5. Linearized Theory of Cavity Flow Past a Hydrofoil of Arbitrary Shape

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Notation

x, z : Catesian coordinates of the point of observation

x', z' : Cartesian coodinates of sigularity point

V : Free-stream velocity

ρ : Fluid density

Φ : Perturbation velocity potential

p : Fluid pressure

p_c : Vapour pressure

σ : Cavitation number $\sigma=-2p_c/(\rho V^2)$

α : Angle of attack (clockwise rotation taken as negative)

c : Half chord length

c_0 : Half cavity length

δ : Half cavity width

z_+, z_- : Upper and lower z-ordinates of the foil-cavity region

z'_+ : Upper z-ordinate of the foil

C_D : Drag coefficient

C_L : Lift coefficient

C_M : Moment coefficient about the rear end of cavity for fully cavitated flow and the middle point of the chord for partially cavitated flow (clockwise moment taken as negative)

$$L=\sqrt{(l_2-l_1)/(l_2'-l_2)}\ ,$$
$$I=\sqrt{(l_2'-l_1)/(l_2-l_2')}\ ,$$
$$\xi=(x-x_0')/c_0\ ,$$
$$\zeta=(x-x_0)/c\ ,$$
$$x_0'=(l_1'+l_2')/2\ ,$$
$$x_0=(l_1+l_2)/2\ .$$

The boundary-values of w and p in approaching the x-axis from the upper or lower half-plane are denoted by w_+, p_+ or w_-, p_- respectively.

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Introduction

Present report is concerned with the problem of steady plane flow of a cavitating hydrofoil, within the frame of the linearized theory. The aim of this study is the development of a theory which connects the gap between the theories of partially cavitated and fully cavitated flows.

When we treat a cavity flow as perfect fluid flow, hypothetical images are needed for various cavitation numbers. In this theory, the idea of relaxing the closure condition of a cavity is introduced. If the measured cavity-length is introduced as a parameter, one finds better experimental agreement for lift. The theory is semiempirical and contains constants or functions to be determined by extensive experiments.

It is desirable to obtain analytical prediction for cavity flow of a hydrofoil, and it may be possible, if experiments are performed systematically. It is hoped that the present analysis may shed some light on future development of the theoretical prediction on a hydrofoil best fitted for cavity flow in characteristics.

1. Velocity Potential

A hydrofoil is placed in the uniform flow of a perfect fluid filling an infinite space. The stream velocity is taken to be V. A sketch of the cavitating hydrofoil is shown in Fig. 1. In linearized theory, the analysis

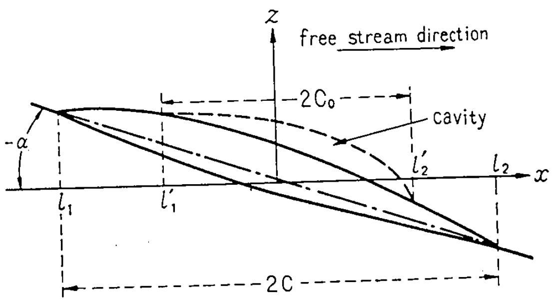

Fig. 1.

is simplified by fulfilling the surface boundary condition on the x-axis rather than on an approximate neighboring shape.

So, the velocity potential is given by

$$\Phi(x,z)=\frac{1}{2\pi V}\int_{l_1}^{l_2}\hat{\phi}\tan^{-1}\frac{x-x'}{z}dx'+\frac{1}{2\pi}\int_{l_1 \text{ or } l_1'}^{l_2'}\frac{d\varphi}{dx}\ln\{(x-x')^2+z^2\}\,dx' \quad (1.1)$$

In these approximations, we will require that the velocity never differs too much from the free-stream velocity and that the slope of the body must be small. Therefore, we may neglect squares and higher powers of small quantities. Subsequently, we obtain the linearized relation between the velocity potential and the fluid pressure

$$\phi=V\frac{\partial\Phi}{\partial x}=-p/\rho \quad (1.2)$$

from Euler's equation of motion.

2. Boundary Condition

If the median plane of the hydrofoil makes a negative angle α with the free-stream direction, the cavity should extend along a portion of the upper side of x-axis. The case that the cavity extends along all portions of the suction side is termed fully cavitated flow. So the case that the cavity extends along a part of the suction side of the hydrofoil may be called partially cavitated flow.

The linearized boundary condition may be stated as follows:

The velocity vector on the hydrofoil surface is parallel to the surface. That is

$$\left.\begin{aligned}\lim_{z\to+0}\frac{\partial\Phi}{\partial z}&=V\frac{dz_+}{dx}\\ \lim_{z\to-0}\frac{\partial\Phi}{\partial z}&=V\frac{dz_-}{dx}\end{aligned}\right\} \quad (2.1)$$

The pressure in the cavity is assumed to be a constant, p_c. That is

$$\lim_{z\to+0}-\rho V\frac{\partial\Phi}{\partial x}=p_c \text{ for } l_1'<x<l_2'. \quad (2.2)$$

The condition at infinity is

$$\left.\frac{\partial\Phi}{\partial x}\right|_{x\to\infty}=0. \quad (2.3)$$

(1.1) satsfies the above condition (2.3).

In order to hold a smooth juncture between the cavity wall and the hydrofoil surface, the cavity should be formed so that the following relationship is fulfilled.

$$\left.\frac{d\delta}{dx}\right|_{x=l_1'}=0\,. \tag{2.4}$$

The condition of smooth flow at the trailing edge is

$$(p_+-p_-)|_{x=l_2}=0\,. \tag{2.5}$$

The author adopts a partly open, linearized cavity model which was introduced by A. G. Fabula [1][1)]. That is

$$\delta|_{x=l_2'}=\delta_2\,. \tag{2.6}$$

The open width δ_2 is determined empirically.

To solve the mixed boundary value problem, we must treat two integral equations simultaneously. The key of the method is to convert the two equations into one equation by eliminating an unknown function among two unknown functions.

3. Integral Equation

In this section, we will derive an integral equation for the solution of the boundary-value problem given in the above section.

Using (1.1) and (1.2), we have the following expressions

$$w_+=\lim_{z\to+0}\frac{\partial\Phi}{\partial z}=-\frac{1}{2\pi V}\oint_{l_1}^{l_2}\frac{\hat\phi}{x-x'}\,dx'+\frac{d\varphi}{dx} \tag{3.1}$$

$$w_-=\lim_{z\to-0}\frac{\partial\Phi}{\partial z}=-\frac{1}{2\pi V}\oint_{l_1}^{l_2}\frac{\hat\phi}{x-x'}\,dx'-\frac{d\varphi}{dx} \tag{3.2}$$

$$p_+=\lim_{z\to+0}-\rho V\frac{\partial\Phi}{\partial x}=-\frac{\rho\hat\phi}{2}-\frac{\rho V}{\pi}\oint_{l_1\text{ or }l_1'}^{l_2'}\frac{d\varphi/dx'}{x-x'}dx' \tag{3.3}$$

1) Numbers between square brackets refer to the literature listed at the end of the paper.

$$p_-=\lim_{z\to-0}-\rho V\frac{\partial\Phi}{\partial x}=\frac{\rho\hat\phi}{2}-\frac{\rho V}{\pi}\oint_{l_1\text{ or }l_1'}^{l_2'}\frac{d\varphi/dx'}{x-x'}dx'\,. \tag{3.4}$$

If we put

$$(z_+-z_-)/2=\bar z\,,\qquad (z_++z_-)/2=\hat z \tag{3.5}$$

$$p_--p_+=\hat p\,,\qquad p_-+p_+=\bar p\,, \tag{3.6}$$

we get the equations

$$V\frac{d\hat z}{dx}=-\frac{1}{2\pi V}\oint_{l_1}^{l_2}\frac{\hat\phi}{x-x'}dx' \tag{3.7}$$

$$V\frac{d\bar z}{dx}=\frac{d\varphi}{dx} \tag{3.8}$$

$$\hat p=\rho\hat\phi \tag{3.9}$$

$$\bar p=-\frac{2\rho V}{\pi}\oint_{l_1\text{ or }l_1'}^{l_2}\frac{d\varphi/dx'}{x-x'}dx' \tag{3.10}$$

from (2.1) and (3.1)~(3.4).

If we write

$$\bar z'=(z_+'-z_-)/2,\qquad \bar z=\bar z'+\delta\,, \tag{3.11}$$

we have from (3.8)

$$\frac{d\varphi}{dx}=V\frac{d\bar z'}{dx}+V\frac{d\delta}{dx}\,. \tag{3.12}$$

We also put

$$\left.\begin{aligned}\hat z'&=(z_+'+z_-)/2=\hat z^*+\alpha x\\ \hat z&=\hat z'+\delta\end{aligned}\right\} \tag{3.13}$$

for the convenience of following analysis.

For the case of the fully cavitated flow, (3.13) becomes

$$\hat z'=z_-=z^*+\alpha x \tag{3.14}$$

as we may put $\bar z'=0$.

(i) Fully cavitated flow ($l_1=l_1'$, $l_2<l_2'$)

If we write

$$p_b=\frac{2\rho V^2}{\pi}\oint_{l_1}^{l_2}\frac{d\bar z'}{dx'}\frac{1}{x-x'}dx' \tag{3.15}$$

$$p^*=\bar p+p_b=\hat p+2p_c+p_b\,, \tag{3.16}$$

we get

$$p^*=-\frac{2\rho V^2}{\pi}\oint_{-1}^{1}\frac{d\delta}{dx'}\frac{1}{\xi-\xi'}d\xi' \qquad (3.17)$$

where

$$\xi=(x-x_0')/c_0, \qquad x_0'=(l_1'+l_2')/2\,. \qquad (3.18)$$

In order to get the solution of the boundary value problem, we must solve two integral equations (3.7) and (3.17) of two unknown functions $\hat{p}$ and $d\delta/dx'$ simultaneously. Fortunately, the analytical solution of the integral equation of the same type with (3.17) is known in the field of thin-airfoil theory.

When we introduce the juncture condition (2.4), it is written

$$2\rho V^2\frac{d\delta}{dx}=\frac{1}{\pi}\sqrt{\frac{1+\xi}{1-\xi}}\oint_{-1}^{1}\sqrt{\frac{1-\xi'}{1+\xi'}}\frac{p^*}{\xi-\xi'}d\xi'\,. \qquad (3.19)$$

If we insert the above result in (3.2) in the place of $d\varphi/dx$, two integral equations (3.7) and (4.17) are converted into an integral equation of $\hat{p}$. The clear expression will be shown in the following section.

(ii) Partially cavitated flow ($l_1=l_1'$, $l_2'<l_2$)

In the case of the partially cavitated flow, it is rather convenient to the analysis to derive an integral equation of $d\delta/dx$ than $\hat{p}$, because $\hat{p}$ is not continuous at the rear end of the cavity.

When we introduce Kutta's condition (2.5), the solution of the integral equation (3.7) is given by

$$\frac{\hat{\phi}}{V^2}=\frac{2}{\pi}\sqrt{\frac{l_2-x}{x-l_1}}\oint_{l_1}^{l_2}\sqrt{\frac{x'-l_1}{l_2-x'}}\frac{d\hat{z}/dx'}{x-x'}dx' \qquad (3.20)$$

Inserting (3.20) in (3.3) in the place of $\hat{\phi}$, we get

$$\frac{p_+}{\rho V^2}=-\frac{1}{\pi}\sqrt{\frac{l_2-x}{x-l_1}}\oint_{l_1}^{l_2}\sqrt{\frac{x'-l_1}{l_2-x'}}\frac{d\hat{z}/dx'}{x-x'}dx'$$

$$-\frac{1}{\pi V}\oint_{l_1}^{l_2}\frac{d\varphi/dx'}{x-x'}dx' \qquad (3.21)$$

As $p_+=p_c$ for $l_1<x<l_2'$, (3.21) is written in the form

$$Z-\frac{\sigma}{2}=-\frac{1}{\pi}\oint_{l_1}^{l_2'}\frac{d\delta/dx'}{x-x'}dx'$$
$$-\frac{1}{\pi}\sqrt{\frac{l_2-x}{x-l_1}}\oint_{l_1}^{l_2'}\sqrt{\frac{x'-l_1}{l_2-x'}}\frac{d\delta/dx'}{x-x'}dx' \qquad (3.22)$$

where

$$Z=\frac{1}{\pi}\sqrt{\frac{l_2-x}{x-l_1}}\oint_{l_1}^{l_2}\sqrt{\frac{x'-l_1}{l_2-x'}}\frac{d\hat{z}'/dx'}{x-x'}dx'$$
$$+\frac{1}{\pi}\oint_{l_1}^{l_2}\frac{d\bar{z}'/dx'}{x-x'}dx\,. \qquad (3.23)$$

(3.22) is an integral equation of $d\delta/dx$.

4. Cavity Shape

Substituting (3.15) in (3.19), we get the expression of the cavity shape. To perform the integrations, we use Poincaré-Bertrand's formula [2].

$$\oint_a^b\frac{dt}{t-t_0}\oint_a^b\frac{\psi(t,t_1)}{t_1-t}dt_1$$
$$=-\pi^2\psi(t_0,t_0)+\oint_a^b dt_1\oint_a^b\frac{\psi(t,t_1)}{(t-t_0)(t_1-t)}dt\,,$$
$$a<t_0<b \qquad (4.1)$$

and an integral formula

$$\int_{-1}^{1}\sqrt{\frac{1-\xi'}{1+\xi'}}\frac{1}{\xi-\xi'}\frac{1}{x_0-x''+c_0\xi'}d\xi'$$
$$=\begin{cases}\pi\sqrt{X+1}/\{c_0(\xi+X)\sqrt{X+1}\} \\ \qquad \text{for } x''<l_1 \text{ or } l_2'<x'' \\ 0 \qquad \text{for } l_1<x''<l_2'\,,\end{cases} \qquad (4.2)$$

where $X=(x_0-x'')/c_0$.

Subsequently, it is written in the form

$$2\rho V^2\left(\frac{d\delta}{dx}+\frac{d\bar{z}'}{dx}\right)=2\rho V\frac{d\varphi}{dx}=\frac{1}{\pi}\sqrt{\frac{1+\xi}{1-\xi}}\oint_{-1}^{1}\sqrt{\frac{1-\xi'}{1+\xi'}}\frac{\hat{p}}{\xi-\xi'}d\xi'+2p_c\sqrt{\frac{1+\xi}{1-\xi}}$$

$$+\frac{2\rho V^2}{\pi}\sqrt{\frac{1+\xi}{1-\xi}}\left\{\int_{l_1}^{l_1'}+\int_{l_2'}^{l_2}\right\}\frac{d\bar{z}'}{dx'}\frac{1}{x-x'}\sqrt{\frac{l_2'-x'}{l_1'-x'}}dx' \quad \text{for} \quad l_1'<x<l_2'. \tag{4.3}$$

When $l_1=l_1'$ and $l_2<l_2'$, the third term of the right-hand side is equal to zero.

When we insert (4.3) in (3.2) in the place of $d\varphi/dx$, the integral equation of $\hat{p}$ is written down as follows:

$$w_-=-\frac{1}{2\pi\rho V}\oint_{l_1}^{l_2}\frac{\hat{p}}{x-x'}dx'-\frac{1}{2\pi\rho V}\sqrt{\frac{1+\xi}{1-\xi}}\oint_{-1}^{1}\sqrt{\frac{1-\xi'}{1+\xi'}}\frac{\hat{p}}{\xi-\xi'}d\xi'$$
$$-\frac{p_c}{\rho V}\sqrt{\frac{1+\xi}{1-\xi}}-\frac{V}{\pi}\sqrt{\frac{1+\xi}{1-\xi}}\left\{\int_{l_1}^{l_1'}+\int_{l_2'}^{l_2}\right\}\frac{d\bar{z}'}{dx'}\frac{1}{x-x'}\sqrt{\frac{l_2'-x'}{l_1'-x'}}dx'. \tag{4.4}$$

This equation is fit for solving the problem of the fully cavitated flow as mentioned in the above section.

We can also get the lower boundary z_- of the cavity by integrating (4.4) with respect to x. In such a case the second term and downward are to be omitted such that $x<l_1'$ and $l_2'<x$.

Integrating (4.3) with respect to x from l_1 to x, we get the expression of $\delta(x)$ as follows:

$$\delta(x)+\bar{z}'(x)=\bar{z}'(l_1')+\frac{c_0\cos^{-1}(-\xi)}{2\pi\rho V^2}\int_{-1}^{1}\sqrt{\frac{1-\xi'}{1+\xi'}}\hat{p}d\xi'$$
$$-\frac{c_0}{4\pi\rho V^2}\int_{-1}^{1}\ln\frac{1-\xi\xi'+\sqrt{1-\xi^2}\sqrt{1-\xi'^2}}{|\xi'-\xi|}\hat{p}d\xi'+\frac{p_c c_0}{\rho V^2}$$
$$\times\{\cos^{-1}(-\xi)-\sqrt{1-\xi^2}\}+\frac{\cos^{-1}(-\xi)}{\pi}\left\{\int_{l_1}^{l_1'}+\int_{l_2'}^{l_2}\right\}$$
$$\times\sqrt{\frac{l_2'-x'}{l_1'-x'}}\frac{d\bar{z}'}{dx'}dx'-\frac{2}{\pi}\left\{\int_{l_1}^{l_1'}+\int_{l_2'}^{l_2}\right\}$$
$$\times\tan^{-1}\left(\sqrt{\frac{l_2'-x'}{l_1'-x'}}\tan\frac{\cos^{-1}(-\xi)}{2}\right)\frac{d\bar{z}'}{dx'}dx'. \tag{4.5}$$

When $l_2<l_2'$, the integral between the range l_2' and l_2 in the last two terms of (4.5) is to omitted. We can calculate the upper boundary of the cavity by using (4.5) and the relation $z_+=z_-+2\bar{z}'+2\delta$.

5. The Solutions of the Integral Equations

We can convert the integral equations in the same type as the integral equation of a thin airfoil. So we readily get the solutions by applying the results of the thin airfoil theory to the integral equations.

(i) Fully cavitated flow ($l_1=l_1'$, $l_2<l_2'$)

Changing the variable from ξ to x, we get without difficulty, from (4.4)

$$\frac{dz_-}{dx}-\frac{\sigma}{2}\sqrt{\frac{x-l_1}{l_2'-x}}=\frac{c_0}{\pi\rho V^2}\oint_{l_1}^{l_2}\frac{\hat{p}}{\sqrt{(x'-l_1)/(l_2'-x')}-\sqrt{(x-l_1)/(l_2'-x)}}$$
$$\times\frac{1}{(l_2'-x)\sqrt{(l_2'-x)(x'-l_1)}}dx'. \tag{5.1}$$

Putting in the above equation

$$\left.\begin{aligned}&(1+\Xi)L/2=\sqrt{(x-l_1)/(l_2'-x)}\,, && L=\sqrt{(l_2-l_1)/(l_2'-l_2)}\\ &A(\Xi)=\{dz_-/dx-\sigma L(1+\Xi)/4\}(l_2'-x), && \frac{l_2'-x}{c_0}=\frac{8/L^2}{(1+\Xi)^2+4/L^2}\\ &P(\Xi)=\hat{p}\cdot(l_2'-x)/(\rho V^2)\end{aligned}\right\} \qquad (5.2)$$

we have

$$A(\Xi)=-\frac{1}{\pi}\oint_{-1}^{1}\frac{P(\Xi')}{\Xi-\Xi'}d\Xi'\,. \qquad (5.3)$$

This is the integral equation of the same type as the thin airfoil theory. If we introduce the trailing edge condition (2.5), the solution is written in the form

$$P(\Xi)=\frac{\hat{p}\cdot(l_2'-x)}{\rho V^2}=\frac{1}{\pi}\sqrt{\frac{1-\Xi}{1+\Xi}}\oint_{-1}^{1}\sqrt{\frac{1+\Xi'}{1-\Xi'}}\frac{A(\Xi')}{\Xi-\Xi'}d\Xi' \qquad (5.4)$$

We can evaluate the lift distribution $\hat{p}$ on the hydrofoil by (5.4), if the relation between the cavity length and the cavitation number is determined.

(ii) Partially cavitated flow ($l_1=l_1'$, $l_2'<l_2$)
Putting in (3.22)

$$\left.\begin{aligned}&l/2\cdot(1+\Theta)=\sqrt{(x-l_1)(l_2-x)}\,, && l=\sqrt{(l_2'-l_1)/(l_2-l_2')}\\ &B(\Theta)=\sqrt{(l_2-x)(x-l_1)}\cdot(Z-\sigma/2)/2, && \frac{l_2-x}{c}=\frac{8/l^2}{(1+\Theta)^2+4/l^2}\\ &\varDelta(\Theta)=\sqrt{(l_2-x)(x-l_1)}\cdot d\delta/dx\,,\end{aligned}\right\} \qquad (5.5)$$

we have

$$B(\Theta)=-\frac{1}{\pi}\oint_{-1}^{1}\frac{\varDelta(\Theta')}{\Theta-\Theta'}d\Theta'\,. \qquad (5.6)$$

This is also the same type as the thin airfoil theory. When we introduce the juncture condition (2.4), the solution is written

$$\varDelta(\Theta)=\frac{1}{\pi}\sqrt{\frac{1+\Theta}{1-\Theta}}\oint_{-1}^{1}\sqrt{\frac{1-\Theta'}{1+\Theta'}}\frac{B(\Theta')}{\Theta-\Theta'}d\Theta'\,. \qquad (5.7)$$

We can evaluate the cavity thickness by (5.7), if the relation between the cavity length and the cavitation number is determined. Substituting the solution (5.7) in $d\hat{z}/dx$ in (3.20), we get the lift distribution $\hat{p}$ on the hydrofoil.

6. Closure Condition—I

Referring (5.4) and (5.7), we see that the hydrofoil characteristics depend only upon the relation between the cavity length and the cavitation number which is determined by the closure condition, except for the foil-shape.

Experience seems to indicate that satisfactory results will not be expected from only one kind of cavity models in the whole range of cavitation number. Therefore, it would be better that we pick up the cavity model or the closure condition which contains some empirical parameters. A partly open cavity model of which closure condition is given by (2.6) seems to be suitable for this purpose.

For the case of the fully cavitated flow, the condition is written down in the form

$$\frac{c_0}{2\rho V^2}\int_{-1}^{1}\sqrt{\frac{1-\xi'}{1+\xi'}}\hat{p}d\xi'-\frac{\pi c_0\sigma}{2}=\delta_2 \qquad (6.1)$$

by using (4.5). Inserting (5.4) in (6.1) in the place of $\hat{p}$ and performing the integration by referring the integral formula, we have

$$\sigma=-2L\alpha-\frac{4}{\pi}\frac{\delta_2}{c}(\sqrt{1+L^2}+1)-\frac{16\sqrt{2}(1+L^2)}{\pi L^3\sqrt{\sqrt{1+L^2}+1}}$$

$$\times\int_{-1}^{1}\sqrt{\frac{1+\Xi}{1-\Xi}}\frac{\{(1+\Xi)(\sqrt{1+L^2}-1)+2\}}{(1+\Xi)^2+4/L^2}\frac{dz^*}{dx}d\Xi\,. \qquad (6.2)$$

This equation (6.2) gives the relationship between the cavity length and the cavitation number for the case of the fully cavitated flow.

For the case of the partially cavitated flow, the closure condition is written

$$\delta_2=\int_{l_1}^{l_2'}\frac{d\delta}{dx}dx=\frac{4}{I}\int_{-1}^{1}\frac{\Delta(\Theta)}{(1+\Theta)^2+4/I^2}d\Theta\,. \qquad (6.3)$$

Inserting (5.7) in (6.3) in the place Δ and performing the integration by referring the integral formulas, we have

$$\frac{\delta_2}{c}=\frac{\pi\sigma(\sqrt{1+I^2}-1)}{4(1+I^2)}-\frac{2\sqrt{2}}{I^2\sqrt{1+I^2}}$$

$$\times\int_{-1}^{1}\sqrt{\frac{1-\Theta}{1+\Theta}}\frac{(1+\Theta)\{(1+\Theta)I\sqrt{\sqrt{1+I^2}-1}-2\sqrt{\sqrt{1+I^2}+1}\}}{\{(1+\Theta)^2+4/I^2\}^2}Zd\Theta\,. \qquad (6.4)$$

The equation (6.4) gives the relationship between the cavity length and the cavitation number for the case of the partially cavitated flow.

7. Drag

The drag coefficient may be defined as

$$C_D=\frac{\text{Drag}}{\rho V^2c}$$
$$=\frac{1}{\rho V^2 c}\left\{-\int_{l_1}^{l_2}p_-\frac{dz_-}{dx}dx+\int_{l_1}^{l_2}p_+\frac{dz_+'}{dx}dx\right\}\,. \qquad (7.1)$$

This may be converted in

$$C_D=\frac{1}{\rho V^2c}\left\{-\int_{l_1}^{l_2}p_-\frac{dz_-}{dx}dx+\int_{l_1}^{l_2}p_+\frac{dz_+}{dx}dx-2\int_{l_1'}^{l_2}p_+\frac{d\delta}{dx}dx\right\}$$

$$=\frac{1}{\rho V^2c}\left\{-\int_{l_1}^{l_2}\hat{p}\frac{d\hat{z}}{dx}dx+\int_{l_1}^{l_2\text{ or }l_2'}\bar{p}\frac{d\bar{z}}{dx}dx-2p_c\delta_2\right\} \qquad (7.2)$$

where the upper range of the integral l_2 or l_2' denotes l_2 for $l_2<l_2'$ and l_2' for $l_2<l_2'$.

Inserting (3.7) in (7.2) in the place of $d\hat{z}/dx$ and (3.10) in (7.2) in the place of $\bar{p}$, we have

$$C_D=\frac{1}{\rho V^2c}\left\{\frac{1}{2\pi\rho V^2}\int_{l_1}^{l_2}\hat{p}dx\int_{l_1}^{l_2}\frac{\hat{p}}{x-x'}dx'\right.$$

$$\left.-\frac{2\rho V^2}{\pi}\int_{l_1}^{l_2\text{ or }l_2'}\frac{d\bar{z}}{dx}dx\int_{l_1}^{l_2\text{ or }l_2'}\frac{d\bar{z}/dx'}{x-x'}dx'-2p_c\delta_2\right\}\,. \qquad (7.3)$$

The integrants of (7.3) being antisymmetric with respect to x and x', the first and second terms of right-hand side of (7.3) vanish, if $\hat{p}$ and $d\bar{z}/dx$ have not singularities in the range $l_1<x<l_2$ or l_2'.

According to the thin airfoil theory, it is verified that the first term vanishes, in spite of the fact that $\hat{p}$ has a singularity at a leading edge in general. The second term does not vanish, because $d\bar{z}/dx$ has a sigularity at the end of the cavity. The improper integral can be performed by means of the similar process to the thin airfoil theory.

When we write

$$d\delta/dx = A_0(\xi)/\sqrt{1-\xi} + A_1(\xi) \qquad (7.4)$$

where $\xi=(x-x_0')/c_0$, $x_0'=(l_1'+l_2')/2$, the final expression of the drag coefficient is written in the form

$$C_D=\pi A_0^2(1)c_0/c+\sigma\delta_2/c \qquad (7.5)$$

8. Hydrofoil Characteristics

(i) Fully cavitated flow

When we perform the integration of (5.4), remembering (5.2), the expression for the lift distribution is written in the form

$$\frac{\hat{p}}{\rho V^2}=\sqrt{\frac{1-\Xi}{1+\Xi}}\frac{\sqrt{\sqrt{1+L^2}+1}}{2\sqrt{2}\sqrt{1+L^2}}\left[\alpha\{(1+\Xi)(\sqrt{1+L^2}-1)-2\}+\frac{\sigma}{2}\left\{L(1+\Xi)+\frac{2(\sqrt{1+L^2}-1)}{L}\right\}\right]$$
$$+\frac{8c_0}{\pi L^2(l_2'-x)}\sqrt{\frac{1-\Xi}{1+\Xi}}\int_{-1}^{1}\sqrt{\frac{1+\Xi'}{1-\Xi'}}\frac{1}{(1+\Xi')^2+4/L^2}\frac{1}{\Xi-\Xi'}\frac{dz^*}{dx'}d\Xi'\,. \qquad (8.1)$$

Integrating (8.1) along the chord, we get the expression for the lift coefficient C_L

$$C_L=\frac{1}{c}\int_{l_1}^{l_2}\frac{\hat{p}}{\rho V^2}dx=-\frac{\pi c_0(\sqrt{1+L^2}-1)}{c(1+L^2)}\cdot\left(\alpha-\frac{\sigma L}{2}\right)-\frac{16c_0\sqrt{\sqrt{1+L^2}-1}}{\sqrt{2}cL^3}$$
$$\times\int_{-1}^{1}\sqrt{\frac{1+\Xi}{1-\Xi}}\frac{(\sqrt{1+L^2}+1)(1+\Xi)-2}{\{(1+\Xi)^2+4/L^2\}^2}\frac{dz^*}{dx}d\Xi\,. \qquad (8.2)$$

In a similar way as the calculation of C_L, we get for the moment coefficient about the rear end of the cavity C_M

$$C_M=\frac{1}{2c^2\rho V^2}\int_{l_1}^{l_2}\hat{p}\cdot(l_2'-x)\,dx=\frac{\pi c_0^2}{8(1+L^2)^2c^2}\{(2+L^2)(\alpha L^2-\alpha+\sigma L)+2(1+L^2)^{3/2}(\alpha-\sigma L)\}$$
$$-\frac{64c_0^2}{L^4c^2}\int_{-1}^{1}\sqrt{\frac{1+\Xi}{1-\Xi}}\frac{1}{\{(1+\Xi)^2+4/L^2\}^2}$$
$$\times\left[\frac{\sqrt{2}\sqrt{\sqrt{1+L^2}-1}\{1-(\sqrt{1+L^2}+1)(1+\Xi)/2\}}{L\{(1+\Xi)^2+4/L^2\}}-\frac{L\sqrt{\sqrt{1+L^2}-1}}{8\sqrt{2}(1+L^2)}\right.$$
$$\left.\times\{\sqrt{1+L^2}(\sqrt{1+L^2}-1)+L^2(2+\sqrt{1+L^2}(1+\Xi)/2)\}\right]\frac{dz^*}{dx}d\Xi\,. \qquad (8.3)$$

If we change the variable x into Ξ by using the relation (5.2), the slope of the upper boundary of the foil-cavity region is expressed in the similar form to (5.3)

$$\frac{dz_+}{dx}=-\frac{\sigma L(1+\Xi)}{4}$$
$$+\frac{1}{\pi(l_2'-x)}\int_{-1}^{1}\frac{P(\Xi')}{\Xi+\Xi'+2}d\Xi'\,. \qquad (8.4)$$

Inserting (5.4) in (5.3) and (8.4) in the place of $P(\Xi)$ and performing the integrations, we get for the slopes of the upper surface and the lower surface

$$\frac{dz_-}{dx}=\sqrt{\frac{\Xi-1}{\Xi+1}}\frac{1}{2\sqrt{2}\sqrt{1+L^2}}[\sqrt{\sqrt{1+L^2}-1}\{\alpha L(1+\Xi)+\sigma\}$$
$$+\sqrt{\sqrt{1+L^2}+1}\{\sigma L(1+\Xi)/2-2\alpha\}]+\alpha$$
$$+\frac{8c_0}{\pi L^2(l_2'-x)}\sqrt{\frac{\Xi-1}{\Xi+1}}\int_{-1}^{1}\sqrt{\frac{1+\Xi'}{1-\Xi'}}\frac{1}{(1+\Xi')^2+4/L^2}\frac{1}{\Xi-\Xi'}\frac{dz^*}{dx'}d\Xi'$$
$$\text{for } \Xi>1 \tag{8.5}$$

$$\frac{dz_+}{dx}=-\sqrt{\frac{\Xi+3}{\Xi+1}}\frac{1}{2\sqrt{2}\sqrt{1+L^2}}[\sqrt{\sqrt{1+L^2}-1}\{\alpha L(1+\Xi)-\sigma\}$$
$$+\sqrt{\sqrt{1+L^2}+1}\{\sigma L(1+\Xi)/2+2\alpha\}]+\alpha$$
$$-\frac{8c_0}{\pi L^2(l_2'-x)}\sqrt{\frac{\Xi+3}{\Xi+1}}\int_{-1}^{1}\sqrt{\frac{1+\Xi'}{1-\Xi'}}\frac{1}{(1+\Xi')^2+4/L^2}\frac{1}{\Xi+\Xi'+2}\frac{dz^*}{dx'}d\Xi' . \tag{8.6}$$

dz_-/dx is known for $\Xi<1$, because $\Xi<1$ corresponds to $x<l_2$.

The cavity shape is obtained by integrating (8.5) and (8.6) with respect to x.

Substracting (8.5) from (8.6) and dividing it by 2, we have

$$\frac{d\bar{z}}{dx}=-\frac{1}{4}\frac{1}{\sqrt{\Xi+1}}\sqrt{\frac{\sqrt{1+L^2}-1}{2(1+L^2)}}\Big[(1+\Xi)(\sqrt{\Xi-1}+\sqrt{\Xi+3})$$
$$\times\left\{\alpha L+\frac{\sigma}{2}(\sqrt{1+L^2}+1)\right\}-2(\sqrt{\Xi-1}-\sqrt{\Xi+3})$$
$$\times\left\{\frac{\alpha(\sqrt{1+L^2}+1)}{L}-\frac{\sigma}{2}\right\}\Big]-\frac{4c_0}{\pi L^2(l_2'-x)}\frac{1}{\sqrt{\Xi+1}}$$
$$\times\int_{-1}^{1}\sqrt{\frac{1+\Xi'}{1-\Xi'}}\frac{dz^*/dx}{(1+\Xi')^2+4/L^2}\left\{\frac{\sqrt{\Xi-1}}{\Xi-\Xi'}-\frac{\sqrt{\Xi+3}}{\Xi+\Xi'+2}\right\}d\Xi' \quad \text{for } \Xi>1 . \tag{8.7}$$

$A_0(1)$ which is needed for the calculation of the drag, is deduced from (8.7). Since $\lim_{x\to l_2'}\sqrt{1-\xi}=2\sqrt{2}/\{L(1+\Xi)\}$ by (5.2), $A_0(1)$ results

$$A_0(1)=\lim_{x\to l_2'}\sqrt{1-\xi}\frac{d\bar{z}}{dx}$$
$$=\lim_{x\to l_2'}\frac{2\sqrt{2}}{L(1+\Xi)}\frac{d\bar{z}}{dx} \tag{8.8}$$

from (7.4). When $x\to l_2'$, $1+\Xi\to\infty$, $\sqrt{\Xi-1}+\sqrt{\Xi+3}\to2\sqrt{\Xi+1}$, $\sqrt{\Xi+1}-\sqrt{\Xi+3}\to0$, $(\Xi+\Xi'+2)(\Xi-\Xi')\to(\Xi+1)^2$ and $\sqrt{\Xi-1}\to\sqrt{\Xi+1}$.

Applying these characteristics to (8.7), we get

$$A_0(1)=-\frac{\sqrt{\sqrt{1+L^2}-1}}{\sqrt{1+L^2}}\left(\alpha+\frac{\sigma}{2}\frac{\sqrt{1+L^2}+1}{L}\right)$$
$$-\frac{2\sqrt{2}}{\pi L}\int_{-1}^{1}\sqrt{\frac{1+\Xi'}{1-\Xi'}}\frac{1}{(1+\Xi')^2+4/L^2}\frac{dz^*}{dx}d\Xi' \tag{8.9}$$

Inserting (8.9) in (7.5) in the place of $A_0(1)$, we get the expression of the drag coefficient.

(ii) Partially cavitated flow

On using (3.20), we have

$$C_L=\frac{1}{cV^2}\int_{l_1}^{l_2}\hat{\phi}dx=-\frac{2}{c}\int_{l_1}^{l_2'}\sqrt{\frac{x-l_1}{l_2-x}}\frac{d\delta}{dx}dx-2\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{d\hat{z}'}{dx}d\zeta \tag{8.10}$$

where $\zeta=(x-x_0)/c$, $x_0=(l_1+l_2)/2$.

Substituting (5.7) for $d\delta/dx$ in (8.10), we get

$$C_L=\frac{\pi\sigma I(\sqrt{1+I^2}-1)}{2(1+I^2)}-2\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{d\hat{z}'}{dx}d\zeta-\frac{8}{\sqrt{2}I^2\sqrt{1+I^2}}$$
$$\times\int_{-1}^{1}\sqrt{\frac{1-\Theta}{1+\Theta}}\frac{(1+\Theta)\{(1+\Theta)I\sqrt{\sqrt{1+I^2}+1}+2\sqrt{\sqrt{1+I^2}-1}\}}{\{(1+\Theta)^2+4/I^2\}^2}Zd\Theta\,. \tag{8.11}$$

For the moment coefficient about the middle point of the chord C_M, from a similar way as the calculation of C_L, it follows that

$$C_M=\frac{1}{2V^2}\int_{-1}^{1}\hat{\phi}\zeta d\zeta=\frac{1}{c^2}\int_{l_1}^{l_2}\Delta dx+\int_{-1}^{1}\sqrt{1-\zeta^2}\frac{d\hat{z}'}{dx}d\zeta$$
$$=2I\int_{-1}^{1}\sqrt{\frac{1-\Theta}{1+\Theta}}\frac{(1+\Theta)i_2^{(2)}(\Theta)}{(1+\Theta)^2+4/I^2}Zd\Theta+\int_{-1}^{1}\sqrt{1-\zeta^2}\frac{d\hat{z}'}{dx}d\zeta$$
$$-\frac{\pi\sigma I^2}{4\sqrt{2}\sqrt{1+I^2}}\{I\sqrt{\sqrt{1+I^2}+1}\,(2m_3^{(2)}-m_3^{(3)})$$
$$+2\sqrt{\sqrt{1+I^2}-1}\,(2m_3^{(1)}-m_3^{(2)})\}+\frac{\pi\sigma I^5}{16}$$
$$\times\left\{m_2^{(2)}(2m_2^{(2)}-m_2^{(3)})+\frac{4}{I^2}(m_1^{(1)}-m_2^{(1)})(2m_2^{(1)}-m_2^{(2)})\right\}\,. \tag{8.12}$$

By making use of (5.7), we can deduce the expression for the cavity shape

$$\frac{d\delta}{dx}=-\frac{\sigma I^2\{(1+\Theta)^2+4/I^2}{16\sqrt{1-\Theta^2}}\{2i_1^{(1)}-i_1^{(2)}$$
$$+\frac{(1+\Theta)^2+4/I^2}{2\pi\sqrt{1-\Theta^2}}\int_{-1}^{1}\sqrt{\frac{1-\Theta'}{1+\Theta'}}\frac{1+\Theta'}{(1+\Theta')^2+4/I^2}\frac{Z}{\Theta-\Theta'}d\Theta'\,. \tag{8.13}$$

$A_0(1)$ is written in the form

$$A_0(1)=\lim_{\xi\to1}\frac{d\delta}{dx}\sqrt{1-\xi}$$
$$=\sqrt{\frac{2}{1+I^2}}\lim_{\Theta\to1}\frac{d\delta}{dx}\sqrt{1-\Theta} \tag{8.14}$$

since

$$1-\xi=(2/I^2)(1-\Theta)(3+\Theta)/\{(1+\Theta)^2+4/I^2\}\,.$$

Substituing (8.13) in (8.14), we get the expression of $A_0(1)$.

9. Calculating Method of the Characteristics of the Hydrofoil of Which Shape is Expressed by Polynomials in x

We can find a simple method for the calculation of the hydrofoil characteristics, if the hydrofoil shapes are expressed by polynomials of the form

$$\frac{dz^*}{dx}=\sum_{n=0}^{m}t_n\cdot\left(\frac{l_2'-x}{b}\right)^n=\sum_{n=0}^{m}t_n\cdot\left(\frac{2}{L^2}+1-\zeta\right)^n \tag{9.1}$$

$$=\sum_{n=0}^{m}t_n\cdot\left\{\frac{8c_0/(cL^2)}{(1+\Xi)^2+4/L^2}\right\}^n \tag{9.2}$$

or

$$\frac{dz^*}{dx}=\sum_{n=0}^{m} a_n\zeta^n$$
$$=\sum_{n=0}^{m} a_n\left(\frac{2c_0}{c}\right)^n\left\{\lambda-\frac{4/L^2}{(1+\Xi)^2+4/L^2}\right\}^n \quad (9.3)$$

for the fully cavitated flow, where $\lambda=(1+L^2/2)/(1+L^2)$ and

$$\frac{d\hat{z}'}{dx}=\sum_{n=0}^{m} t_n'\zeta^n+\alpha \quad (9.4)$$

$$\frac{d\bar{z}'}{dx}=\frac{1}{\sqrt{1-\zeta^2}}\sum_{n=0}^{r} s_n\zeta^n \quad (9.5)$$

or

$$\frac{d\bar{z}'}{dx}=\sum_{n=0}^{r} o_n\zeta^n \quad (9.6)$$

for the partially cavitated flow.

(i) Fully cavitated flow.

Inserting (9.2) in (6.2), (8.2), (8.3), (8.9) and (8.1) in the place of dz^*/dx and performing the integrations by using the characteristic functions shown in the Appendix, we have

$$\sigma=-2\alpha L-\frac{4}{\pi}\frac{\delta_2}{c}(\sqrt{1+L^2}-1)-\sqrt{2}(1+L^2)\sqrt{\sqrt{1+L^2}-1}$$
$$\times\left[(\sqrt{1+L^2}-1)\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+2}^{(2)}+2\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+2}^{(2)}\right], \quad (9.7)$$

$$C_L=-\pi/L^2\cdot(\sqrt{1+L^2}-1)(\alpha-\sigma L/2)-\pi(1+L^2)\sqrt{\sqrt{1+L^2}-1}/(\sqrt{2}L)$$
$$\times\left[(\sqrt{1+L^2}+1)\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+2}^{(2)}-2\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+2}^{(1)}\right], \quad (9.8)$$

$$C_M=\pi/8L^4)\cdot[(L^2+2)\{\alpha(L^2-1)+\sigma L\}+2(1+L^2)^{3/2}(\alpha-\sigma L)]$$
$$-4\pi(1+L^2)^2/L^4\left[\frac{L}{2\sqrt{2}}\sqrt{\sqrt{1+L^2}-1}\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+3}^{(1)}\right.$$
$$-\frac{L^2}{4\sqrt{2}}\sqrt{\sqrt{1+L^2}+1}\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+3}^{(2)}$$
$$-\frac{L(\sqrt{1+L^2}-1)^{3/2}}{8\sqrt{2}\sqrt{1+L^2}}\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+2}^{(1)}$$
$$\left.-\frac{L^3\sqrt{\sqrt{1+L^2}-1}(\sqrt{1+L^2}+2)}{16\sqrt{2}(1+L^2)}\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+2}^{(2)}\right], \quad (9.9)$$

$$A_0(1)=-\sqrt{\sqrt{1+L^2}-1}/\sqrt{1+L^2}\cdot\{\alpha+\sigma(\sqrt{1+L^2}+1)/(2L)\}$$
$$-\frac{L}{\sqrt{2}}\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n m_{n+1}^{(1)}, \quad (9.10)$$

and

$$\frac{\hat{p}}{\rho V^2}=\frac{2c_0}{l_2'-x}\sqrt{\frac{1-\Xi}{1+\Xi}}\sum_{n=0}^{m} t_n\left(\frac{2c_0}{c}\right)^n i_{n+1}^{(1)}(\Xi)+\frac{1}{\sqrt{2}\sqrt{1+L^2}}\sqrt{\frac{1-\Xi}{1+\Xi}}$$
$$\times[\sqrt{\sqrt{1+L^2}-1}\{\alpha L(1+\Xi)+\sigma\}/2+\sqrt{\sqrt{1+L^2}+1}\{\sigma L/4\cdot(1+\Xi)-\alpha\}]. \quad (9.11)$$

We can get the similar expressions of dz_-/dx and dz_+/dx in a similar way as above.

If m is not a small integer, the above expressions are not suitable for the numerical calculation. The expression (9.3) is needed for this case and, for example, the lift coefficient is expressed in the form

$$C_L=-\pi/L^2\cdot(\sqrt{1+L^2}-1)(\alpha-\sigma L/2)-\pi(1+L^2)\sqrt{\sqrt{1+L^2}-1}/(\sqrt{2}L)$$
$$\times\left\{(\sqrt{1+L^2}+1)\sum_{n=0}^{m}a_nS_{n,2}^{(2)}-2\sum_{n=0}^{m}a_nS_{n,2}^{(1)}\right\} \tag{9.12}$$

where

$$S_{k,r}^{(j)}=\frac{1}{\pi}\left(\frac{2c_0}{c}\right)^k\int_0^\pi\frac{(4/L^2)^r(1+\cos\theta)^j}{\{(1+\cos\theta)^2+4/L^2\}^r}\left\{\lambda-\frac{4/L^2}{(1+\cos\theta)^2+4/L^2}\right\}^k d\theta\,. \tag{9.13}$$

It is hoped that the tables of $S_{k,r}^{(j)}$ will be constructed in future.

(ii) Partially cavitated flow

We know the integral formulas

$$\left.\begin{aligned}
&\frac{1}{\pi}\int_{-1}^{1}\frac{\zeta'^n}{\sqrt{1-\zeta'^2}(\zeta'-\zeta)}d\zeta'=b_{n-1}+b_{n-2}\zeta+\cdots+b_1\zeta^{n-2}+b_0\zeta^{n-1}\\
&\frac{1}{\pi}\int_{-1}^{1}\frac{\zeta'^n}{\zeta'-\zeta}d\zeta'=h_n+h_{n-1}\zeta+\cdots+h_1\zeta^{n-1}+\frac{\zeta^n}{\pi}\ln\frac{1-\zeta}{1+\zeta}\\
&\text{where}\\
&b_n=\frac{1}{\pi}\int_{-1}^{1}\frac{\zeta^n}{\sqrt{1-\zeta^2}}d\zeta=\frac{1}{2}\{1+(-)^n\}\frac{1\cdot3\cdot5\cdots(n-1)}{2\cdot4\cdot6\cdots n}\\
&h_n=\frac{1}{\pi n}\{1-(-)^n\}\,.
\end{aligned}\right\} \tag{9.14}$$

Inserting (9.4) and (9.5) or (9.6) in (3.23) in the place of $d\hat{z}'/dx$ and $d\bar{z}'/dx$, and performing the integrations by using (9.14), we have

$$Z=-\alpha\sqrt{\frac{l_2-x}{x-l_1}}-\sqrt{\frac{l_2-x}{x-l_1}}\sum_{n=0}^{m}d_n\zeta^n-\sum_{n=0}^{r-1}f_n\zeta^n \tag{9.15}$$

or

$$Z=-\alpha\sqrt{\frac{l_2-x}{x-l_1}}-\sqrt{\frac{l_2-x}{x-l_1}}\sum_{n=0}^{m}d_n\zeta^n-\sum_{n=0}^{r-1}g_n\zeta^n$$
$$-\frac{1}{\pi}\ln\frac{l_2-x}{x-l_1}\sum_{n=0}^{r}o_n\zeta^n \tag{9.16}$$

where

$$\left.\begin{aligned}
d_k&=\sum_{j=k+1}^{m}t_j'\,b_{j-k-1}+\sum_{j=k}^{m}t_j'\,b_{j-k}\\
f_k&=\sum_{j=k+1}^{r}s_j\,b_{j-k-1}\\
g_k&=\sum_{j=k+1}^{r}o_j\,h_{j-k}\,,\qquad g_r=0\,.
\end{aligned}\right\} \tag{9.17}$$

Inserting (9.15) or (9.16) in (6.4) and (8.11) in the place of Z, we have

$$\sigma=-\frac{2\alpha I(\sqrt{1+I^2}+1)}{\sqrt{1+I^2}-1}-\frac{4}{\pi}\frac{\delta_2}{c}\frac{1+I^2}{\sqrt{1+I^2}-1}+\frac{\sqrt{2}I\sqrt{1+I^2}}{\sqrt{1+I^2}-1}\sum_{n=0}^{m}d_nT_n^{(0)}$$
$$+\frac{I^2\sqrt{1+I^2}}{\sqrt{2}(\sqrt{1+I^2}-1)}\sum_{n=0}^{r-1}f_nT_n^{(1)} \tag{9.18}$$

or

$$\sigma=-\frac{2\alpha I(\sqrt{1+I^2}+1)}{\sqrt{1+I^2}-1}-\frac{4}{\pi}\frac{\delta_2}{c}\frac{1+I^2}{\sqrt{1+I^2}-1}+\frac{\sqrt{2}\,I\sqrt{1+I^2}}{\sqrt{1+I^2}-1}\sum_{n=0}^{m}d_nT_n^{(0)}$$
$$+\frac{I^2\sqrt{1+I^2}}{\sqrt{2}(\sqrt{1+I^2}-1)}\left\{\sum_{n=0}^{r}G_nT_n^{(1)}-2\sum_{n=0}^{r}o_nL_n^{(1)}\right\} \qquad (9.19)$$

and

$$C_L=\frac{\pi\sigma I(\sqrt{1+I^2}-1)}{2(1+I^2)}-\frac{\pi\alpha(\sqrt{1+I^2}+1)}{1+I^2}-2\pi\sum_{n=0}^{m}t_n'(b_n+b_{n+1})$$
$$+\frac{\pi I}{\sqrt{2}\sqrt{1+I^2}}\sum_{n=0}^{m}d_n\tilde{T}_n^{(0)}+\frac{\pi I^2}{2\sqrt{2}\sqrt{1+I^2}}\sum_{n=0}^{r-1}f_n\tilde{T}_n^{(1)} \qquad (9.20)$$

or

$$C_L=\frac{\pi\sigma I(\sqrt{1+I^2}-1)}{2(1+I^2)}-\frac{\pi\alpha(\sqrt{1+I^2}+1)}{1+I^2}-2\pi\sum_{n=0}^{m}t_n'(b_n+b_{n+1})$$
$$+\frac{\pi I}{\sqrt{2}\sqrt{1+I^2}}\sum_{n=0}^{m}d_n\tilde{T}_n^{(0)}+\frac{\pi I^2}{2\sqrt{2}\sqrt{1+I^2}}\left\{\sum_{n=0}^{r}G_n\tilde{T}_n^{(1)}-2\sum_{n=0}^{r}o_n\tilde{L}_n^{(1)}\right\} \qquad (9.21)$$

where

$$\left.\begin{aligned}
T_n^{(j)}&=I\sqrt{\sqrt{1+I^2}-1}\,(2T_{2,n}^{(j+1)}-T_{2,n}^{(j+2)})-2\sqrt{\sqrt{1+I^2}+1}\,(2T_{2,n}^{(j)}-T_{2,n}^{(j+1)})\\
L_n^{(j)}&=I\sqrt{\sqrt{1+I^2}-1}\,(2L_{2,n}^{(j+1)}-L_{2,n}^{(j+2)})-2\sqrt{\sqrt{1+I^2}+1}\,(2L_{2,n}^{(j)}-L_{2,n}^{(j+1)})\\
\tilde{T}_n^{(j)}&=I\sqrt{\sqrt{1+I^2}+1}\,(2T_{2,n}^{(j+1)}-T_{2,n}^{(j+2)})+2\sqrt{\sqrt{1+I^2}-1}\,(2T_{2,n}^{(j)}-T_{2,n}^{(j+1)})\\
\tilde{L}_n^{(j)}&=I\sqrt{\sqrt{1+I^2}+1}\,(2L_{2,n}^{(j+1)}-L_{2,n}^{(j+2)})+2\sqrt{\sqrt{1+I^2}-1}\,(2L_{2,n}^{(j)}-L_{2,n}^{(j+1)})\\
G_n&=g_n+2/\pi\cdot\ln(2/I)\cdot o_n
\end{aligned}\right\} \qquad (9.22)$$

and

$$\left.\begin{aligned}
T_{r,k}^{(j)}&=\frac{1}{\pi}\left(\frac{4}{I^2}\right)^r\int_0^\pi\frac{(1+\cos\theta)^j}{\{(1+\cos\theta)^2+4/I^2\}^r}\left\{1-\frac{8/I^2}{(1+\cos\theta)^2+4/I^2}\right\}^k d\theta\\
L_{r,k}^{(j)}&=\frac{1}{\pi}\left(\frac{4}{I^2}\right)^r\int_0^\pi\frac{(1+\cos\theta)^j}{\{(1+\cos\theta)^2+4/I^2\}^r}\left\{1-\frac{8/I^2}{(1+\cos\theta)^2+4/I^2}\right\}^k \ln(1+\cos\theta)\,d\theta\,.
\end{aligned}\right\} \qquad (9.23)$$

If we introduce $m_n^{(j)}$ and a function defined by

$$\chi_n^{(j)}=\frac{1}{\pi}\left(\frac{4}{I^2}\right)^n\int_0^\pi\frac{(1+\cos\theta)^j\ln(1+\cos\theta)}{\{(1+\cos\theta)^2+4/I^2\}^n}d\theta \qquad (9.24)$$

$T_{r,k}^{(j)}$ and $L_{r,k}^{(j)}$ are given by

$$\left.\begin{aligned}
T_{r,k}^{(j)}&=\sum_{p=0}^{k}(-2)^p{}_kC_p m_{r+p}^{(j)}\\
L_{r,k}^{(j)}&=\sum_{p=0}^{k}(-2)^p{}_kC_p\chi_{k+p}^{(j)}
\end{aligned}\right\} \qquad (9.25)$$

where ${}_kC_p$ denotes the binomial coefficient.

If k is not small (e.g. $k>5$), it is better to perform the integration of (9.23) by means of numerical computation than to use (9.25).

If it is assumed that the hydrofoil is a flat plate and then $\delta_2=0$, it follows from the above expressions that

$$\left.\begin{aligned}
\sigma&=-2L\alpha\\
C_L&=-\pi\alpha c_0/c\cdot(\sqrt{1+L^2}-1)
\end{aligned}\right\} \qquad (9.26)$$

for the fully cavitated flow and

$$\left.\begin{aligned}-\alpha/\sigma&=(\sqrt{1+I^2}-1)/\{2I(\sqrt{1+I^2}+1)\}\\ C_L&=-\pi\alpha(\sqrt{1+I^2}+1)\end{aligned}\right\}\tag{9.27}$$

for the partially cavitated flow.

Geurst [3], [4], [5] solved the similar problem using the technique of conformal mapping and he assumed that the cavity is closed. His results are as follows:
for the fully cavitated flow

$$\left.\begin{aligned}-\alpha&=\frac{\sigma}{2}\tan\frac{\gamma}{2}\\ C_L&=\frac{-\pi\alpha}{\sin\gamma/2\cdot(1+\sin\gamma/2)}\end{aligned}\right\}\tag{9.28}$$

where

$$\frac{c}{c_0}=\frac{1+\cos\gamma}{2}=\cos^2\frac{\gamma}{2}$$

and for the partially cavitated flow

$$\left.\begin{aligned}-\frac{\alpha}{\sigma}&=\frac{1}{2}\tan\frac{\gamma}{2}\cdot\frac{1-\sin\gamma/2}{1+\sin\gamma/2}\\ C_L&=-\pi\alpha\left(1+\frac{1}{\sin\gamma/2}\right)\end{aligned}\right\}\tag{9.29}$$

where

$$\frac{c_0}{c}=\cos^2\frac{\gamma}{2}.$$

Since

$$\frac{c}{c_0}=\frac{l_2-l_1}{l_2'-l_1}=\frac{L^2}{1+L^2}=\cos^2\frac{\gamma}{2}$$

and

$$\sin\frac{\gamma}{2}=\frac{1}{\sqrt{1+L^2}}$$

for the fully cavitated flow and

$$\frac{c_0}{c}=\frac{l_2'-l_1}{l_2-l_1}=\frac{I^2}{1+I^2}=\cos^2\frac{\gamma}{2}$$

and

$$\sin\frac{\gamma}{2}=\frac{1}{\sqrt{1+I^2}}$$

for the partially cavitated flow, (9.26) and (9.27) agree with (9.28) and (9.29) respectively.

10. Closure Condition—II

Since flows with cavities of finite length do not exist in the frame of inviscid flow theory, several flow models have been proposed in an attempt to represent accurately the physical flow. As they have the same boundary conditions, except for the closure condition, the various theories produce similar results, regardless of the model used. However experimental results seem to indicate that the use of a sort of model is not physically justifiable in the whole range of cavitation number. Especially, the flow model theories seem to break down for cavity lengths near to foil chords. It rather seems to be realistic and suitable to aid the designer that we develop the theory of which closure conditions are selected as a function of the cavitation number to agree with experimental data.

In this paper, the author proposes to select the cavity open width δ_2 to agree with experimental data.

For the case of a flat plate, we have

$$-\frac{\alpha}{\sigma}=\frac{1}{2L-4/\pi\cdot(\sqrt{1+L^2}-1)\delta^*}=\frac{\sqrt{1+L^2}+1}{2L(\sqrt{1+L^2}+1)-4\pi L^2\delta^*}\tag{10.1}$$

for $c_0/c>1$, and

$$-\frac{\alpha}{\sigma}=\frac{\sqrt{1+I^2}-1}{2I(\sqrt{1+I^2}+1)-4/\pi(1+I^2)\delta^*}\tag{10.2}$$

for $c_0/c<1$, from (6.2) and (6.4), where $\delta^*=-\delta_2/(c\alpha)$.

If we put

$$\delta^*=\frac{\pi}{4}F_1(c_0/c)\cdot\left\{2+\frac{kF_2(c_0/c)}{1+L^*}\right\}\tag{10.3}$$

and insert it in (10.1) and (10.2), we get

$$-\frac{\alpha}{\sigma}=\frac{\sqrt{1+L^2}+1}{2L(\sqrt{1+L^2}+1)-L^2F_1\{2+kF_2/(1+L^*)\}}\quad\text{for } c_0/c>1\tag{10.4}$$

$$-\frac{\alpha}{\sigma}=\frac{\sqrt{1+I^2}-1}{2I(\sqrt{1+I^2}+1)-(1+I^2)F_1\{2+kF_2/(1+L^*)\}} \quad \text{for } c_0/c<1 \tag{10.5}$$

where

$$F_1(1)=1,\quad F_2(1)=1,$$

$$L^*=\sqrt{(l_2-l_1)/|l_2'-l_2|}$$

and F_1, F_2 and k denote empirical factors.

So we have

$$-\frac{\alpha}{\sigma}\bigg|_{c_0/c=1}=\frac{1}{2-k}\,. \tag{10.6}$$

k is determined by contrasting (10.6) with reliable experimental data.

11. Numerical Examples

In Fig. 2 the lengths of the cavities measured by a few researchers are plotted as a function of $\sigma/(\alpha_0-\alpha)$, in which α_0 is the angle of attack at zero lift. If F_1 and F_2 take the values indicated in Fig. 4 and k takes -8, the equations (10.4) and (10.5) agree with the experimental results, and then lift$\sim\sigma$ curve agrees with experimental information in outline as shown in Fig. 3.

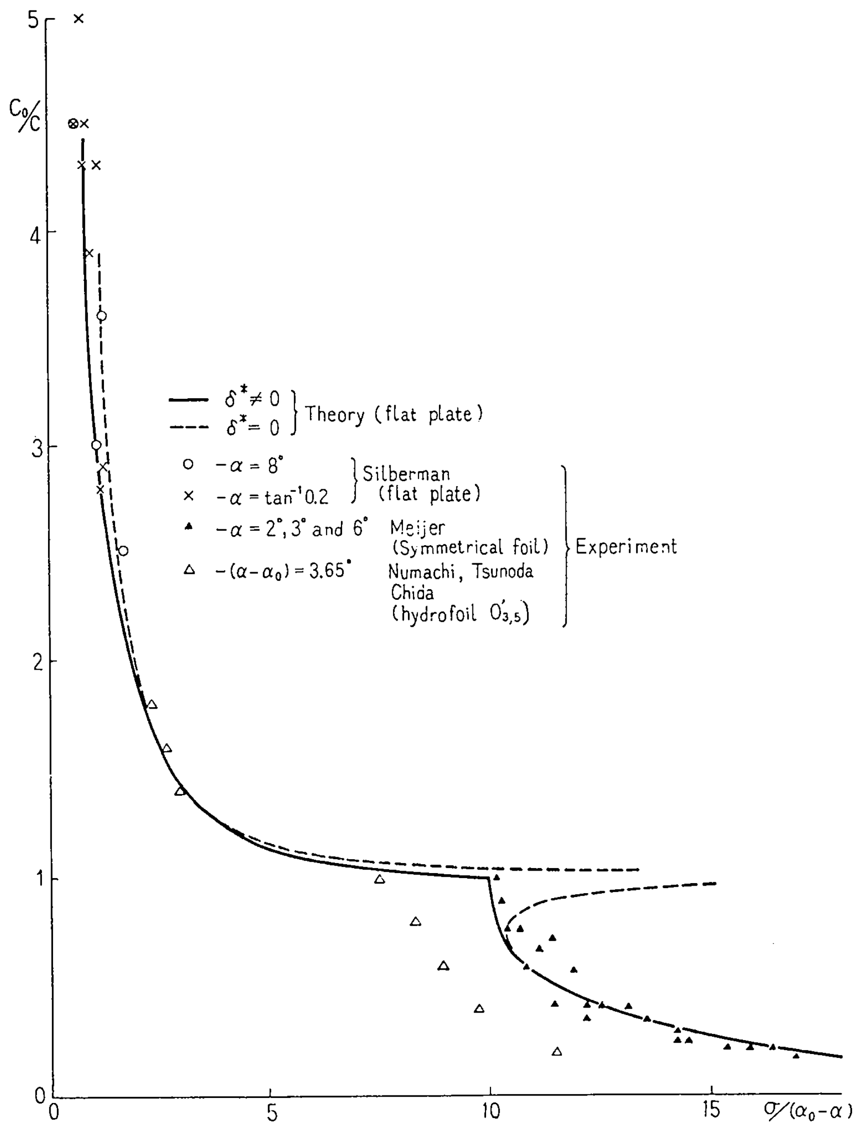

Fig. 2.

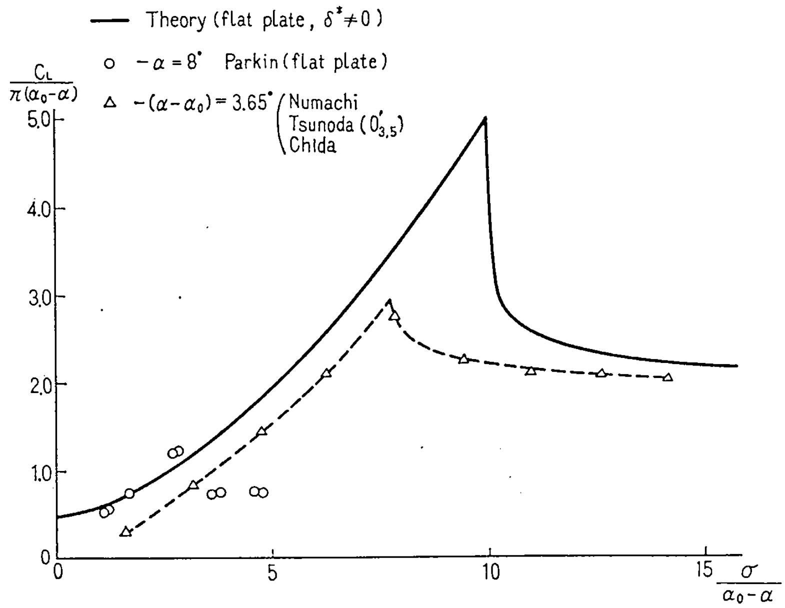

Fig. 3.

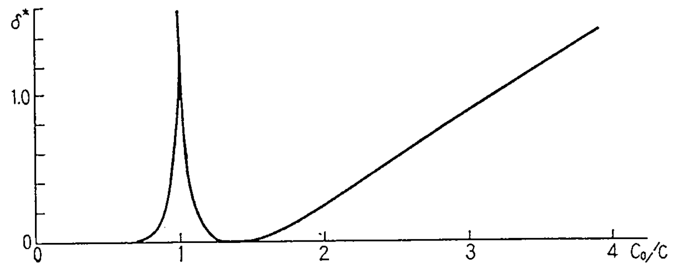

Fig. 4.

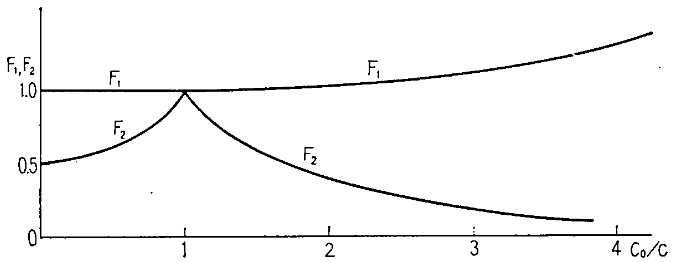

Fig. 5.

δ_2 which is evaluated by using the values of F_1 and F_2 indicated in Fig. 5 is shown in Fig. 4.

As the wake thickness will correspond to instability of cavitation, the estimation of wake width will be important in real problem of cavitation. The evaluated value of δ_2 will be useful for the estimation of the wake width of real flow.

When the hydrofoils have camber and thickness, we may write, referring to (9.7) and (9.18) or (9.19),

$$\sigma=-2\alpha L-\frac{4}{\pi}\frac{\delta_2}{c}(\sqrt{1+L^2}-1)+2L\mu(L)\,,\qquad \text{for } c_0/c>1 \tag{11.1}$$

$$\sigma=-\frac{2\alpha I(\sqrt{1+I^2}+1)}{\sqrt{1+I^2}-1}-\frac{4}{\pi}\frac{\delta_2}{c}\frac{1+I^2}{\sqrt{1+I^2}-1}+\frac{2\mu(I)I(\sqrt{1+I^2}+1)}{\sqrt{1+I^2}-1}\,,\quad \text{for } c_0/c<1\,. \tag{11.2}$$

We can evaluate μ by the rest terms of the first and the second terms in the right-hand side of (9.7) and (9.18) or (9.19). μ is a function of L or I, and also it tends to a finite value, when $c_0/c\to 1$, because it is proved, through the recursions formulas, that $m_n^{(j)}(L)$ and $T_{r,k}^{(j)}(L)$ tend to $1/L^{j+1/2}$ as $L\to\infty$.

If we insert $\alpha-\mu$ in (10.1) and (10.2) in the place of α, we can carry out the evaluation for a hydrofoil with camber and thickness by the same way with the case of a flat plate.

In Fig. 6 we show a result of the evaluation with respect to a hydrofoil expressed by the equations

$$\bar{z}'/c=0.04\{0.3\sqrt{1-\zeta^2}+0.7(1-\zeta^2)^{3/2}\},$$

$$\hat{z}'/c=0.04(1-\zeta^2).$$

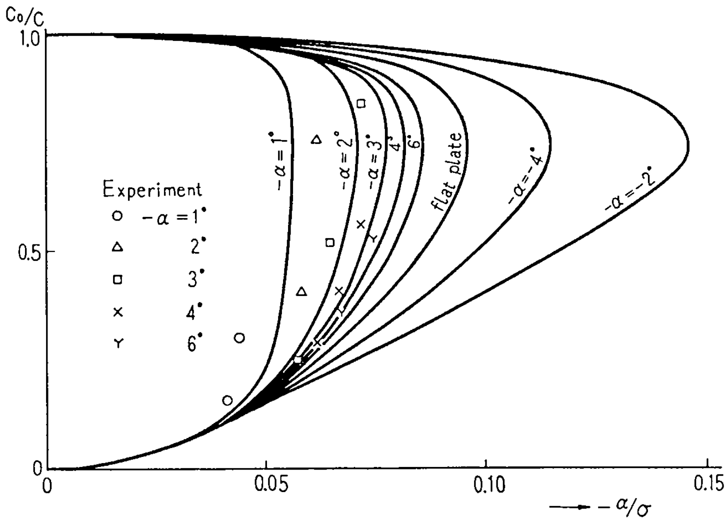

Fig. 6.

In the calculation it is assumed that $\delta_2=0$. This hydrofoil is similar to the profile which was tested by Meijer [6]. The theoretical results agree with experimental data in outline, except for cavity lengths near to foil chord.

In order to know the tendency of δ_2, much more experiments are wanted.

12. Other Partially Cavitated Flow ($l_1<l_1'$)

In this section we treat the cavity flow problem that the cavitation occurs at an intermediate position of a chord. In these cases we know three kinds as shown in Fig. 7 (a), (b), (c).

We omit the detail descriptions regarding the case (c), because the problem can be solved by the same way with the fully cavitated flow, if we regard the length $\overline{l_1l_1'}$ as the chord.

In the following analysis, it is assumed

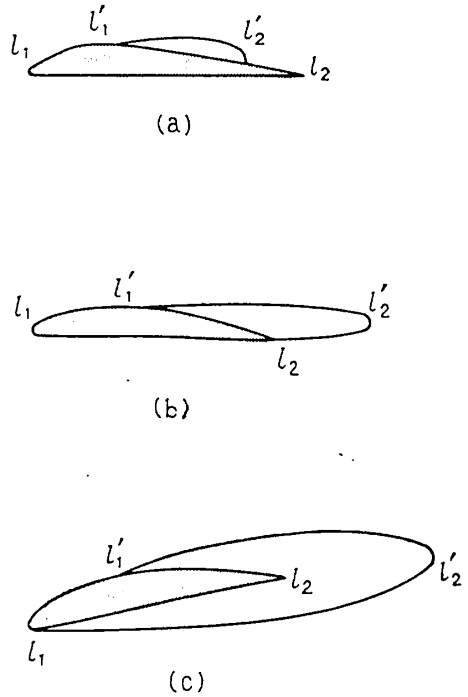

Fig. 7.

that the location of the separation point l_1' is already known.

(a) $l_2'<l_2$

If we write

$$(1+s)\,l/2=\sqrt{(l_2'-l_1)/(l_2-l_1')} \tag{12.1}$$

we have, from (5.6)

$$B(\Theta)=-\frac{1}{\pi}\oint_{s}^{1}\frac{\Delta(\Theta')}{\Theta-\Theta'}d\Theta' \tag{12.2}$$

since $d\delta/dx=0$ for $-1<\Theta<s$.

Introducing a new variable λ defined by

$$(\Theta-s^*)/s_0=\lambda \tag{12.3}$$

where

$$(1+s)/2=s^*\,,\qquad (1-s)/2=s_0\,,$$

we have

$$B(\lambda)=-\frac{1}{\pi}\oint_{-1}^{1}\frac{\Delta(\lambda')}{\lambda-\lambda'}d\lambda' \tag{12.4}$$

where

$$B(\lambda)\equiv B(\Theta=s_0\lambda+s^*)\,,\quad \Delta\equiv\Delta(\Theta=s_0\lambda+s^*)\,.$$

As the equation (12.4) is the same type with the equation (5.6), the problem can be solved by the same way with the partially cavitated flow shown in the preceding sections.

(b) $l_2'>l_2$

We may write l_2' for the upper range of the integral equation (3.7), because $d\hat{z}/dx\neq0$ for the range $l_2<x<l_2'$. The solution of the integral equation is written in the form

$$\frac{\hat{\phi}}{V^2}=\frac{2}{\pi}\sqrt{\frac{l_2'-x}{x-l}}\oint_{l_1}^{l_2'}\sqrt{\frac{x'-l_1}{l_2'-x'}}\frac{d\hat{z}/dx'}{x-x'}dx'\,. \tag{12.5}$$

We must notice that this solution is applied for the range $l_1<x<l_2'$.

Inserting (12.5) in (3.3) in the place of $\hat{\phi}$, we have

$$\frac{p_+}{\rho V^2}=-\frac{1}{\pi}\sqrt{\frac{l_2'-x}{x-l_1}}\oint_{l_1}^{l_2'}\sqrt{\frac{x'-l_1}{l_2'-x'}}\frac{d\hat{z}/dx'}{x-x'}dx' -\frac{1}{\pi V}\oint_{l_1}^{l_2'}\frac{d\varphi/dx'}{x-x'}dx'\,. \tag{12.6}$$

If we write

$$Z^*=\frac{1}{\pi}\sqrt{\frac{l_2'-x}{x-l_1}}\oint_{l_1}^{l_2'}\sqrt{\frac{x'-l_1}{l_2'-x'}}\frac{d\hat{z}'/dx'}{x-x'}dx' +\frac{1}{\pi}\oint_{l_1}^{l_2'}\frac{d\bar{z}'/dx'}{x-x'}dx' \tag{12.7}$$

and

$$\left.\begin{aligned} &J=\sqrt{(l_2'-l_1')/(l_1'-l_1)}\,, && (1-\Theta)J/2=\sqrt{(l_2'-x)/(x-l_1)}\\ &B^*(\Theta)=1/2(Z^*-\sigma/2)(x-l_1)\,, && \Delta^*(\Theta)=(x-l_1)d\delta/dx\,, \end{aligned}\right\} \tag{12.8}$$

we get the integral equation of $d\delta/dx$ from (12.6)

$$B^*(\Theta)=-\frac{1}{\pi}\oint_{-1}^{1}\frac{\Delta^*(\Theta')}{\Theta-\Theta'}d\Theta'\,. \tag{12.9}$$

This is the same type with the equation (5.6). So the problem can be solved by the same way with the partially cavitated flow shown in the preceding sections.

In order to simplify the calculation of the case that the cavitation occurs from the intermediate point of the chord, tables of new characteristic functions must be costructed.

The actual location of the separation point depends on several physical parameters. However, in order to predict the location, we may adopt the rather simple condition that the pressure must be a minimum in the foil-cavity-wake region. As the minimum pressure generally occurs at the leading edge in the linearized theory, some contrivance must be done for the prediction of the separation point. The support of non-linear theory will be required for the weak point of the linearized theory, after all.

Appendix. Characteristic Functions

$$m_n^{(j)}(L)=\frac{1}{\pi}\left(\frac{4}{L^2}\right)^n\int_{-1}^{1}\frac{1}{\sqrt{1-\Xi^2}}\frac{(1+\Xi)^j}{\{(1+\Xi)^2+4/L^2\}^n}d\Xi$$

$$i_n^{(j)}(\Xi)=\frac{1}{\pi}\left(\frac{4}{L^2}\right)^n\int_{-1}^{1}\frac{1}{\sqrt{1-\Xi'^2}}\frac{(1+\Xi')^j}{\{(1+\Xi')^2+4/L^2\}^n}\frac{1}{\Xi-\Xi'}d\Xi' .$$

$$m_1^{(0)}(L)=\sqrt{\frac{\sqrt{1+L^2}+1}{2(1+L^2)}} \qquad m_1^{(1)}(L)=\frac{2}{L}\sqrt{\frac{\sqrt{1+L^2}-1}{2(1+L^2)}}$$

$$m_2^{(0)}(L)=\frac{3L^2+5-\sqrt{1+L^2}}{4(1+L^2)}m_1^{(0)}, \qquad m_2^{(1)}(L)=\frac{L^2+3+\sqrt{1+L^2}}{4(1+L^2)}m_1^{(1)}$$

$$m_3^{(0)}(L)=\frac{21L^4+55L^2+46-2(4L^2+7)\sqrt{1+L^2}}{32(1+L^2)^2}m_1^{(0)}$$

$$m_3^{(1)}(L)=\frac{5L^4+15L^2+22+2(2L^2+5)\sqrt{1+L^2}}{32(1+L^2)^2}m_1^{(1)}$$

$$i_0^{(1)}(\Xi)=-1 , \quad i_0^{(2)}(\Xi)=-(2+\Xi) \quad \text{for } |\Xi|<1 .$$

$$i_0^{(1)}(\Xi)=-1+(\Xi+1)/\sqrt{\Xi^2-1} , \quad i_0^{(2)}(\Xi)=-(2+\Xi)+(\Xi+1)^2/\sqrt{\Xi^2-1} \quad \text{for } \Xi>1 .$$

recursions formulas of $m_k^{(j)}$ and $i_k^{(j)}$

$$m_k^{(1)}=-8/L^2\cdot(k-1)m_{k-1}^{(0)}+\{4(2k-1)(1+2/L^2)+2\}m_k^{(0)}-8k(1+1/L^2)m_{k+1}^{(0)} .$$

$$8/L^2\cdot(2k-5)(k-3)m_{k-3}^{(0)}-\{16(k-2)^2/L^2-1+4(2k-5)^2(1+2/L^2)\}m_{k-2}^{(0)}$$
$$+8(k-2)\{(2k-5)(1+1/L^2)+(2k-3)(1+2/L^2)\}m_{k-1}^{(0)}-16(k-2)(k-1)(1+1/L^2)m_k^{(0)}=0$$

$$m_k^{(j)}=4/L^2\cdot\{m_{k-1}^{(j-2)}-m_k^{(j-2)}\}$$

$$i_k^{(1)}(\Xi)=\frac{4/L^2\cdot i_{k-1}^{(1)}(\Xi)}{(1+\Xi)^2+4/L^2}+\frac{1}{(1+\Xi)^2+4/L^2}\{(1+\Xi)m_k^{(1)}+m_k^{(2)}\}$$

$$i_k^{(2)}(\Xi)=\frac{4/L^2\cdot i_{k-1}^{(2)}(\Xi)}{(1+\Xi)^2+4/L^2}+\frac{1}{(1+\Xi)^2+4/L^2}\{(1+\Xi)m_k^{(2)}+4/L^2\cdot m_{k-1}^{(1)}-4/L^2\cdot m_k^{(1)}\} .$$

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