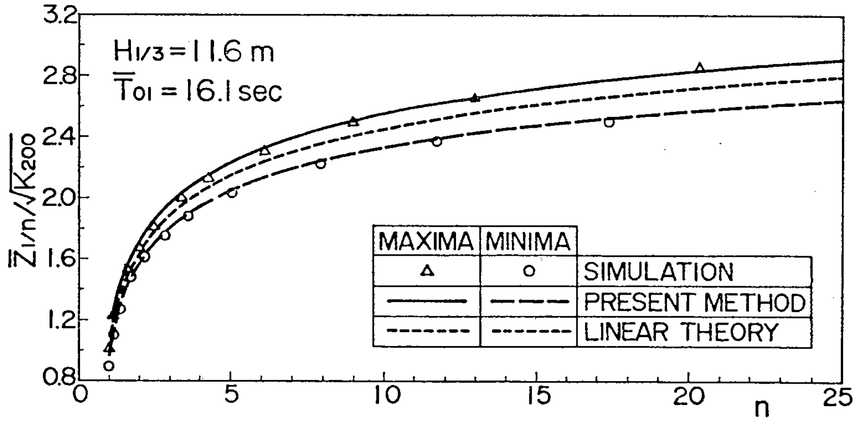

Fig. 6 $1/n$ th highest expected wave amplitude

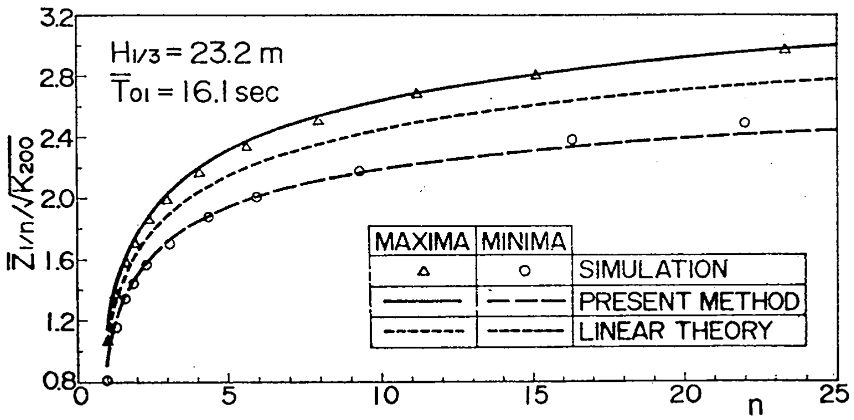

Fig. 7 $1/n$ th highest expected wave amplitude

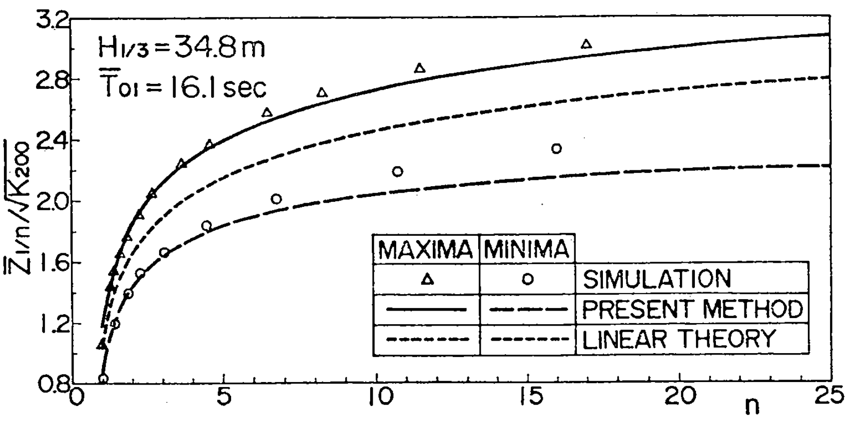

Fig. 8 $1/n$ th highest expected wave amplitude

NII-Electronic Library Service

The Society of Naval Architects of Japan

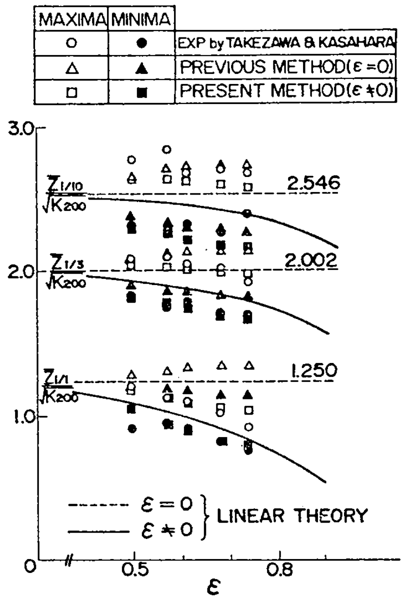

Fig. 9 Relation between the band width parameter ε and statistical values of maxima or minima

for several values of band width parameter. The statistical characteristics of irregular waves generated in a experimental tank have been investigated in detail by Takezawa and Kasahara[9]. One of their studies is on the relation of band width parameter and the $1/n$ th highest expected wave amplitudes which are maxima and minima, shown in Fig. 9. As may be noted, the analytical results of the experimental data are denoted by white circles for maxima and black ones for minima. The solid line shows the calculated ones by linear theory, taking into account the effect of band width parameter, and the dashed line shows the results calculated also by linear theory where the band width parameter is zero. In order to compare with these results, the calculations were carried out by two methods including the nonlinear effect. One is by the present method, shown by squares. The other is by the previous method which was derived for narrow band spectrum, shown by triangles. It can be seen that the calculated ones by the previous method are distributed around the dashed line. The present results, on the other hand, are distributed around the solid line, which means that the present approximation includes the effect of the band width parameter correctly. Different from the calculated results with the assumption of narrow band, the present method explains reasonably well the experimental ones. From this comparison, it can be seen that the present method is applicable to the weakly nonlinear problem whose spectrum is arbitrarily wide banded, and is a useful approach at least to the irregular waves in most cases of engineering interest. This method will be available for other responses. For example, the low frequency motion of a semisubmersible in irregular waves, one of the typical nonlinear responses in seakeeping problems, will be explained by the present formulation. The quantitative accuracy of the present method depends on the strength of the nonlinearity of the problem. The stronger the nonlinear component, it may be necessary to take into account the higher order terms in the derivation which will bring a much more cumbersome manipulation. It will be a further study for the present method to investigate the applicability to the other nonlinear problems and the relation with the extent of nonlinearity.

4. Concluding remarks

An approximate method to calculate the statistical distribution of a quadratic nonlinear response, taking into account the band width of the spectrum, was proposed. The formulation was derived in the form expressed by the Hermite polynomial without using the two dimensional Hermite polynomial. The calculations were carried out for the elevation of irregular waves and compared with the simulated and the experimental results. The main conclusions of the present work are summarized as follows:

(1) The asymmetrical distributions of the maxima and the minima of a weakly nonlinear response can be obtained by the present method with the input spectrum and the frequency response functions of up to 2 nd order.

(2) The present method explains more accurately the statistical distributions of a weakly nonlinear response whose spectrum is arbitrarily wide banded than the previous approximate method with the assumption of narrow band.

(3) The average period between successive maxima for the quadratic response is identical to the one for a linear case.

(4) The statistical properties of the irregular waves in deep water, such as probability densities and $1/n$ th highest expected values of the maxima and the minima, may be correctly calculated by the present method.

It is necessary to investigate the applicability of the present method to the other nonlinear responses and to study whether this method can be useful even to relatively stronger nonlinear responses.

Acknowledgement

The author is grateful to Dr. Y. Yamanouchi, Technical Adviser and Dr. M. Kobayashi, Head of Ocean Engineering Research Division of Akishima Laboratory, Mitsui Engineering & Shipbuilding Co., Ltd. for their stimulating discussions.

NII-Electronic Library Service

The Society of Naval Architects of Japan

He is also grateful to Prof. Takezawa of Yokohama National University and Mr. Kasahara of University of Tokyo for allowing him to quote their experimental results.

References

1) Hineno, M.: A Calculation of the Statistical Distribution of the Maxima of Nonlinear Responses in Irregular Waves, Jour. of the Soc. of Naval Arch. of Japan, Vol. 156 (Dec. 1984).
2) Vinje, T.: On the Calculation of Maxima of Non-linear Waveforces and Wave Induced Motions, International Shipbuilding Progress, Vol. 23, No. 268 (Dec. 1976).
3) Kato, S. and Ando, S.: On the Statistical Prediction of Horizontal Motions of Moored Floating Structures in Random Waves, Jour. of the Soc. of Naval Arch. of Japan, Vol. 158 (Dec. 1985).
4) Dalzell, J. F.: Approximations to the Probability Density of Maxima and Minima of the Response of a Nonlinear System, U. S. Naval Academy, Report No. EW-22-84 (Oct. 1984).
5) Cartwright, D. E. and Longuet-Higgins, M. S.: The statistical distribution of the maxima of a random function, Proc. of the Royal Society, A, Vol. 237 (1956).
6) Price, W. G. and Bishop, R. E. D.: Probabilistic Theory of Ship Dynamics, Chapman & Hall Ltd. (1974).
7) Goda, Y.: Numerical Experiments on Wave Statistics with Spectral Simulation, Rept. of the Port & Harbour Res. Institute, Vol. 9, No. 3 (Sept. 1970).
8) Takezawa, S. and Wakiya, H.: On the Modified Truncated Wave Spectra for the Artificial Irregular Water Wave Generation, Jour. of the Soc. of Naval Arch. of Japan, Vol. 149 (June 1981).
9) Takezawa, S. and Kasahara, A.: On Statistical Properties of Artificial Irregular Water Waves, Jour. of the Kansai Soc. of Naval Arch., Japan, No. 195 (Dec. 1984).

NII-Electronic Library Service