A HeI-HeII Interface and Second Sound Shock Waves near the Superfluid Transition

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A HeI-HeII interface and second sound shock waves are examined in ^{4}He near the superfluid transition on the basis of a phenomenological time-dependent Ginzburg-Landau model. They are kink solutions of the equations. The former is induced by heat flow and has a structure similar to that of the interface in type I superconductors in a magnetic field. This interface exists when the fluid is inhomogeneous with some regions above the transition and others below the transition. It generally moves slowly and can be stopped for some special boundary conditions. In simple one-dimensional cases its motion is described by a scheme of a modified version of the Stefan problem. On the other hand, in superfluid states finite thermal disturbances can propagate as a shock wave. This paper is the first attempt to examine its properties fully in the nonlinear regime including the dissipation. Our theory can be used for aribitrary temperature discontinuity. As a by-product we calculate in Appendix A the dispersion and attenuation of the linear second sound modes in the presence of arbitrary thermal counterflow. We take into account a nonlinear coupling between the order parameter and the entropy in the free energy. This coupling, the so-called dissipative coupling, is indispensable for our problems. We also discuss in detail a transition from a normal fluid state to a coexisting state, which is predicted to be discontinuous sufficiently near the λ point.

§ 1. Introduction

Systems near critical points are very sensitive to external disturbances. The author has been interested in nonlinear effects of dissipative perturbations. There are numerous, potentially interesting examples of such phenomena and most of them remain unnoticed. In this paper we investigate two such nonlinear effects induced by heat flow in ^{4}He near the superfluid transition. In the λ region we can start with a relatively simple dynamic model to a great advantage. The first and very general equations were proposed by Pitaevskii[1] for a complex order parameter $\psi(\boldsymbol{r}, t)$ and the hydrodynamic variables. They are, however, too general and detailed and have never been solved in nonlinear regimes out of equilibrium.[2),3)] We stress that they have some inhomogeneous solutions as in the case of the famous Ginzburg-Landau equations for superconductors in a magnetic field.[4),5)] However, with the progress of the study of (linear) critical dynamics it has been shown that Pitaevskii's model can be much simplified in the λ region if only relevant terms are retained in the dynamic equations.[6)] We thus use the simplest model, the so-called F model,[6)] to investigate inhomogeneous structures.

Recently the author has shown the existence of a HeI-HeII interface in ^{4}He near the superfluid transition under heat flow.[7)~10)] That is, a superfluid region and a normal fluid region can coexist and be separated by a thin interface.[11)~13)] The temperature in the superfluid region T_∞ is slightly below the transition temperature T_λ as $1-T_\infty/T_\lambda \sim 10^{-8} Q^{3/4}$ with heat flow Q in erg/cm^{2}sec. The thickness of the interface ξ_c is of the order of the correlation length in the superfluid region and is given by $3\times10^{-3}Q^{-1/2}$ cm. The temperature gradient in the interfacial region is of the following order:

$$(T_\lambda - T_\infty)/\xi_c \sim 10^{-5}Q^{5/4}\,\mathrm{deg/cm}\,, \tag{1·1}$$

where Q is in erg/cm²sec.

However, in real helium, as a complicated effect, vortices can be generated at a relatively small heat flow. In our case of the two phase coexistence they give rise to a small temperature gradient in the superfluid region. It is necessary to estimate its magnitude near the interface. We use the Gorter-Mellink relation[14)]

$$\frac{dT}{dx} = (A\rho/s)|v_s - v_n|^3\,. \tag{1·2}$$

Close to the interface the quantities on the right hand sides are constants dependent only on Q because $T_\lambda - T \cong T_\lambda - T_\infty \propto Q^{3/4}$. A is proportional to Hall and Vinen's coefficent B and behaves as $(1-T/T_\lambda)^{-2/3+x_\lambda}$ as $T\to T_\lambda$,[15)] x_λ being the critical exponent of the thermal conductivity.[16)] The difference of the superfluid velocity v_s and the normal fluid velocity v_n is nearly equal to $Q/\rho_s s T_\lambda$, where ρ_s is the superfluid density and s is the entropy. After some calculations we obtain

$$\frac{dT}{dx} \cong \varepsilon_{\mathrm{th}}(T_\lambda - T_\infty)/\xi_c\,, \tag{1·3}$$

where $\varepsilon_{\mathrm{th}}$ is only weakly dependent on Q and is of order 10^{-3}. Thus, the temperature lowering due to the vortices is negligible if the distance from the interface is less than $\varepsilon_{\mathrm{th}}^{-1}\xi_c \sim 10^3\xi_c$. Therefore, the temperature depression $T_\lambda - T_\infty$, which is intrinsic to the interface, can in principle be observed if the lowering due to vortices can be separated. Note that the vortex resistance can be neglected if the cell size L is less than $10^3\xi_c \sim 10\,Q^{-1/2}$cm where Q is in eg/cm²sec. This condition is realized only at very small Q.

In this paper we also report a preliminary theory of second sound shock waves. Lots of experiments have shown the existence of a rich field of nonlinear wave phenomena of temperature variations in superfluid helium.[17)~22)] Its theoretical study is still in its infancy, however. We may expect considerable progress also in this problem particularly in the λ region.

First theories of the second sound shock fronts were due to Temperley[23)] and Khalatnikov.[3),24)] They neglected the dissipation but could calculate the shock velocity u_2 in the case of weak shock waves with small heat flow. Note that u_2 is determined only from the conservation laws if the shock front region is regarded as a geometrical surface where discontinuity occurs.[25)] Khalatnikov expanded u_2 with respect to the temperature discontinuity ΔT as

$$u_2 = u_{20}\left\{1+\frac{1}{2}\Delta T\frac{\partial}{\partial T}\log\left(u_{20}^3\frac{\partial\sigma}{\partial T}\right)+O\left(\Delta T^2\right)\right\}, \tag{1·4}$$

where u_{20} and $\partial\sigma/\partial T$ are the second sound velocity and the specific heat far ahead of the shock front. Here the fluid ahead of the front is assumed to be in equilibrium with temperature T, whereas the rear region has a temperature $T+\Delta T$ and a thermal counterflow.

Experiments supported Khalatnikov's theory at small heat flow.[18)~22)] When a heat pulse is injected into superfluid helium, the leading edge of the pulse steepens into a front if the temperature derivative in (1·4) is positive, whereas the trailing edge becomes a front if the derivative is negative. Very interestingly the derivative changes its sign at three temperatures below T_λ. Its experimental consequences were summarized in Ref. 22). Particularly, the derivative is negative in the λ region. From $u_{20} \propto \rho_s^{1/2} \propto (T_\lambda - T)^{1/3}$, we calculate

$$u_2 = u_{20}\left\{1 - \frac{1}{2}\frac{\Delta T}{T_\lambda - T} + \cdots\right\}, \tag{1·5}$$

where the weak singularity of the specific heat is neglected. More interesting features would be nonlinear effects which appear with increasing heat flow. For example, Cummins et al.[19] and also Turner[22] observed that u_2 attains its maximum as a function of heat flow in a temperature region where the temperature derivative in (1·4) is positive. Also in our case near the superfluid transition, u_2 has a maximum as we vary the heat flows on the two sides separated by the front. However, we do not know whether it is observable or not because it is realized only at very large heat flow. Finally we must also mention a beautiful work of Kitabatake and Sawada,[20] who derived the Burgurs equation to describe nonlinear second sound waves and investigated the shape evolution of a heat pulse.[21]

The purpose of the present study is not only to calculate u_2 but also to examine the structure of the shock front including the dissipative effect. Its thickness will be found to be of the order of $\xi/(1-\rho_{s2}/\rho_{s1})$ where ξ is the correlation length and ρ_{s1} and ρ_{s2} are the superfiuid densities in the rear and forward regions of the front.

This paper is organized as follows. In §2 we present the F model equations and make them dimensionless in a convenient form for our purpose. In §3 we show inhomogeneous stationary solutions representing the interface and boundary profiles under heat flow. In §4 we examine propagating solutions in the ordered phase which represent second sound shock fronts. We can prove generally that the superfluid density ahead of the front must be smaller than that behind the front. This is required from the positivity of the heat production in the front region. We also examine a well-known stability criterion of shock waves, the subsonic-supersonic criterion,[25] in our case. However, we cannot give complete stability analysis of the second sound shock front at present. In §5 an explicit solution of the shock front is obtained in the case where a parameter $\varkappa$, which corresponds to the GL parameter in superconductors,[4] is much greater than 1. In §6 the interfacial motion is examined in the two phase coexisting case.[8] In §7 we analyze how a superfluid region appears at a cooler boundary as the boundary temperatures and the heat flow are changed slowly. This is a transition from a normal fiuid state to a coexisting state and is discontinuous sufficiently close to the λ point. We believe that the arguments are more persuasive than those of Ref. 7). In §7 we give a summary together with objections to Turner's work.[22] In Appendix A we calculate the eigen frequencies of the linear collective modes for the F model in the presence of thermal counterflow.

§ 2. Model equations

Let $\psi(\boldsymbol{r}, t)$ be a complex order parameter and $m(\boldsymbol{r}, t)$ be the entropy density per unit mass multiplied by ρ_0/k_B, where ρ_0 is the average mass density. The simplest dynamic model is given by the following equations:[6)]

$$\frac{\partial}{\partial t}\psi = ig_0\frac{\delta H}{\delta m}\psi - \Gamma_0[\tau - \nabla^2 + 4u_0|\psi|^2]\psi\ , \tag{2·1}$$

$$\frac{\partial}{\partial t}m = g_0\mathrm{Im}(\psi^*\nabla^2\psi) + \lambda_0\nabla^2\frac{\partial H}{\partial m}\ , \tag{2·2}$$

where τ represents a temperature deviation defined by

$$\tau = \bar{r}_0 + 2\gamma_0\chi_0\frac{\delta H}{\delta m} = \bar{r}_0 + 2\gamma_0(m + \gamma_0\chi_0|\psi|^2). \tag{2·3}$$

Here H is a Ginzburg-Landau free energy defined by

$$H = \int d\boldsymbol{r}\left[\frac{1}{2}\bar{r}_0|\psi|^2 + \frac{1}{2}|\nabla\psi|^2 + u_0|\psi|^4 + \frac{1}{2\chi_0}(m + \gamma_0\chi_0|\psi|^2)^2\right], \tag{2·4}$$

where $\bar{r}_0$, u_0, χ_0 and γ_0 are static parameters. In equilibrium we have $\delta H/\delta m = 0$ and τ is a constant, whereas in nonequilibrium cases τ can depend on space and time due to the coupling $\gamma_0 m|\psi|^2$ in H. We should require $\gamma_0 > 0$ because the temperature should increase as the entropy is increased. The coupling $\gamma_0 m|\psi|^2$ is essentially important for the problems in this paper and also for that of the vortex motion near the λ point.[15)] Γ_0 and λ_0 are the kinetic coefficients and g_0 is a reversible mode coupling constant fixed as

$$g_0 = T_\lambda s_0/\hbar\ , \tag{2·5}$$

where s_0 is the average entropy per particle and $\hbar$ is the Planck constant. Γ_0 is a complex number with $\mathrm{Re}\Gamma_0 > 0$ and $\mathrm{Im}\ \Gamma_0/\mathrm{Re}\Gamma_0 \sim 1$ near the λ point.

In this paper we neglect the deviations of the pressure and the mass density. Gravity effects will also be neglected, which should be important, however, when the cell size is long and the heat flow is small. The mass flow may be assumed to be zero. Its discontinuity across the shock front is negligibly small near the λ point. As a result the normal fluid velocity $\boldsymbol{v}_n$ is much smaller than the superfluid velocity $\boldsymbol{v}_s$ and nearly equal to $-(1/\rho_0)\boldsymbol{J}_s$, $\boldsymbol{J}_s$ being the superfluid current. In our notation the superfluid density ρ_s and $\boldsymbol{J}_s = \rho_s\boldsymbol{v}_s$ are related to ψ by $\rho_s = (m_H{}^2k_BT_\lambda/\hbar^2)|\psi|^2$ and $\boldsymbol{J}_s = (m_Hk_BT_\lambda/\hbar)\mathrm{Im}(\psi^*\nabla\psi)$, m_H being the helium mass.[6)]

We also omit the random source terms usually present on the right-hand sides of (2·1) and (2·2). In our nonequilibrium situations deviations occur only at wavelengths longer than or at least comparable with the local thermal correlation length ξ. We can thus coarse-grain the small-scale fluctuations smaller than ξ at the starting equations (2·1) and (2·2). We will be interested only in average profiles of ψ and m.

We make (2·1) and (2·2) dimensionless by the following transformations. New spatial coordinate and time are defined by

$$x' = k_0x\ , \qquad t' = (g_0k_0{}^2/\sqrt{4u_0\chi_0}\,)t\ . \tag{2·6}$$

The wave number k_0 will be chosen to be the value of $|\tau|^{1/2}$ at $x = -\infty$ or at $x = \infty$. Then it is of the order of the inverse correlation length at $x = -\infty$ or ∞. New dynamic variables are given

$$\Phi = (4u_0/k_0{}^2)\psi,\ A = \tau/k_0{}^2,\ M = (2\gamma_0/k_0{}^2)m\ . \tag{2·7}$$

The dimensionless temperature A and entropy M are related by

$$A=M+\frac{1}{2}a^2|\psi|^2\,, \tag{2·8}$$

where $\bar{r}_0$ in (2·3) can be set equal to zero without loss of generality and

$$a=(\gamma_0{}^2\chi_0/u_0)^{1/2}\,. \tag{2·9}$$

The dynamic equations are of the following forms:

$$\frac{\partial}{\partial t'}\Phi=ia^{-1}A\Phi-b_1\Big[A-\Big(\frac{\partial}{\partial x'}\Big)^2+|\Phi|^2\Big]\Phi\,, \tag{2·10}$$

$$\frac{\partial}{\partial t'}M=a\frac{\partial}{\partial x'}\mathcal{J}+b_2\Big(\frac{\partial}{\partial x'}\Big)^2A\,, \tag{2·11}$$

where

$$b_1=(4u_0\chi_0)^{1/2}\Gamma_0/g_0,\ b_2=(4u_0/\chi_0)^{1/2}\lambda_0/g_0\,. \tag{2·12}$$

$\mathcal{J}$ is the dimensionless superfluid current:

$$\mathcal{J}=\mathrm{Im}\Big(\Phi^*\frac{\partial}{\partial x'}\Phi\Big)=|\Phi|^2\frac{\partial}{\partial x'}\theta\,, \tag{2·13}$$

where θ is the phase of ψ (or Φ).

It would be convenient to relate the superfluid velocity v_s and the shock front velocity u_2 to their corresponding dimensionless quantities because the proportionality constants are different. That is,

$$v_s=\frac{\hbar}{m_H}\frac{\partial}{\partial x}\theta=(\hbar k_0/m_H)\frac{\partial}{\partial x'}\theta\,, \tag{2·14}$$

$$u_2=dx_s/dt=[g_0k_0/(4u_0\chi_0)^{1/2}]dx_s'/dt'\,, \tag{2·15}$$

where x_s denotes the position of the shock front and dx_s'/dt' will be written as u in §4 and is of order 1. Thus,

$$v_s/u_2=\hbar(4u_0\chi_0)^{1/2}/(g_0m_H)\Big(\frac{\partial\theta}{\partial x'}\Big/\frac{dx_s'}{dt'}\Big), \tag{2·16a}$$

$$\cong 3.2(1-T/T_\lambda)^{1/3}\Big(\frac{\partial\theta}{\partial x'}\Big/\frac{dx_s'}{dt'}\Big). \tag{2·16b}$$

The second line is the estimation at saturated vapor pressure, where $u_0=u^*\xi^{-1}$ with u^* being a universal number.[26] The temperature T in (2·16b) represents the typical value in the fluid. The derivative $\partial\theta/\partial x'$ can be at most of order 1 (see (4·6) below), so that we always have $v_s\ll u_2$ as $T\to T_\lambda$.

We also express the heat flow $\rho s v_n\cong -sJ_s\equiv -Q$ in superfluids in terms of $\mathcal{J}$ in (2·13). That is,

$$Q=k_BT_\lambda g_0\mathrm{Im}\Big(\psi^*\frac{\partial}{\partial x}\psi\Big)\,, \tag{2·17a}$$

$$\cong 0.82\times10^{11}(1-T/T_\lambda)^{4/3}(k_0\xi)^3\mathcal{J}\,,\qquad \text{(in cgs)} \tag{2·17b}$$

where $\xi \equiv 1.4(1-T/T_\lambda)^{-2/3}$ Å. We will choose k_0 such that $k_0\xi \sim 1$, so that the proportionality constant decreases as $T \to T_\lambda$. Hereafter we suppress the primes of x' and t' for simplicity unless confusion can occur.

In (2·10) and (2·11) there are three dimensionless parameters a, b_1 and b_2. All of them depend on $T_\lambda - T$ rather weakly. The static parameter a is of order 1,[16] $b_1/b_2 = \chi_0 \mathrm{Re}\Gamma_0/\lambda_0$ is the ratio of the time scale of the entropy to that of the order parameter,[6] and $1/b_1b_2 (\propto g_0^2)$ represents the strength of the reversible coupling between the two dynamic variables.

§ 3. Stationary solutions

We consider stationary solutions of (2·10) and (2·11). As trivial solutions in normal fluid states we find $\Phi=0$ and $A=A_0+B_0x$, where A_0 and B_0 are constants. On the other hand, in homogeneous superfluid states we have $A=A_0$ and $\Phi=(|A_0|-K^2)^{1/2}\exp(iKx + A_0t/\alpha_0)$, where A_0 is a negative constant. The dimensionless heat flow is $-b_2B_0$ and $-aK(|A_0|-K^2)$, respectively.

General solutions can be obtained by setting

$$\frac{\partial}{\partial t}\Phi = i\omega_0\Phi\,, \qquad \frac{\partial}{\partial t}M=0\,. \tag{3·1}$$

First we notice that the heat flow is constant:

$$a\mathcal{J} + b_2\frac{d}{dx}A \equiv Q^* = (a/k_0^3)(4u_0Q/g_0k_BT_\lambda), \tag{3·2}$$

where use has been made of (2·17a). Equation (2·10) can be rewritten as

$$\frac{d^2}{dx^2}\Phi = [A_0 + (1-i/ab_1)(A-A_0)+|\Phi|^2]\Phi\,, \tag{3·3}$$

where

$$A_0 = aw_0\,. \tag{3·4}$$

We multiply (3·3) by Φ^* and take the imaginary part to obtain

$$\frac{d}{dx}\mathcal{J} = -\left(\frac{1}{a}\mathrm{Re}\frac{1}{b_1}\right)|\Phi|^2(A-A_0). \tag{3·5}$$

Thus (3·2) can be transformed into

$$\frac{d^2}{dx^2}A = \varkappa^{-2}|\Phi|^2(A-A_0), \tag{3·6}$$

where

$$\varkappa = \left(b_2/\,\mathrm{Re}\frac{1}{b_1}\right)^{1/2} = (4u_0\lambda_0|\Gamma_0|^2/g_0^2\mathrm{Re}\,\Gamma_0)^{1/2}\,. \tag{3·7}$$

(A) HeI-HeII Interface

We note that (3·3) and (3·6) are very similar to the Ginzburg-Landau equations for superconductors in a magnetic field.[4] The correspondences between the two cases are as follows: temperature$\leftrightarrow$ vector potential $\boldsymbol{A}$, temperature gradient $\leftrightarrow$ magnetic induction $\boldsymbol{B}$ $=\mathrm{rot}\ \boldsymbol{A}$, and heat flow $\leftrightarrow$ external magnetic field $\boldsymbol{H}$. The $\varkappa$ is called the GL parameter in the superconducting case. It is the ratio of the spatial scale of $|\Phi|$ to that of A and $\mathcal{J}$. Ginzburg and Landau calculated the normal-superconducting interface in the case $\varkappa \ll 1$. In our case we can also construct interfacial solutions in the two limiting cases $\varkappa \ll 1$ and $\varkappa \gg 1$. In real helium $\varkappa$ depends on $T-T_\lambda$ rather weakly. Very close to the λ point, say, for $|T-T_\lambda - 1| \lesssim 10^{-3}$, we find $\varkappa^2 \sim 1/10$, whereas far from the λ point it is considerably larger than 1.[7),8),16)] See (iii) of §8.

The boundary conditions for the interface solution are of the forms

$$A \to A_0 = -1, \quad \Phi \to [1-K^2]e^{iKx+i\omega_0 t} \quad \text{as } x \to -\infty, \tag{3·8}$$

$$\frac{d}{dx}A \to Q^*/b_2 = \text{const}, \quad \Phi \to 0 \quad \text{as } x \to +\infty, \tag{3·9}$$

where

$$Q^* = K(1-K^2). \tag{3·10}$$

We have set $A_0=-1$. Then k_0 in (2·6) is of the order of the inverse correlation length on the superfluid side ξ^{-1}. The interfacial thickness is naturally of order ξ. The solution can be obtained particularly simply in the case $\varkappa \gg 1$. That is, we may assume the local equilibrium relation

$$|\Phi|^2 \cong -A \qquad \text{for } \varkappa \gg 1. \tag{3·11}$$

See Ref. 7) and also §5 of this paper. Then (3·6) becomes

$$\varkappa^2 \frac{d^2}{dx^2}A + A(A+1) = 0. \tag{3·12}$$

The solution which satisfies $A(-\infty)=-1$ is uniquely of the form

$$A = \frac{1}{2} - \frac{3}{2}\tanh^2\left(\frac{1}{2}\varkappa^{-1}x + y_0\right), \tag{3·13}$$

where y_0 is a constant. The superfluid current is obtained from (3·2) in the form

$$\mathcal{J} = a^{-1}Q^* + \frac{3}{2}(b_2/a\varkappa)\sinh\left(\frac{1}{2}\varkappa^{-1}x + y_0\right)/\cosh^3\left(\frac{1}{2}\varkappa^{-1}x + y_0\right). \tag{3·14}$$

We require that $A \to 0$ as $x \to 0$. Then $\mathcal{J}$ must also go to 0 as $x \to 0$. These two conditions lead to $y_0 = \frac{1}{2}\log(2-\sqrt{3}) = -0.66$ and

$$Q^* = 3^{-1/2}\varkappa^{-1}b_2. \tag{3·15}$$

This means that, if the temperature on the superfluid side is given, the heat flow is uniquely determined. Equivalently, the temperature on the superfluid side is a unique function of the heat flow.*)

On the other hand, in the case $\varkappa \ll 1$, we can use the mathematical techniques of GL. The calculations are rather complicated and the details can be found in Refs. 7) and 8).

In any case we find a unique relation between the heat flow and the temperature on

*) In the superconducting case the external megnetic field, which corresponds to the heat flow, must be equal to the critical magnetic field proportional to $T_c - T$ if the interface is one-dimensional and at rest.[4)]

the superfluid side. Let $\tau_\infty = k_0^2 A_0 = -k_0^2$. Then, for any value of $\varkappa$ we obtain

$$|\tau_\infty| = C_\infty (4u_0 Q/g_0 k_B T_\lambda)^{2/3}\,. \tag{3·16}$$

The proportionality constant C_∞ can be calculated in the two cases $\varkappa \gg 1$ and $\varkappa \ll 1$. For $\varkappa \gg 1$, (3·2) and (3·15) yield

$$C_\infty \cong (\sqrt{3}a\varkappa/b_2)^{2/3} \qquad \text{for } \varkappa \gg 1\,. \tag{3·17}$$

For $\varkappa \ll 1$ we use results of Ref. 8). Equation (52) of Ref. 8) means

$$C_\infty = \varepsilon_S^{-2/3} = (2B)^{1/2}(1+c_0^2)^{1/6}\varkappa^{-1/3} \quad \text{for } \varkappa \ll 1\,, \tag{3·18}$$

where ε_S is defined in (54) and

$$c_0 = \mathrm{Im}\, b_1/\mathrm{Re}\, b_1 = \mathrm{Im}\, \Gamma_0/\mathrm{Re}\, \Gamma_0\,. \tag{3·19}$$

B in (3·18) is a function of c_0 of order 1 and is calculated numerically below (48) of Ref. 8).

Now it is convenient to express b_1, b_2 and C_∞ in terms of $\varkappa$, (3·7), c_0, (3·19), and [6),16),27),28)]

$$w \equiv \mathrm{Re}\, b_1/b_2 = \chi_0 \mathrm{Re}\, \Gamma_0/\lambda_0\,. \tag{3·20}$$

Then,

$$b_1 = \frac{1+ic_0}{\sqrt{1+c_0^2}}\varkappa w^{1/2} \sim \varkappa w^{1/2}\,, \quad b_2 = \frac{1}{\sqrt{1+c_0^2}}\varkappa w^{-1/2} \sim \varkappa w^{-1/2}\,, \tag{3·21}$$

$$C_\infty \sim \begin{cases} a^{3/2} w^{1/3} & \text{for } \varkappa \gg 1\,, \\ \varkappa^{-1/3} & \text{for } \varkappa \ll 1\,. \end{cases} \tag{3·22}$$

To get the real value of the temperature on the HeII side T_∞ we must take into account the temperature dependence of u_0. We estimate T_∞ by setting $u_0 = u^* \xi^{-1}$ and $|\tau_\infty| = \xi^{-2} = \xi_0^{-2}(1 - T_\infty/T_\lambda)^{4/3}$. The result is

$$1 - T_\infty/T_\lambda = A_\infty Q^{3/4}\,, \tag{3·23}$$

where $A_\infty \sim 10^{-8}$ if Q is erg/cm²sec. The exponent 3/4 in (3·23) is unchanged even if the correction to the dynamic scaling law[16),27),28)] is taken into account as long as $\Gamma_0 \lambda_0 \sim \xi$.

(B) Boundary Profiles

At the boundary wall Ψ must go to zero even in the superfluid state. In particular, $\Psi = \tanh(x/\sqrt{2})$ and $A = -1 = $const in equilibrium, where the boundary wall is placed at $x = 0$. In the presence of heat flow this profile must be modified. Such a calculation was performed at very small Q by Ginzburg and Sobaynin.[2)] In our scheme their results can be reproduced simply from (3·6). Namely, we may set $|\Psi|^2 = \tanh^2(x/\sqrt{2})$ in (3·6) to first order in Q to obtain

$$\frac{d^2}{dx^2} A = \varkappa^{-2} \tanh^2(x/\sqrt{2})(A+1)\,, \tag{3·24}$$

where $A(\infty) = -1$. The solution of (3·24) can be expressed in terms of the hypergeometric function $F(\alpha, \beta, \gamma, z)$ as*)

$$A+1=C_1(1-\zeta^2)^{\varepsilon_1/2}F\left(2\alpha_+, 2\alpha_-, \varepsilon_1+1, \frac{1-\zeta}{2}\right), \tag{3·25}$$

where

$$\varepsilon_1=\sqrt{2}/x, \quad \alpha_\pm=\varepsilon_1/2+1/4\pm\left(\varepsilon_1{}^2+\frac{1}{4}\right)^{1/2}/2, \quad \zeta=\tanh(x/\sqrt{2}). \tag{3·26}$$

The constant C_1 is determined from $dA/dx=Q^*/b_2$ at $x=0$, (3·2). Then, the value of A at $x=0$, A_0, is related to Q^* by

$$A_0+1=-C_B(Q^*/b_2), \tag{3·27}$$

$$C_B=\Gamma(\alpha_+)\Gamma(\alpha_-)/\sqrt{2}\Gamma\left(\alpha_++\frac{1}{2}\right)\Gamma\left(\alpha_-+\frac{1}{2}\right). \tag{3·28}$$

Use has been made of the identiy $F(2\alpha, 2\beta, \alpha+\beta+1/2, 1/2)=\pi^{1/2}\Gamma(\alpha+\beta+1/2)/\Gamma(\alpha+1/2)\Gamma(\beta+1/2)$. Here $C_B\cong x$ for $x\gg1$ and $C_B\cong1.76\sqrt{x}$ for $x\ll1$.

Equation (3·27) means that there is a boundary resistance arising from the condition $\Psi=0$ at the boundary. Let us denote the temperature at $x=0$ by $T+(\delta T)_B$, T being the temperature far from the boundary. Then assuming $A\cong-\xi_0{}^{-2}(1-T/T_\lambda)^{2\nu}$ we have

$$2\nu(\delta T)_B/(T_\lambda-T)\cong-C_B(4u^*a/b_2g_0k_BT_\lambda)\xi^2Q\,. \tag{3·29}$$

A rough estimation can be made as follows:

$$R_i\equiv(\delta T)_B/Q\sim10^{-3}(1-T/T_\lambda)^{-1/3}\ \text{deg cm}^2\,W^{-1}\,. \tag{3·30}$$

In most experimental conditions R_i is much smaller than the Kapitza resistance $R_K(\sim1$ deg cm$^2W^{-1}$, typically),[2] although R_i diverges as $T\to T_\lambda$.

Next we calculate the temperature at $x=0$ in the two-phase coexistence case (see Fig. 1). For simplicity we assume a very small heat flow and neglect the vortex resistance. In this case the value of τ in the flat region, τ_∞, is given by (3·16). The value of τ at $x=0$ will be denoted by $\tau_0=k_0{}^2A_0=|\tau_\infty|A_0$. To estimate τ_0 we may use (3·27) to

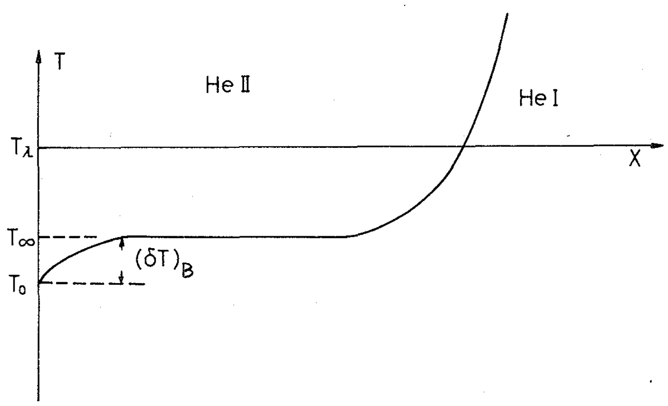

Fig. 1. The temperature profile in the coexisting case. We are neglecting the temperature lowering due to vortices.

*) In Ref. 2) the temperature is claimed to change over a distance $l=l_0(T_\lambda-T)^{-1/2}$, whose temperature dependence is different from that of the correlation length $(\propto(T_\lambda-T)^{-2/3})$. This is clearly an error originating from a wrong order estimation in (4·33) of Ref. 2).

a good approximation, although it is the formula in the linear response regime. Then, we find

$$|\tau_0|/|\tau_\infty|-1=C_B Q^*/b_2=C_B k_0^{-3}(4u_0 a/b_2 g_0 k_B T_\lambda). \tag{3·31}$$

Using (3·16) we obtain

$$|\tau_0|/|\tau_\infty|-1=aC_B/b_2 C_\infty^{3/2}\cong\begin{cases}1/\sqrt{3} & \text{for } \varkappa\gg 1\,, & (3·22\text{a})\\ C_2 a\sqrt{w} & \text{for } \varkappa\ll 1\,, & (3·32\text{b})\end{cases}$$

where C_2 is a number of order 1. Therefore the lowering at the boundary $|\tau_0|-|\tau_\infty|$ is considerably smaller than the intrinsic lowering $|\tau_\infty|$. As a rough approximation we may assume the temperature at $x=0$, T_0, equal to T_∞.

§ 4. Propagating solutions

Next we assume that Φ and A depend on space and time in the forms

$$\Phi=\Phi(x-ut)e^{i\omega_0 t}\,, \qquad A=A(x-ut). \tag{4·1}$$

We may assume $u>0$ without loss of generality. If $u<0$, we perform the inversion transformation $x\to -x$. Then, (2·10) and (2·11) are transformed into

$$i\left(\frac{1}{a}A-\omega_0\right)\Phi+u\frac{d}{dx}\Phi=b_1\left[A-\frac{d^2}{dx^2}+|\Phi|^2\right]\Phi\,, \tag{4·2}$$

$$uM+a\mathcal{J}+b_2\frac{d}{dx}A=\text{const}\,. \tag{4·3}$$

Equation (4·3) represents the conservation of energy.

Further we assume that the system tends to homogeneous superfluid states both for $x\to-\infty$ and for $x\to\infty$. Namely, we are seeking step-wise variations moving with a constant velocity in superfluid (shock fronts). The boundary conditions are of the forms

$$A\to A_1\,,\quad \Phi\to\eta_1^{1/2}e^{iK_1 x} \qquad \text{as } x\to-\infty\,, \tag{4·4}$$

$$A\to A_2\,,\quad \Phi\to\eta_2^{1/2}e^{iK_2 x+i\theta_0} \qquad \text{as } x\to\infty\,, \tag{4·5}$$

where A_1 and A_2 are negative constant temperature deviations and θ_0 is a constant. The amplitudes η_1 and η_2 in (4·4) and (4·5) and the superfluid currents at $x=\pm\infty$ are written as

$$\eta_1=|A_1|-K_1^2,\quad \eta_2=|A_2|-K_2^2\,,\quad \mathcal{J}_1=\eta_1 K_1\,,\quad \mathcal{J}_2=\eta_2 K_2\,. \tag{4·6}$$

We derive some relations among the parameters at $x=\pm\infty$. Taking the imaginary parts of (4·2) at $x=\pm\infty$, we find

$$\omega_0=a^{-1}A_1+uK_1=a^{-1}A_2+uK_2\,. \tag{4·7}$$

Eliminating ω_0 we obtain

$$A_1-A_2=-au(K_1-K_2). \tag{4·8}$$

This relation means that the temperature discontinuity at the shock front results in the acceleration (or deceleration) of the superfluid velocity $(\partial/\partial x)\theta$. It cannot be modified by the details of the dissipative processes occurring in the front region. From (4·3) the energy conservation leads to

$$u(M_1-M_2)+a(\mathcal{J}_1-\mathcal{J}_2)=0\,, \tag{4·9}$$

where

$$M_1=(1+\frac{1}{2}a^2)A_1+\frac{1}{2}a^2K_1{}^2\,,\quad M_2=(1+\frac{1}{2}a^2)A_2+\frac{1}{2}a^2K_2{}^2\,. \tag{4·10}$$

Experimentally we can measure A_1, A_2, $\mathcal{J}_1$, $\mathcal{J}_2$ and u. The above relations constitute two relations among these five quantities. Especially elimination of u from (4·8) and (4·9) yields a relation among A_1, A_2, $\mathcal{J}_1$ and $\mathcal{J}_2$,*)

$$a^2(\mathcal{J}_1-\mathcal{J}_2)/(A_1-A_2)=(M_1-M_2)/(K_1-K_2). \tag{4·11}$$

Here we show a convenient equation of u derived from (4·9) with the aid of (4·8) and (4·10):

$$\left(1+\frac{1}{2}a^2\right)u^2-a(K_1+K_2)u+\left[\frac{1}{2}(A_1+A_2)+K_1{}^2+K_1K_2+K_2{}^2\right]=0\,. \tag{4·12}$$

If we assume that the last term of (4·12) is negative, then the positive solution of (4·12) is given by

$$u=\frac{a(K_1+K_2)}{2+a^2}+\left(1+\frac{1}{2}a^2\right)^{-1/2}\left[\frac{1}{4}a^2\left(1+\frac{1}{2}a^2\right)^{-1}(K_1+K_2)^2 -\frac{1}{2}(A_1+A_2)-K_1{}^2-K_1K_2-K_2{}^2\right]^{1/2}. \tag{4·13}$$

As ought to be the case, in the limit $A_1\to A_2$ and $K_1\to K_2$, the shock velocity tends to the (linear) second sound velocity c_{II} in the homogeneous current-carrying state.[29)~32)] It is the solution of the equation (Appendix A):

$$\left(1+\frac{1}{2}a^2\right)c_{\text{II}}^2-2aKc_{\text{II}}+(A+3K^2)=0\,. \tag{4·14}$$

If the superfluid velocity K is zero, we obtain the equilibrium relation $c_{\text{II}}^2=\eta/(1+\frac{1}{2}a^2)$.

Note that the fluid far from the front must be stable with respect to small disturbances. Obviously, the solution of (4·14) must be real and $[3-a^2/(1+\frac{1}{2}a^2)]K_j{}^2<|A_j|$ for $j=1,2$.[33)] However, in Appendix A we show that a stronger condition is necessary for the stability of the second sound mode propagating in the reverse direction of the superfluid current. It is independent of a and simply of the form**)

$$K_1{}^2<|A_1|/3,\qquad K_2{}^2<|A_2|/3\,. \tag{4·15}$$

This criterion was derived from purely thermodynamic arguments by some authors.[34),35)]

In the case of shock waves of ordinary sounds the entropy behind the shock front must

*) This relation is analogous to the so-called Hugoniot adiabatic in the theory of shock waves of ordinary sounds[25)] (see Fig. 1).

**) In experiments, however, the shock fronts have been observed at the leading or trailing edges of a finite heat pulse. In such cases (4·15) would be too strong.

always be greater than that ahead of the front.[25] The difference of the entropies is produced by the dissipation in the thin front region. An equivalent relation should also hold in the case of second sound shock waves. In Appendix B we calculate the entropy production rate $R_{\rm dis}$ in the shock front region, which is defined by[1]

$$R_{\rm dis}=\int_{-\infty}^{\infty}dx\left[b_2\left(\frac{d}{dx}A\right)^2+({\rm Re}\,b_1)|D_{\rm dis}|^2\right], \tag{4·16}$$

where

$$D_{\rm dis}=\left[A-\frac{d^2}{dx^2}+|\Phi|^2\right]\Phi\,. \tag{4·17}$$

We are neglecting the dissipation associated with the shear viscosity. In terms of the parameters at $x=\pm\infty$, $R_{\rm dis}$ is expressed in a surprisingly simple form,

$$R_{\rm dis}=\frac{1}{4}u(\eta_1-\eta_2)(K_1-K_2)^2\,. \tag{4·18}$$

Thus if $u>0$, we must always require

$$\eta_1>\eta_2\,. \tag{4·19}$$

The superfluid density behind the front must be greater than that ahead of the front. Otherwise, there is no solution satisfying the boundary conditions.

We also notice that, if $R_{\rm dis}=0$, we have $A=$const and $D_{\rm dis}=0$. Then $d\theta/dx=-(A/a-\omega_0)/u=$const from (4·2) and hence $|\Phi|^2=$const from (4·3). Namely, if either of $\eta_1=\eta_2$ or $K_1=K_2$ is assumed, there remains only a trivial homogeneous solution. In particular, when A_2-A_1 is very small, the right-hand side of (4·18) is of order $(A_2-A_1)^3$ and (4·16) means that the thickness of the front should grow as $|A_2-A_1|^{-1}$. It should be noted that (4·18) is not the sufficient condition for the shock front stability. To examine its stability, we must examine the time-evolution of corrugation-like disturbances on the front surface.

Further using (4·6) and (4·8) the condition (4·19) is rewritten as

$$\eta_1-\eta_2=\frac{1}{2}(A_2-A_1)[1-(K_1+K_2)/au]>0\,. \tag{4·20}$$

Therefore there can be the following two cases:

(i) $A_2>A_1$ and $K_1+K_2<au$, (4·21)

(ii) $A_2<A_1$ and $K_1+K_2>au$. (4·22)

In the first case (4·21) the shock front propagates into a warmer spatial region in accord with the prediction of Khalatnikov which is valid for small values of K_1 and K_2, whereas in the second case (4·22) the reverse phenomenon occurs.

In the following we will fix the parameters ahead of the front, A_2 and K_2, and examine a relation between A_1 and K_1 (or $\mathcal{J}_1$), We may set $A_2=-1$ without loss of generality. In Fig. 2 trajectories of A_1 and K_1 satisfying (4·19) are shown. They start from points on the line $A_1=-1$ in the direction determined by $\eta_1>\eta_2$, (4·19). The starting points, where $A_1=A_2$ and $K_1=K_2$, represent homogeneous states. As the distance from the starting point increases, the discontinuity of the shock wave increases. We notice that

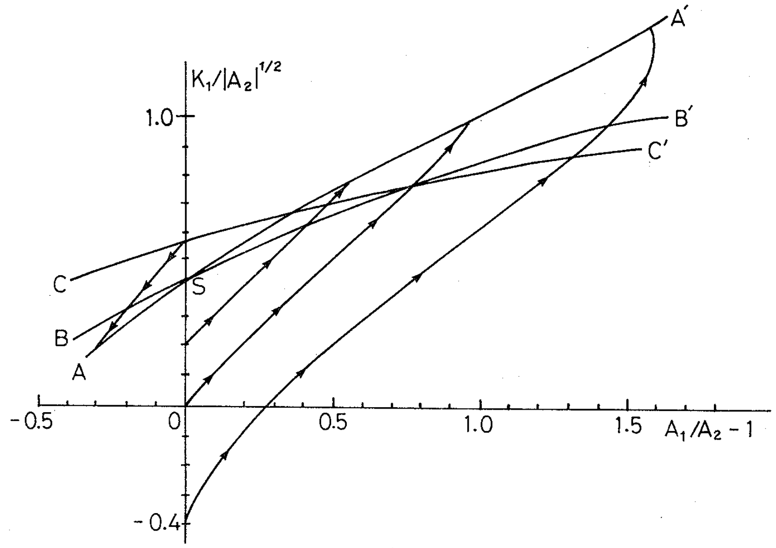

Fig. 2. The shock adiabatic for $a=1$, where we fix the temperature deviation A_2 and the superfluid velocity K_2 ahead of the front. Each point on the ordinate is a starting point of an adiabatic curve and represents a homogeneous state in which $A_1=A_2$ and $K_1=K_2$. The curves AA′ and BB′ are the trajectories determined by $\eta_1=\eta_2$ and by $u=c_1$, respectively, and $K_1=\sqrt{|A_1|/3}$ on the curve CC′. The adiabatic curve starting at $K_1=0$ happens to cross the curves BB′ and CC′ at the same point, which is fortuitous due to the special choice of a.

the case (4·21) is realized for $K_2<K_c$, whereas the case (4·22) for $K_2>K_c$, where

$$K_c=a(4+a^2)^{-1/2}|A_2|^{1/2}. \tag{4·23}$$

From (4·15) we notice that a must be less than $\sqrt{2}$ in order that the fluid ahead of the front is stable even at $K_2=K_c$. The trajectories end at the marginal curve AA′ on which $\eta_1=\eta_2$. From (4·6)~(4·13) the condition $\eta_1=\eta_2$ yield

$$K_1=-K_2+a(1-K_2{}^2)^{1/2}, \tag{4·24}$$

$$A_1=-1-a^2(1-K_2{}^2)+2aK_2(1-K_2{}^2)^{1/2}. \tag{4·25}$$

This marginal curve and the starting line $A_1=-1$ crosses at a singular point S in Fig 2, where we $K_1=K_2=K_c$ from (4·24). The limiting behaviour near this singular point is as follows:

$$\eta_1-\eta_2\cong\Delta\left[\left(2+\frac{1}{2}a^2\right)(K_c-K_2)-\Delta\right],\ A_1\cong-1-a\left(1+\frac{1}{4}a^2\right)^{-1/2}\Delta, \tag{4·26}$$

where $\Delta\equiv K_1-K_2$ and $|K_2-K_c|$ are assumed to be small. The first equation of (4·26) means that $|\Delta|<(2+\frac{1}{2}a^2)|K_c-K_2|$ near the singular point.

Here we must refer to the so-called subsonic and supersonic criterion.[25] It requires that stable shock fronts should travel with a velocity smaller than the sound velocity behind the front c_1 and greater than that ahead of the front c_2. In the case of ordinary sounds this criterion is equivalent to the positive entropy production in the front region if $(\partial^2 V/\partial p^2)_s>0$*) where $V=1/\rho$. In our case we compare u with the second sound velocities at $x=\pm\infty$, c_1 and c_2:

$$c_j=\left(1+\frac{1}{2}a^2\right)^{-1}aK_j+\left[\left(1+\frac{1}{2}a^2\right)^{-2}(aK_j)^2+\left(1+\frac{1}{2}a^2\right)^{-1}(|A_j|-3K_j{}^2)\right]^{1/2}, \tag{4·27}$$

*) This inequality does not hold generally. Particularly, it is invalid near the gas-liquid critical point.[36),37)]

where $j=1$, 2 and use has been made of (4·14). We consider only the sound modes propagating in the positive x direction.*) In Appendix C we show that the supersonic condition $u>c_2$ can be derived only from (4·19) and hence holds as long as $\eta_1>\eta_2$, whereas the subsonic condition $u<c_1$ holds only when $|K_1-K_2|$ is less than a certain value for fixed A_2 and K_2. In Fig. 2 we display the marginal curve BB′ on which $u=c_1$. In the two regions ASB and A′SB′ we find $\eta_1>\eta_2$ and $u>c_1$. As known from Fig. 2, the two conditions hold at least when A_2-A_1 is small. In fact, to first order in A_2-A_1 we have

$$u-c_2\cong c_1-u$$

$$\cong\frac{3}{4}\left[a^2K_2^2-\left(1+\frac{1}{2}a^2\right)(A_2+3K_2^2)\right]^{-1/2}(\eta_1-\eta_2)>0\,. \quad (4\cdot 28)$$

In particular, if we assume $K_2=0$, we find

$$u/c_2-1\cong\frac{3}{4}(\eta_1/\eta_2-1). \quad (4\cdot 29)$$

As $T\to T_\lambda$, (4·29) is asymptotically equivalent to the result of Khalatnikov, (1·4), where u_2/u_{20} in his notation corresponds to u/c_2 in our notation.

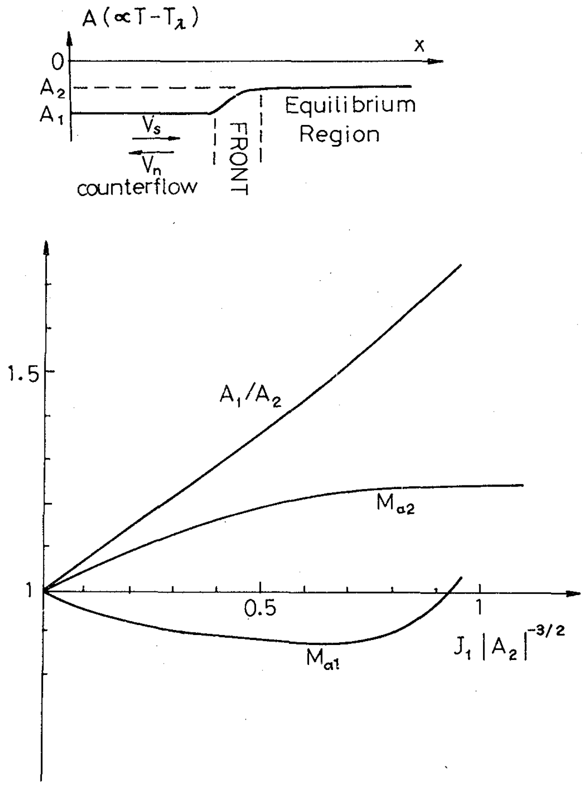

Fig. 3. In the inset a "rarefacation" shock wave is shown schematically. The region ahead of the front is in thermal equilibrium and there is a heat flow in the negative x direction in the back region. In the figure we fix the temperature deviation A_2. The horizontal axis denotes the superfluid current density in the back region, $\mathcal{J}_1$, divided by $|A_2|^{3/2}$, where $\mathcal{J}_1$ is proportional to the heat flow. The two dimensionless numbers $M_{\alpha 1}$ and $M_{\alpha 2}$ are the Mach numbers with respect to the rear and forward media, respectively.

Keeping A_2 and K_2 held fixed and changing K_1 from K_2, we can also find that u increases with increasing K_1 for $K_2<K_c$ or with decreasing K_1 for $K_2>K_c$ untill u/c_1 approaches 1 from above 1. Namely, u attains a maximum when $u/c_1=1$. In Appendix C we calculate the maximum shock velocity $u_{\max}$ as a function of K_2 in the form

$$u_{\max}=-\frac{aK_2}{8\left(1-\frac{1}{16}a^2\right)}+\left[\left\{\frac{aK_2}{8\left(1-\frac{1}{16}a^2\right)}\right\}^2+\frac{1-\frac{3}{4}K_2^2}{1-\frac{1}{16}a^2}\right]^{1/2}, \quad (4\cdot 30)$$

where we have set $A_2=-1$. The ratio $u_{\max}/c_2$ is the maximum of the Mach number with respect to the medium ahead of the front, where c_2 is defined by (4·27). In particular,

*) Thus, if $u>c_2$, small perturbations on the front surface will not penetrate into the fluid ahead of the front. If $u>c_1$, they can be propagated behind the front as linear combinations of the two second sound modes, whereas, if $u<c_1$, only as the second sound mode propagating in the negative x direction. If the two conditions $u>c_1$ and $u<c_2$ are satisfied, the conservation laws require that the perturbations cannot separate from the surface. This argument[25] is very plausible, but is not rigorous owing to the highly nonlinear nature of the shock front.

when $K_2=0$, we have a simple expression

$$u_{\max}/c_2=\left[\left(1+\frac{1}{2}a^2\right)/\left(1-\frac{1}{16}a^2\right)\right]^{1/2}. \tag{4·31}$$

At this maximum we also have $K_1=\frac{3}{4}a/(1-\frac{1}{16}a^2)^{1/2}$ and $|A_1|=1+\frac{3}{4}a^2/(1-\frac{1}{16}a^2)$. In Fig. 3 we write the two Mach numbers $M_{a1}=u/c_1$ and $M_{a2}=u/c_2$, and $|A_1|$ as functions of ϑ_1 in the case $K_2=0$ for $a=1$.

§ 5. Shock profiles

The shock profile can be calculated very simply in the case $\varkappa^2\gg1$ as in the case of the interface profile, where $\varkappa^2\sim(\mathrm{Re}\,b_1)b_2$ is defined by (3·7). In this section we assume $\varkappa^2\gg1$ and also $\mathrm{Im}\,b_1=0$, since $\mathrm{Im}\ b_1$ gives rise to no essential difference in the final results.

Let us first set up the equations for η and ϑ from (4·2) as

$$A+\left(\frac{d}{dx}\theta\right)^2+\eta=\frac{1}{2\eta}\frac{d^2}{dx^2}\eta-\frac{1}{4\eta^2}\left(\frac{d}{dx}\eta\right)^2+\frac{1}{2}b_1{}^{-1}\frac{u}{\eta}\frac{d\eta}{dx}, \tag{5·1}$$

$$\left(b_1\frac{d}{dx}+u\right)\vartheta+(a^{-1}A-\omega_0)\eta=0. \tag{5·2}$$

We will first neglect the right-hand side of (5·1) and will check self-consistently that it is in fact small. Note that the magnitude of the deviation of A from the average is of order A_2-A_1 and that of $(d\theta/dx)^2$ from the average is at most of order $2K(K_1-K_2)=2K(A_1-A_2)/au$, where K is the greater of $|K_1|$ and $|K_2|$. Thus, if $2K\ll au$, we simply have

$$\eta\cong-A. \tag{5·3}$$

Then (5·2) and (4·3) are rewritten as

$$\left(b_1\frac{d}{dx}+u\right)\vartheta-a^{-1}(\eta+a\omega_0)\eta=0, \tag{5·4}$$

$$\left[b_2\frac{d}{dx}+\left(1+\frac{1}{2}a^2\right)u\right]\eta-a\vartheta=\text{const}. \tag{5·5}$$

ϑ can be eliminated to give

$$b_1b_2\frac{d^2}{dx^2}\eta+\left[b_2+\left(1+\frac{1}{2}a^2\right)b_1\right]u\frac{d}{dx}\eta$$
$$=\eta^2+\left[a\omega_0-\left(1+\frac{1}{2}a^2\right)u^2\right]\eta+\text{const}. \tag{5·6}$$

The right-hand side of (5·6) should be of the form $(\eta-\eta_1)(\eta-\eta_2)$ since the left-hand side vanishes as $x\to\pm\infty$. The width of the front region is known to be given by

$$l_s=u\left[b_2+\left(1+\frac{1}{2}a^2\right)b_1\right]/(\eta_1-\eta_2). \tag{5·7}$$

We introduce y and $F(y)$ by

$$y=x/l_s,\quad F(y)=(\eta_1-\eta)/(\eta_1-\eta_2). \tag{5·8}$$

Then (5·6) reads

$$\varepsilon_D \frac{d^2}{dy^2}F + \frac{d}{dy}F = (1-F)F\ , \tag{5·9}$$

where

$$\varepsilon_D = (\eta_1-\eta_2)(b_1b_2/u^2)\left[b_2+\left(1+\frac{1}{2}u^2\right)b_1\right]^{-2}. \tag{5·10}$$

The boundary conditions are $F(-\infty)=0$ and $F(\infty)=1$.

If $\varepsilon_D \leq 1/2$, F approaches 1 monotonically, whereas, if $\varepsilon_D > 1/2$, it oscillates around 1 for large y. In real helium we have $b_2 \gtrsim b_1$*) and

$$\varepsilon_D \cong (\eta_1-\eta_2)(b_1/b_2u^2) \sim (1-\eta_2/\eta_1)w\ . \tag{5·11}$$

Therefore, ε_D appears to be considerably smaller than 1 in any case. If $\varepsilon_D \ll 1$, (5·9) is solved to give

$$F \cong 1/(1+e^y). \tag{5·12}$$

Thus,

$$\eta(x) = \frac{1}{2}(\eta_1+\eta_2) - \frac{1}{2}(\eta_1-\eta_2)\tanh(x/2l_s). \tag{5·13}$$

A similar profile is known for the pressure variation in the case of weak shock waves of ordinary sounds.[25] The shock thickness is roughly of the following order in the original units:

$$k_0^{-1}l_s \sim \xi(4u_0/\chi_0)^{1/2}(\lambda_0/g_0)/(\eta_1-\eta_2) \sim \xi/(1-\rho_{s2}/\rho_{s1}), \tag{5·14}$$

where ξ is the correlation length assumed to be of the same order on the two sides of the front and ρ_{s1} and ρ_{s2} are the superfluid densities on the two sides. Equations (4·16) and (4·18) means that the shock thickness grows as $1/(\eta_1-\eta_2)$ irrespective of the value of $\varkappa$. Thus (5·14) will be valid even if $\varkappa \ll 1$.

Now we can estimate the right-hand side of (5·1): the first term $\sim(1-\eta_2/\eta_1)l_s^{-2}$, the second term $\sim(1-\eta_2/\eta_1)^2 l_s^{-2}$ and the third term $\sim(1-\eta_2/\eta_1)b_1^{-1}l_s^{-1}$. Because $b_1 \lesssim l_s$ from (5·7), the third term is largest and is more precisely of order $(b_1b_2)^{-1}(1-\eta_2/\eta_1)(A_2-A_1)$. Thus the right-hand side is smaller than the deviation of $A(\sim A_2-A_1)$ by the factor $\varkappa^{-2}$ even if $1-\eta_2/\eta_1$ is not small.

§ 6. Interfacial motion

Thermal disturbances in superfluids propagate rapidly in the form of second sound waves or shock waves. Those in normal fluids, on the contrary, obey the thermal diffusion equation and their time scales are much slower than those in superfluids. Then, what is the time scale of temperature variations in the coexisting case? The answer is that the interfacial motion is slowed down by the slow relaxation in normal fluids.[8]

In most cases we can neglect the rapid and small temperature variations on the superfluid side as compared to those on the normal fluid side. The problem is reduced to

*) This is because $w \equiv (\mathrm{Re}\, b_1)/b_2 = \chi_0(\mathrm{Re}\,\Gamma_0)/\lambda_0 \lesssim 1$.[16),27),28)]

a modified version of the Stefan problem.*) Let us consider one-dimensional cases with a superfluid region in $0<x<x_i(t)$ and a normal fluid region in $x_i(t)<x<L$, where $x_i(t)$ is the interfacial position and L is the cell size. First, we assume that the characteristic temperature variation in the normal fluid region is much greater than that in the superfluid region. Then we have

$$T(x,t)\cong T_\lambda \qquad \text{for} \quad x\leq x_i(t). \tag{6·1}$$

Second, note that the heat flow in the superfluid region Q is nearly constant over the region and that the latent heat across the interface is negligibly small, because the entropy gap implied by (3·16) is proportional to $T_\lambda - T_\infty$.[8] Thus we obtain the second boundary condition at the interface:

$$\left(\lambda\frac{\partial T}{\partial x}\right)_i = Q\,, \tag{6·2}$$

where λ is the thermal conductivity and the left-hand side is the heat current extrapolated to the interface on the normal fluid side. Note that the right-hand side is replaced by $-T\Delta S dx_i/dt$ in the usual Stefan problem, ΔS being the entropy gap. We call (6·2) the modified Stefan condition. Of course T obeys the thermal diffusion equation in $x_i<x<L$:

$$\frac{\partial}{\partial t}T=\frac{\partial}{\partial x}(\lambda/\chi)\frac{\partial}{\partial x}T\,, \tag{6·3}$$

where χ is the specific heat. If we give the value of Q and one additional boundary condition at $x=L$, the equations are then complete and we can in principle calculate $x_i(t)$ and $T(x,t)$.

The simplest example is the case in which the superfluid region expands with a constant velocity $v_i=dx_i/dt$. Then $\partial T/\partial t=-v_i\partial T/\partial x$ and (6·3) is integrated to give

$$\lambda\frac{dT}{dx}=-v_i\chi(T-T_\lambda)+Q\,. \tag{6·4}$$

Here $\lambda\cong\lambda^*(T/T_\lambda-1)^{-x_\lambda}$ where λ^* and $x_\lambda(\cong 0.42)$[16] are constants and the weak critical singularity of χ may be neglected. If the cell size is sufficiently long, T tends to a constant T_2 far from the interface. It is given by

$$T_2-T_\lambda=Q/v_i\chi\,. \tag{6·5}$$

Experimentally, if Q and T_2 are given, the velocity v_i is uniquely determined by (6·5).

On the other hand, a normal fluid region expands in a quite different way. Let us supply a heat flow Q_w greater than Q from the warmer boundary $x=L$. If the fluid is in the superfluid state for $t<0$, a normal fluid region emerges for $t>0$ as

$$y_i(t)\sim[(Q_w-Q)t]^{(1-x_\lambda)/(2-x_\lambda)}\,, \tag{6·6}$$

where $y_i=L-x_i$ is the thickness of the normal fluid region. Details of the calculation can be found in Ref. 8).

*) This is a problem of a diffusion equation with moving boundaries. One additional boundary condition is necessary to determine the boundary motion as compared to the usual case of fixed boundaries.[38]

§ 7. Transition from normal to coexisting states

We are interested in the stability of a stationary normal fluid state under heat flow.[7),9),39)] The heat flow is in the negative x direction and the temperatures at the boundary walls, $x=0$ and L, are denoted by T_0 and T_1 with $T_0 < T_1$. A superfluid region will emerge from the cooler boundary $x=0$ as the three parameters Q, T_0 and T_1 are varied slowly. For simplicity, we consider only stationary states and the boundary conditions will be assumed to be changed much slower than the thermal relaxation time of the system ($\sim L^2/D_T$, D_T being the characteristic value of the thermal diffusivity).

Let us consider the following situations:

(i) We fix the heat flow being subtracted outside at $x=0$, Q_0 and that being supplied inside at $x=L$, Q_1. If $\delta Q \equiv Q_0 - Q_1$ is positive and very small, the fluid will be cooled as a whole quasi-stationarily. A HeII layer will emerge when T_0 is lowered below a certain value.

(ii) T_1 is fixed above T_λ. Then there is a unique relation between T_0 and Q in each stationary state. If T_0 is lowered below T_λ (or if Q is decreased with T_0 slightly below T_λ), a HeII layer will emerge at a certain value of T_0.

(iii) T_0 is fixed slightly below T_λ and T_1 (or Q) is varied. A HeII region will emerge if T_1 is smaller than a certain value. This case is essentially the same as (ii).

We shall see that the normal fluid state will become linearly unstable if T_0 is smaller than a critical temperature T_{sc}, where $T_\lambda - T_{sc} = \text{const } Q^{3/4} > 0$. Our main results are as follows: (1) $T_\lambda - T_{sc} \ll T_\lambda - T_\infty$ for $\varkappa \gg 1$, whereas $T_\lambda - T_{sc} \gg T_\lambda - T_\infty$ for $\varkappa \ll 1$. Then the fluid is bistable for $\varkappa \ll 1$ (see Fig. 5 below). (2) Moreover, for $\varkappa \ll 1$, the transition at $T \sim T_{sc}$ is an inverted bifurcation. The fluid will jump from a normal fluid state to a coexisting state with an interface separating the two phases. Consequently, T_0 changes discontinuously in the first case (i), whereas Q increases discontinuously in the cases (ii) and (iii). In the following we give the detailed calculations and discussions.

For simplicity we consider the case in which the thermal resistance due to vortices can be neglected. Equation (1·3) indicates that the vortex resistance is always negligible when

$$L \ll \varepsilon_{\rm th}^{-1} \xi_c \sim 10\, Q^{-1/2} \mathrm{cm}\,, \tag{7·1}$$

where Q is in erg/sec cm^2. We further neglect the boundary lowering from the estimation (3·32). Under (7·1) the temperature at $x=0$ is nearly equal to T_∞. For example, in Bhagat's experiment Q was of order 10^5 cgs and the right-hand side of (7·1) was of order 10^{-2}cm, which was probably even smaller than the size of his thermometer. On the other hand, in Ahlers' first experiment,[11)] $Q \sim 1$ cgs and $L \sim 1$ cm, and (7·1) was satisfied.

We first investigate the linear stability analysis of the normal fluid state. We linearize (2·10) around the solution $\Psi = 0$ and $A = A_0 + Gx$, where A_0 represents the temperature deviation at $x=0$. G is related to Q^* in (3·2) by $Q^* = b_2 G$. Therefore,

$$G = Q^*/b_2 = (a/k_0{}^3 b_2)(4u_0 Q/g_0 k_B T_\lambda). \tag{7·2}$$

We analyze the linear equation for the small deviation Ψ varying only along the x axis:

$$\frac{\partial}{\partial t}\Psi = ia^{-1}A_0\Psi - b_1[A_0+\mathscr{H}]\Psi\ , \tag{7·3}$$

where

$$\mathscr{H} = -\frac{\partial^2}{\partial x^2}+(1-i/ab_1)Gx\ . \tag{7·4}$$

We are interested in disturbances localized at $x=0$. Then the system size L may be pushed to infinity if L is of a macroscopic size. We seek a growing solution of (7·3) under the boundary conditions $\Psi(0)=\Psi(\infty)=0$. In Appendix D we shall see that the eigen functions Ψ_n and the eigen values E_n of $\mathscr{H}$ can be expressed in terms of the Airy function $\mathrm{Ai}(z)$[40),41)] in the forms

$$\Psi_n(x)=\sigma^{1/2}\beta_n\mathrm{Ai}(\sigma x-\alpha_n), \tag{7·5}$$

$$E_n=\alpha_n\sigma^2\ , \tag{7·6}$$

where $n=0,\ 1,\ 2,\ \cdots$ and

$$\sigma=(1-i/ab_1)^{1/3}G^{1/3}\quad \text{with}\quad |\arg\sigma|<\pi/3\ . \tag{7·7}$$

The series $\alpha_0<\alpha_1<\cdots$ represents the zeros of $\mathrm{Ai}(-z)$ and $\beta_n=(-1)^n/\mathrm{Ai}'(-\alpha_n)$. In particular, $\alpha_0\cong 2.34$ and $\beta_0\cong 1.43$. Because $\mathrm{Im}\ \sigma^3=-(G/a)\mathrm{Re}(1/b_1)<0$, we find $0>\arg\sigma>-(\pi/3)$. Then, $\Psi_n(x)$ goes to zero rapidly as $x\to+\infty$ as shown in Appendix D. The $\Psi_n(x)$ are orthogonal in the following sense:

$$\int_0^\infty dx\,\Psi_n(x)\Psi_m(x)=\delta_{nm}\ . \tag{7·8}$$

Let $\mu_n\equiv\int_0^\infty dx\,\Psi_n(x)\Psi(x,0)$. Then,

$$\Psi(x,\ t)=\sum_{n=0}^{\infty}u_n\Psi_n(x)\exp[ia^{-1}A_0t-b_1A_0t-b_1E_nt]. \tag{7·9}$$

In Appendix D we will show $\mathrm{Re}\ (b_1\sigma^2)>0$. Then, the deviation proportional to Ψ_0 first becomes unstable as A_0 is lowered. Therefore, we find a critical value of A_0, which will be denoted by A_{sc} and is given by

$$A_{\mathrm{sc}}=-\alpha_0\mathrm{Re}(b_1\sigma^2)/\mathrm{Re}\,b_1=-\alpha_0\mathrm{Re}[(1+ic_0)(1-i/ab_1)^{2/3}]G^{2/3}\ , \tag{7·10}$$

where $c_0\equiv\mathrm{Im}\,\Gamma_0/\mathrm{Re}\,\Gamma_0$.

In the original units $\tau_{\mathrm{sc}}\equiv k_0{}^2A_{\mathrm{sc}}$ is written as

$$\tau_{\mathrm{sc}}=-C_{\mathrm{sc}}(4u_0Q/g_0k_BT_\lambda)^{2/3}\ , \tag{7·11}$$

where use has been made of (7·1) and

$$C_{\mathrm{sc}}=\alpha_0b_2{}^{-2/3}\mathrm{Re}[(1+ic_0)(a-i/b_1)^{2/3}]. \tag{7·12}$$

This value of τ_{sc} should be compared with the value of τ on the HeII side in the case of the two-phase coexistence. The latter value was denoted by τ_∞ and given by (3·16). The proportionality constants C_∞ and C_{sc} behave quite differently as x (or $T_\lambda-T$) is varied. For $x\gg1$ we may assume $a\gg1/|b_1|$ and $C_{\mathrm{sc}}\cong 2.34(a/b_2)^{2/3}$. Hence,

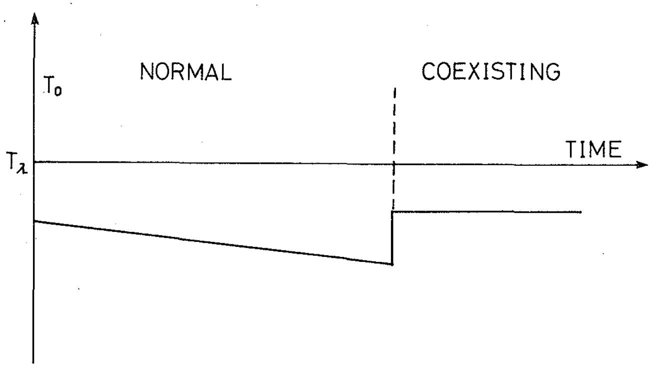

Fig. 4. The expected discontinuous change of the temperature at the cooler boundary, T_0, for $\varkappa \ll 1$ in case (i).

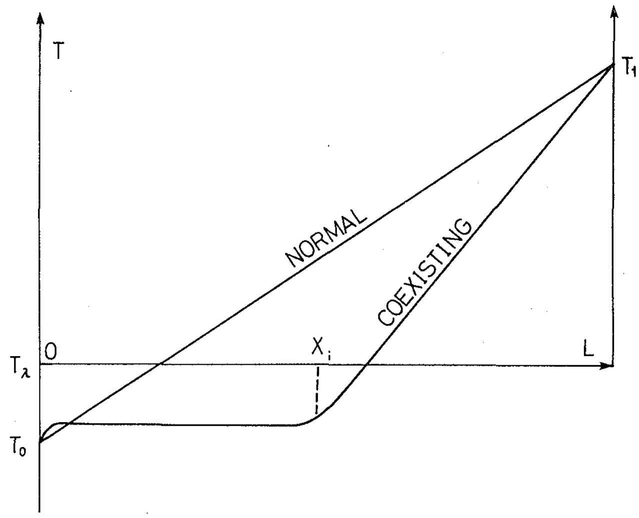

Fig. 5. The bistability for $\varkappa \ll 1$ in the case of fixed boundary temperatures. The fluid will jump from the normal fluid state to the coexisting state.

$$C_{sc}/C_\infty \cong 2.34/(\sqrt{3}\varkappa)^{2/3} \ll 1 . \tag{7·13}$$

For $\varkappa \ll 1$, on the other hand, we have $a \ll 1/|b_1| \sim \varkappa^{-1}|w|^{-1/2}$ and $C_{sc} \cong a_0 \varkappa^{-4/3}$. Thus,

$$C_{sc}/C_\infty \cong 2.34(2\mathrm{B})^{-1/2}\varkappa^{-1} \gg 1 . \tag{7·14}$$

The above two relations suggest a continuous change for $\varkappa \gg 1$ and a discontinuous change for $\varkappa \ll 1$ in the situations (i)~(iii). The expected behavior of T_0 is schematically shown in Fig. 4 for the case of fixed heat flow (i). In the cases of fixed boundary temperatures (ii) and (iii) the temperature profile changes discontinuously for $\varkappa \ll 1$, resulting in a discontinuous increase of the heat flow, as is evident from Fig. 5. If $C_{sc}/C_\infty \gg 1$, the fluid is bistable for fixed boundary temperatures.

To confirm the above expectation we derive the Landau equation for the amplitude μ_0 in (7·9) and examine the type of the bifurcation occurring at $A_0 \cong A_{sc}$. The definition of μ_n is given by

$$\mu_n(\boldsymbol{r}_\perp, t) = \int_0^\infty dx\, \Psi_n(x) \Psi(\boldsymbol{r}, t), \tag{7·15}$$

which are generally dependent on $\boldsymbol{r}_\perp = (y, z)$ and t. We define

$$\delta \equiv -A_0 + A_{sc} . \tag{7·16}$$

We assume $a_0 \sim |\delta|^{1/2}$, $a_n \sim |\delta|^{3/2}$ for $n \geq 1$, and $|\delta A| \sim \delta$, where δA is the temperature deviation defined by

$$A = A_0 + Gx + \delta A . \tag{7·17}$$

As the boundary conditions for δA we assume

$$\delta A(0, t) = \delta A'(0, t) = 0 . \tag{7·18}$$

The prime means the spatial differentiation. The heat flow at $x=0$ is fixed. The

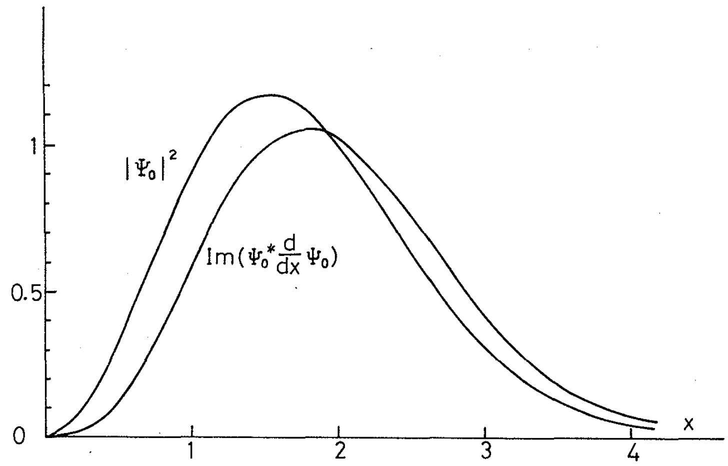

Fig. 6. $|\Psi_0(x)|^2$ and $\mathrm{Im}(\Psi_0^*(x)d/dx\ \Psi_0(x))$. Here $ab_1=0.35+0.23i$, which is the value of the F model to first order in $\varepsilon=4-d$.

superfluid density and current are $|\mu_0|^2|\Psi_0(x)|^2$ and $|\mu_0|^2\mathrm{Im}(\Psi_0^*(\partial/\partial x)\Psi_0)$, respectively, to first order in δ. In Fig. 6 we plot $|\Psi_0(x)|^2$ and $\mathrm{Im}(\Psi_0^*(\partial/\partial x)\Psi_0)$. The two functions are positive-definite for $x>0$ (see Appendix D).

Multiplying (2·10) by $\Psi_0(x)$ and integrating over x we obtain

$$\frac{\partial}{\partial t}\mu_0=[(ia^{-1}-b_1)A_0-b_1\alpha_0\sigma^2+b_1\nabla_\perp{}^2]\mu_0-|\sigma|b_1F_1|\mu_0|^2\mu_0$$

$$+(ia^{-1}-b_1)\left[\int_0^\infty dx\,\Psi_0(x)^2\delta A(\boldsymbol{r},t)\right]\mu_0\,, \qquad (7\cdot19)$$

where $\nabla_\perp{}^2=\partial^2/\partial y^2+\partial^2/\partial z^2$ and

$$F_1=\beta_0{}^4\int_0^\infty dxAi(z)^3Ai(z^*). \qquad (7\cdot20)$$

Here $z=e^{i\theta}x-\alpha_0$ with $\theta=\arg\sigma$ and F_1 is a function of θ. The first linear term on the right-hand side of (7·19) is rewritten as $(i\omega_1+b_1\delta+b_1\nabla_\perp{}^2)\mu_0$ with $\omega_1\equiv(a^{-1}-\mathrm{Im}\,b_1)A_0-\alpha_0\mathrm{Im}(b_1\sigma^2)$. We redefine $\mu_0e^{i\omega_1t}$ as μ_0 and then the term $i\omega_1\mu_0$ disappers in (7·19). The second cubic term $(\propto|\mu_0|^2\mu_0)$ always serves to suppress the growing of μ_0. In fact, we can numerically verify $\mathrm{Re}(b_1F_1)>0$.

However, the third nonlinear term has a destabilizing effect for $a^{-1}\gg|b_1|$. We can show that the third term becomes dominant over the second term for $x\ll1$ and

$$\int_0^\infty dx\,\mathrm{Im}\,\Psi_0(x)^2\delta A(\boldsymbol{r},t)<0\,. \qquad (7\cdot21)$$

To show this we express δA in terms of μ_0 for $|\delta|\ll1$. We can assume that the time scale of μ_0 and δA is of order $|\delta|^{-1}$ and the spatial scale on the yz-plane is of order $|\delta|^{-1/2}$. Then $(\partial/\partial t)M$ is of order δ^2 and can be set to zero in (2·11). Thus δA follows the variation of $\mathcal{J}$ instantaneously as

$$\delta A(\boldsymbol{r},t)\cong-(a/b_2)\int_0^x dx\,\mathcal{J}(\boldsymbol{r},t)$$

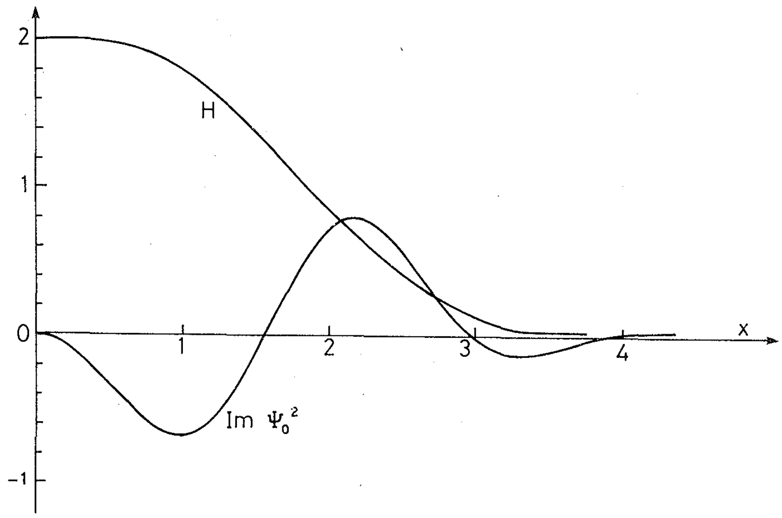

Fig. 7. $H(x)$, (7·23) and Im $\Psi_0(x)^2$ for $ab_1=0.35+0.23i$. The inequality (7·24) can be seen to hold.

$$\cong -(a/b_2)|\mu_0|^2\int_0^x dx\,\mathrm{Im}\left(\Psi_0{}^*\frac{d}{dx}\Psi_0\right), \tag{7·22}$$

where the corrections of order δ^2 have been neglected. The right-hand side of (7·22) is negative-definite for $x>0$. Namely, the temperature inside the fluid must be lowered with the appearance of the HeII layer if the temperature at $x=0$ and the heat flow is fixed. Let us define $H(x)$ by

$$H(x)=\int_x^\infty dx\,\mathrm{Im}\left(\Psi_0{}^*\frac{\partial}{\partial x}\Psi_0\right). \tag{7·23}$$

Then, (7·21) is rewritten as

$$I_0\equiv -\int_0^\infty dx\,[\mathrm{Im}\,\Psi_0(x)^2]H(x)>0\,. \tag{7·24}$$

Use has been made of $\int_0^\infty dx\,\mathrm{Im}\,\Psi_0(x)^2=0$ which is derivable from (7·8). Figure 7 confirms (7·24).

After some manipulations we arrive at the desired Landau equation

$$\frac{\partial}{\partial t}\mu_0=b_1[\delta+\nabla_\perp{}^2]\mu_0-b_1Z_0|\mu_0|^2\mu_0\,, \tag{7·25}$$

where

$$Z_0=|\sigma|F_1-|\sigma|(i-ab_1/b_2)(b_1b_2)^{-1}F_2\,, \tag{7·26}$$

where

$$F_2=\beta_0{}^4\int_0^\infty dx\,[zAi(z)^2-Ai'(z)^2]\mathrm{Im}[Ai(z)^*Ai'(z)e^{i\theta}]. \tag{7·27}$$

The definition of z is the same as in (7·20). Note that F_1 and F_2 are complex numbers of order 1. For $\varkappa\gg1$, we have $Z_0\cong|\sigma|F_1$ and $\mathrm{Re}(b_1Z_0)>0$. For $\varkappa\ll1$, on the contrary, $Z_0\cong -i|\sigma|(b_1b_2)^{-1}F_2$ and

$$\mathrm{Im}(b_1 Z_0) \cong |\sigma| b_2^{-1} (\mathrm{Im}\, F_2) = -b_2^{-1} I_0 < 0\,, \tag{7·28}$$

where I_0 is defined by (7·24). In Ref. 7) F_2 is calculated numerically as a function of θ. The author has not yet succeeded in proving the inequality (7·21) or (7·24) analytically, but there should be a general proof.

In summary, the mode coupling terms, those proportional to g_0 in (2·1) and (2·2), decrease the surface critical temperature T_{sc} and result in the bistability sufficiently close to the criticality as shown in Fig. 5.*) The importance of these terms increases as $T \to T_\lambda$ with the increase of $\varkappa^{-1} (\propto g_0)$. We have found that the transition is an inverted bifurcation, but we cannot determine in our scheme the exact temperature or heat flow at which the jump from the normal to coexisting states occurs. In the following we give some discussion which might be relevant to experiments, assuming that the transition takes place at a value of $T_\lambda - T_0$ considerably greater than $T_\lambda - T_\infty$.

We first estimate the value of $T_\lambda - T_{sc}$ from (7·11). If u_0 in (7·11) and u_0 in (3·16) are assumed to be the same, then $\tau_{sc}/\tau_\infty = [(T_\lambda - T_{sc})/(T_\lambda - T_\infty)]^{4/3} = C_{sc}/C_\infty$ with Q common in the two states. If $u_0 \propto (T_\lambda - T_{sc})^{2/3}$ in (7·11) and $u_0 \propto (T_\lambda - T_\infty)^{2/3}$ in (3·16), we find a slightly different result, $(T_\lambda - T_{sc})/(T_\lambda - T_\infty) = (C_{sc}/C_\infty)^{9/8}$. In any case we have

$$R_{\max} \equiv (T_\lambda - T_{sc})/(T_\lambda - T_\infty) = (C_{sc}/C_\infty)^\gamma \quad \text{with} \quad \gamma \sim 1\,. \tag{7·29}$$

We cannot determine γ exactly because our arguments are based on the mean field calculations.

In the cases of fixed boundary temperatures (ii) and (iii) we denote the heat flow in the normal fluid state by Q_N and that in the coexisting state by Q_S. We assume that the transition takes place for $T_\lambda - T_0 = R(T_\lambda - T_\infty)$, where R is greater than 1 and smaller than $R_{\max}$, (7·29). Then at the transition we find

$$T_\lambda - T_0 = R T_\lambda A_\infty Q_N^{3/4} = T_\lambda A_\infty Q_S^{3/4}\,. \tag{7·30}$$

Therefore,

$$Q_S/Q_N = R^{4/3}\,. \tag{7·31}$$

On the other hand, the temperature in the normal fluid region not close to the cooler boundary and the interface is given by[10)]

$$T - T_\lambda = T_\lambda [(1-x_\lambda) Q_N x/\lambda^* T_\lambda]^{1/(1-x_\lambda)} \qquad \text{for } 0 < x < L\,, \tag{7·32a}$$

$$= T_\lambda [(1-x_\lambda) Q_S (x - x_i)/\lambda^* T_\lambda]^{1/(1-x_\lambda)} \text{for } x_i < x < L\,. \tag{7·32b}$$

The first expression is valid for the normal fluid state and the second for the coexisting state. The constant x_i in (7·32b) represents the interfacial position. The two constants x_λ and λ^* are defined by the expression for the thermal conductivity above T_λ, $\lambda = \lambda^*(T/T_\lambda - 1)^{-x_\lambda}$. The boundary temperature at $x = L$, T_1, can be obtained from (7·32) by setting $x = L$. Therefore, $Q_N L = Q_S (L - x_i)$ or

$$x_i = L(1 - Q_N/Q_S) = L(1 - R^{-4/3})\,. \tag{7·33}$$

We can also calculate the entropy difference between the two states for the cases (ii) and (iii). Let us assume that the specific heat per unit volume χ is a constant, for

*) A similar result was found for a one-dimensional superconductor under electric field.[42)]

simplicity. Then the entropy difference ΔS per unit volume is determined by

$$L\Delta S=\chi\int_0^L dx[T(x)-T_\lambda]-\chi\int_{x_i}^L dx[T(x)-T_\lambda], \tag{7·34}$$

where $T(x)$ in the first term is given by (7·32a) and $T(x)$ in the second term by (7·32b). After some calculations we find a very simple expression

$$\Delta S=S_N\left(\frac{x_i}{L}\right)=S_N(1-R^{-4/3}) \tag{7·35}$$

with

$$S_N=\left(\frac{1-x_\lambda}{2-x_\lambda}\right)\chi(T_1-T_\lambda). \tag{7·36}$$

Here S_N is the first term of (7·34) divided by L and represents the entropy in the normal fluid state measured from the criticality. From (7·31), (7·33) and (7·35) we can eliminate R to obtain

$$1-Q_n/Q_S=x_i/L=\Delta S/S_N\,. \tag{7·37}$$

§ 8. Summary and remarks

(A) HeI-HeII Interface

There are three aspects of the problem of the two-phase coexistence as noted in the final section of Ref. 8).

(i) The structure of the interface as explained in §3. The intrinsic lowering $T_\lambda-T_\infty$ ($\sim 10^{-8}Q^{3/4}$, Q in cgs) can be detected only by special thermometry. One possibility is to use a thin thermometer which can detect temperatures in a very narrow region. Another possibility is to use a very precise thermometer working at small heat flow under the condition $L\ll 10Q^{-1/2}$cm with Q in cgs.

(ii) The motion of the interface as explained in §6.

(iii) The phase changes among the normal fluid, the coexisting, and the superfluid states under heat flow as discussed in §7. In particular, emergence of a HeII region from the cooler boundary is the most interesting phenomenon. Its transition changes over from a continuous one to a discontinuous one as the criticality is approached (or the heat flow is decreased). The two cases are characterized by $\varkappa\gg 1$ and $\varkappa\ll 1$, where $\varkappa$ is defined by (3·7). In the recent literature[6),16),27),28)] the strength of the mode coupling has been represented by a dimensionless number $f\equiv K_d g_0{}^2\xi/\lambda_0\mathrm{Re}\Gamma_0$ where K_d is a constant. In terms of f we have $\varkappa^2\cong 0.1/f$. Here f increases from numbers of order 10^{-2} to a number of order 1 as $|T/T_\lambda-1|$ is decreased. The saturation occurs for $|T/T_\lambda-1|\lesssim 10^{-3}$. We notice that the intrinsic lowering reduced temperature is in most cases much less than 10^{-3}, so that the condition $\varkappa^2\sim 0.1$ will be realized at the transition near the cooler boundary if experiments are performed.

(B) Shock Waves

(i) In the λ region the shock relations and the shock velocity are dependent essentially only on the static parameter a, (2·9), if the physical quantities are appropriately scaled.

Their behavior is quite different for the two cases $a \gtrsim 1$ and $a \ll 1$. Hence it is highly desirable to calculate the parameter a precisely as a function of $T_\lambda - T$ by the renormalization group method.[26] We also notice that Khalatnikov's theory of the second sound entrainment[29] can give us a method to estimate a. He expanded the second sound velocity c_2 in counterflow as $c_2 = c_{20} + \gamma \boldsymbol{v}_n \cdot \boldsymbol{k}/k + \cdots$ where c_{20} is the equilibrium velocity and $\boldsymbol{k}$ is the wave vector of the sound. The entrainment coefficient γ can be expressed in terms of thermodynamic quantities. We can determine a by comparing his expression and (4·14). We must also remark that the entrainment coefficient γ and the steepening coefficient in (1·4) behave quite similarly in the whole temperature region below T_λ. See a striking resemblance between the figure of Ref. 24) (or Fig. 1 of Ref. 19)) and the figure of Ref. 29). This suggests that the second sound velocity in counterflow and the shock velocity are very similar as functions of the temperature and the heat flow.

(ii) As can be known from Fig. 2 a cooler region can expand into a warmer region as a shock wave in the λ region unless the superfluid velocity far ahead of the front is very large and in the same direction as the shock wave. This means that the trailing edge steepens into a front in the case of a heat pulse. If a negative heat pulse can be realized, the leading edge becomes a front. Then it might be possible to observe the maximum of the shock velocity which has been discussed at the end of §4.

(iii) We have discussed on the supersonic-subsonic criterion. The supersonic condition $u > c_2$ is always satisfied, whereas the subsonic condition $u < c_1$ can be violated for large counterflow behind the front. Interestingly, when the marginal condition $u = c_1$ is attained, the shock velocity u takes a maximum. Furthermore, if $u > c_1$, large disturbance will develop in the back region of the front. As a definite result we have found that the superfluid density ahead of the front must be smaller than that behind the front. However, this condition is not sufficient and our analysis of the front stability is still inadequate. We should examine the time-development of corrugation-like disturbances on the front starting with the dynamic equations (2·1) and (2·2).

(iv) We have neglected effects of vortices on the shock wave, expecting their role as secondary. This is in contradiction with Turner's analysis[22] (see below).

We comment on Turner's analysis of his own experiment. He send a heat pulse into equilibrium helium and observed a maximum of the shock velocity with increasing heat power at $T \cong 1.65$°K. There, Khalatnikov's steeping coefficient in (1·4) is positive and a front appears at the leading edge. He interpreted this maximum as indicating vortex nucleation[35] in the pulse region where a large counterflow exists. We disagree with this interpretation. We consider simply that the shock velocity will increase first linearly with increasing heat power by the entrainment effect and then it will saturate and decrease by the decrease of ρ_s in the pulse region. The maximum is attained when these two effects balance. It should be noted that the second sound velocity c_{II} behaves in the same way with increasing counterflow. Turner also found that the pulse shape is distorted from a simply formed trapezoidal shape if the heat power is greater than a breakpoint value at which Khalatnikov's relation breaks down. He considered this as arising from "a breakdown in superfluidity of HeII", whereas we expect that this phenomenon should be caused by the breakdown of the subsonic condition. Vortices would not distort the pulse shape so strongly. We of course admit that our objections to Turner's interpretation have been obtained from the calculation near the λ point. We must extend our results outside the λ region. Some of them should be general and remain unchanged.

Acknowledgements

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Appendix A

We examine linear modes varing along the x axis in homogeneous current-carrying state. We assume that deviations $\delta\Phi=\Phi-\Phi_0$ and $\delta A=A-A_0$ are infinitesimal, where $\Phi_0=[|A_0|-K^2]e^{iKx+i\omega_0 t}$. Then (2·10) and (2·11) may be linearized as

$$\frac{\partial}{\partial t}\delta\Phi=ia^{-1}A_0\delta\Phi+ia^{-1}\Phi_0\delta A-b_1\left[A_0-\left(\frac{\partial}{\partial x}\right)^2+|\Phi_0|^2\right]\delta\Phi$$

$$-b_1\Phi_0[\delta A+\Phi_0\delta\Phi^*+\Phi_0{}^*\delta\Phi], \tag{A·1}$$

$$\frac{\partial}{\partial t}\delta M=a\frac{\partial}{\partial x}\delta\mathcal{J}+b_2\left(\frac{\partial}{\partial x}\right)^2\delta A\,. \tag{A·2}$$

Let us define $W=W_1+iW_2$ by

$$\delta\Phi=\Phi_0 W\exp(ia^{-1}A_0 t). \tag{A·3}$$

Then (A·1) becomes

$$\frac{\partial}{\partial t}W=ia^{-1}\delta A+b_1\left[\frac{\partial^2}{\partial x^2}+2iK\frac{\partial}{\partial x}\right]W-b_1[\delta A+2|\Phi_0|^2W_1]. \tag{A·4}$$

The deviations δM and $\delta\mathcal{J}$ are of the forms

$$\delta M=\delta A-a^2|\Phi_0|^2W_1\,,\quad \delta\mathcal{J}=|\Phi_0|^2\left(2KW_1+\frac{\partial}{\partial x}W_2\right). \tag{A·5}$$

At long wavelengths the spatial derivatives in (A·2) and (A·4) are small and we obtain two oscillating modes and one relaxational mode whose decay rate does not vanish even in the long wavelength limit. The calculations are rather complicated. We first consider the oscillating modes in the long wavelength limit. Then we can self-consistently check that the amplitude deviation W_1 is determined by

$$\delta[A-\nabla^2+|\Phi|^2]\Phi\cong\Phi_0\left(\delta A+2|\Phi_0|^2W_1+2K\frac{\partial W_2}{\partial x}\right)\cong 0\,. \tag{A·6}$$

Namely, the amplitude correction satisfies the local equilibrium relation[3),29)]

$$2|\Phi_0|^2W_1\cong-\delta A-2K\frac{\partial}{\partial x}W_2\,. \tag{A·7}$$

Then,

$$\delta M=\left(1+\frac{1}{2}a^2\right)\delta A+a^2K\frac{\partial}{\partial x}W_2\,, \tag{A·8}$$

$$\delta \mathcal{G} = -K\delta A + (|A_0| - 3K^2)\frac{\partial}{\partial x}W_2 . \tag{A·9}$$

Further we have $(\partial/\partial t)W_2 \cong a^{-1}\delta A$ and $(\partial/\partial t)\delta M \cong a(\partial/\partial x)\delta\mathcal{G}$ from (2·10) and (2·11) neglecting the dissipation to obtain (4·14).

We can obtain complete solutions of the linearized equations (A·1) and (A·2) by assuming the forms

$$W_1 = \mathrm{Re}(\alpha_1 e^{iqx+pt}), \qquad W_2 = \mathrm{Re}(\alpha_2 e^{iqx+pt}),$$
$$\delta A = \mathrm{Re}(\beta e^{iqx+pt}), \tag{A·10}$$

where α_1, α_2 and β are appropriate complex numbers. Then we obtain a matrix equation of the form

$$p\begin{Bmatrix} W_1 \\ W_2 \\ \delta A \end{Bmatrix} = \overleftrightarrow{H}\cdot\begin{Bmatrix} W_1 \\ W_2 \\ \delta A \end{Bmatrix}. \tag{A·11}$$

Note that the derivative $\partial/\partial x$ in $\overleftrightarrow{H}$ can be replaced by iq, whereas this procedure is not allowable in (A·4) because W is complex. After some calculations the determinant of the 3×3 matrix $\overleftrightarrow{H} - p\overleftrightarrow{I}$ can be successfuly transformed into the following cubic equation of p:

$$(p + b_2q^2)[2b_{11}(\eta + q^2)p + 2|b_1|^2\bar{\eta}q^2 + 2ib_{12}Kqp]$$
$$+ b_{11}\eta[a^2p^2 + 2\bar{\eta}q^2 + 4iaqKp] + (p + b_2q^2)p^2 + |1 + iab_1|^2\eta p q^2 = 0 , \tag{A·12}$$

where $b_1 = b_{11} + ib_{12}$, $\eta = |A_0| - K^2$ and $\bar{\eta} = |A_0| - 3K^2 + \frac{1}{2}q^2$. Equation (A·12) is very general and has never been obtained. It gives the dispersion and attenuation of the three modes for arbitrary q, K and a.

We expand p with respect to q as

$$p = -ic_{\mathrm{II}}q - \frac{1}{2}D_{\mathrm{II}}q^2 + \cdots. \tag{A·13}$$

Then we find

$$2b_{11}\eta\left[\left(1 + \frac{1}{2}a^2\right)c_{\mathrm{II}} - aK\right]D_{\mathrm{II}}$$
$$= c_{\mathrm{II}}[2|b_1|^2(\eta - 2K^2) + 2b_{11}b_2\eta + |1 + iab_1|^2\eta] - c_{\mathrm{II}}^3 + 2b_{12}Kc_{\mathrm{II}}^2 . \tag{A·14}$$

In particular, if $K=0$, we obtain the damping coefficient near equilibrium for the F model:

$$D_{\mathrm{II}} = |b_1|^2/b_{11} + b_2/\left(1 + \frac{1}{2}a^2\right) + a^2/\left[4\left(1 + \frac{1}{2}a^2\right)^2 b_{11}\right] - ab_{12}/\left[\left(1 + \frac{1}{2}a^2\right)b_{11}\right]. \tag{A·15}$$

The last two terms are specific to the F model and vanish if $a=0$. The ratios of the second term and the others are estimated as follows: the first/ the second $\sim w$, the third / the second $\sim \varkappa^{-2}a^2/4(1 + a^2/2)$, the fourth/ the second $\sim -a(\mathrm{Im}\,\Gamma_0/\mathrm{Re}\,\Gamma_0)w^{1/2}/\varkappa$, where $w = b_{11}/b_2$ and $\varkappa^2 = |b_1|^2 b_2/b_{11} \sim b_{11}b_2$. Sufficiently close to T_λ the existing theories indicate $a \sim 1$, $\varkappa^2 \sim 1/10$, $w \sim 1/10$, and $\mathrm{Im}\,\Gamma_0/\mathrm{Re}\,\Gamma_0 \sim 1$. Thus, the third and fourth terms appear to be important unless $a \ll 1$. Note that experimental results of the second sound damping have been analyzed in terms of the expression for the E model (where $a=0$).[43)~45)] Our calculation suggests that better agreement would be obtained if analysis is made on the basis of the F model.

Next we assume $|A_0| \cong 3K^2$ or $\eta \cong 2K^2$. Then one solution of (4·14) tends to zero as

$$c_{\rm II} \cong -(|A_0|-3K^2)/2aK\,. \tag{A·16}$$

The corresponding sound is propagating in the reverse direction of the superfluid current if $3k^2<|A_0|$. Its damping coefficient is also small as

$$D_{\rm II} \cong \frac{1}{4(aK)^2}(|A_0|-3K^2)\left[2b_2+\frac{1}{b_{11}}|1+iab_1|^2\right]. \tag{A·17}$$

The right-hand side changes its sign when K^2 exceeds $|A_2|/3$. Thus we obtain the criterion (4·15).

Appendix B

Let us consider the following (dimensionless) free energy density:

$$\mathcal{F}=\frac{1}{2}a^{-2}A^2+\frac{1}{2}|\boldsymbol{\nabla}\Phi|^2+\frac{1}{4}|\Phi|^4\,. \tag{B·1}$$

Here Φ and A obey (2·10) and (2·11). Some manipulations yield

$$\frac{\partial}{\partial t}\mathcal{F}=-(\mathrm{Re}\,b_1)|(A-\boldsymbol{\nabla}^2+|\Phi|^2)\Phi|^2-(b_2/a)|\boldsymbol{\nabla}A|^2-\boldsymbol{\nabla}\cdot\boldsymbol{J}_f\,, \tag{B·2}$$

where the current $\boldsymbol{J}_f$ is given by

$$\boldsymbol{J}_f=-(b_2/a)A\boldsymbol{\nabla}A-\frac{1}{2}\left(\frac{\partial}{\partial t}\Phi\cdot\boldsymbol{\nabla}\Phi^*+\frac{\partial}{\partial t}\Phi^*\cdot\boldsymbol{\nabla}\Phi\right). \tag{B·3}$$

In the case of propagating solutions $\mathcal{F}$ depends on space and time only through $x-ut$ and it satisfies

$$\frac{\partial}{\partial t}\mathcal{F}=-u\frac{\partial}{\partial x}\mathcal{F}\,. \tag{B·4}$$

The x-component of $\boldsymbol{J}_f$, which will be simply written as J_f, is of the form

$$J_f=-(b_2/a)A\frac{\partial}{\partial x}A-\omega_0\mathcal{J}+u\left|\frac{\partial}{\partial x}\Phi\right|^2, \tag{B·5}$$

where use has been made of (4·1). Integrations of (B·2) over space now gives

$$R_{\rm dis}=\int_{-\infty}^{\infty}dx\left[u\frac{\partial}{\partial x}\mathcal{F}-\frac{\partial}{\partial x}J_f\right]=u[W_2-W_1] \tag{B·6}$$

with

$$W_j=F_j-\frac{1}{a}\omega_0 M_j-\left|\frac{\partial}{\partial x}\Phi\right|_j^2, \tag{B·7}$$

where the subscript $j(=1,2)$ denotes the values at $x=\pm\infty$ and use has been made of (4·9).

Using (4·4)~(4·6) and (4·10) we eliminate M_j and η_j and express W_j in terms of A_j and K_j as

$$W_j=\left(\frac{1}{4}+\frac{1}{2a^2}\right)(A_j^2-2a\omega_0A_j)+K_j^2\left[A_j+\frac{3}{4}K_j^2-\frac{1}{2}a\omega_0\right]. \tag{B·8}$$

Equations (4·7) and (4·8) imply that

$$A_j^2-2a\omega_0A_j=a^2u^2K_j^2+a^2\omega_0^2\,, \tag{B·9}$$

$$au(K_2^2-K_1^2)=(A_1-A_2)(K_1+K_2). \tag{B·10}$$

Thus,

$$W_2-W_1=\left[\frac{1}{2}\left(1+\frac{1}{2}a^2\right)u^2-\underline{\frac{1}{2}au(K_1+K_2)}+\frac{3}{4}(K_1^2+K_2^2)\right](K_2^2-K_1^2)$$

$$+\underline{\frac{1}{2}(A_1-A_2)(K_1+K_2)^2}+A_2K_2^2-A_1K_1^2-\frac{1}{2}a\omega_0(K_2^2-K_1^2). \tag{B·11}$$

The two underlined terms cancel each other due to (B·10). Now we can eliminate u using (4·12) and ω_0 using

$$a\omega_0(K_2^2-K_1^2)=\frac{1}{2}(A_1+A_2)(K_2^2-K_1^2)-\frac{1}{2}(A_2-A_1)(K_1+K_2)^2. \tag{B·12}$$

Then we obtain (4·18).

Appendix C

Let us define the following polynomials:

$$F_j(x)=\left(1+\frac{1}{2}a^2\right)x^2-2aK_jx+A_j+3K_j^2\,, \tag{C·1}$$

where $j=1, 2$. The second sound velocity c_j is defined as the greater of the two solutions of $F_j(x)=0$. First we will show $F_2(u)>0$ as follows:

$$F_2(u)=(u-c_2)\left[\left(1+\frac{1}{2}a^2\right)(u+c_2)-2aK_2\right]$$

$$=\frac{3}{2}(\eta_1-\eta_2)+\frac{1}{2}(K_1-K_2)^2. \tag{C·2}$$

The second line has been obtained using (4·6) and (4·8). From (4·15) the smaller solution of $F_2(x)=0$ is negative and we have $u>c_2$. Thus $u>c_2$ as long as $\eta_1>\eta_2$.

On the other hand, the sign of $F_1(u)$ is not definite:

$$F_1(u)=-\frac{3}{2}(\eta_1-\eta_2)+\frac{1}{2}(K_1-K_2)^2$$

$$=(K_2+K_1)\left(\frac{3}{2}au-2K_1-K_2\right). \tag{C·3}$$

In the marginal subsonic case $u=c_1$ we have

$$au=\frac{2}{3}(2K_1+K_2). \tag{C·4}$$

Next we fix A_2 and K_2 and calculate the derivative du/dK_1. Differentiations of (4·8) and (4·12) with respect to K_1 yield equations for du/dK_1 and dA_1/dK_1. They are solved to give

$$\left[\left(1+\frac{1}{2}a^2\right)u-\frac{3}{4}aK_1-\frac{1}{4}aK_2\right]\frac{du}{dK_1}=\frac{3}{4}au-K_1-\frac{1}{2}K_2\,. \tag{C·5}$$

The right-hand side of (C·5) vanishes under the condition (C·4) and we have $du/dK_1=0$. We can easily check that u is maximum under (C·4). The maximum $u_{\max}$ can be obtained by eliminating K_1 from (C·4) and (4·12). It is the solution of the equation

$$\left(1-\frac{1}{16}a^2\right)u_{\max}^2+\frac{1}{4}aK_2u_{\max}=1-\frac{3}{4}K_2^2\,. \tag{C·6}$$

The solution of (C·6) greater than c_2 is given by (4·30).

Appendix D

The Airy function $\mathrm{Ai}(z)$ is the solution of the differential equation

$$\mathrm{Ai}''(z)-z\,\mathrm{Ai}(z)=0\,. \tag{D·1}$$

Its asymptotic behavior at large $|z|$ is given by[41)]

$$\mathrm{Ai}(z)\cong\frac{1}{2}\pi^{-1/2}z^{-1/4}\exp\left[-\frac{2}{3}z^{3/2}\right], \tag{D·2}$$

where $|\arg z|<\pi$ and $|z|\gg1$. Then $\Psi_n(x)$ defined by (7·5) decays as $\exp[-\frac{2}{3}\sigma^{3/2}x^{3/2}]$ as $x\to\infty$ if $|\arg\sigma|<\pi/3$. Next we show (7·8). The orthogonality is evident from the symmetric nature of $\mathscr{H}$. Note the relation

$$\mathrm{Ai}(z)^2=\frac{d}{dz}[z\,\mathrm{Ai}(z)^2-\mathrm{Ai}'(z)^2]. \tag{D·3}$$

Integration of (D·3) from $-\alpha_n$ to ∞ yields

$$\int_{-\alpha_n}^{\infty}dz\,\mathrm{Ai}(z)^2=\mathrm{Ai}'(-\alpha_n)^2=1/\beta_n^2\,. \tag{D·4}$$

Thus Ψ_n is normalized if β_n is chosen as in (D·4).

In (7·10) we have assumed $|\arg(b_1\sigma^2)|<\pi/2$. This can be proved if we notice the relation

$$\arg(b_1\sigma^2)=\frac{1}{3}\arg b_1+\frac{2}{3}\arg(b_1-i/a). \tag{D·5}$$

Here $|\arg b_1|<\pi/2$ and $|\arg(b_1-i/a)|<\pi/2$ because the two complex members have a common positive real part.

We also show the positivity of the function $J_0(x)\equiv\mathrm{Im}(\Psi_0^*(d/dx)\Psi_0)$ for $-\pi/3<\arg\sigma<0$. From $\Psi_0''=\sigma^2(\sigma x-\alpha_0)\Psi_0$ we obtain

$$\frac{d}{dx}J_0=[(\mathrm{Im}\,\sigma^3)x-\alpha_0(\mathrm{Im}\,\sigma^2)]|\Psi_0|^2\,. \tag{D·6}$$

Since $-\pi/3<\arg\sigma<0$, we find $\mathrm{Im}\sigma^3<0$ and $\mathrm{Im}\,\sigma^2<0$. Thus J_0 increases from zero up to a maximum $J_0(\alpha_0\mathrm{Im}\,\sigma^3/\mathrm{Im}\,\sigma^2)$ in the region $0<x<\alpha_0\mathrm{Im}\,\sigma^3/\mathrm{Im}\,\sigma^2$ and decreases to zero in the region $\alpha_0\mathrm{Im}\,\sigma^3/\mathrm{Im}\,\sigma^2<x<\infty$. It cannot be zero for $x>0$.

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