Wronskian Structures of Solutions for Soliton Equations

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Solutions for the equations of the Kadomtsev-Petviashvili hierarchy are given in terms of Wronskian. The two-dimensional Toda lattice equation and the two-dimensional Toda molecule equation are investigated and their solutions are expressed in the form of two-directional Wronskian and double Wronskian. These solutions have common structure so that we can construct systems of these equations and their Wronskian solutions.

§ 1. Introduction

In recent twenty years many investigations have been done in the field of soliton theory and various nonlinear partial differential equations with soliton solutions have been found, including the Korteweg-de Vries (KdV) equation, the Kadomtsev-Petviashvili (KP) equation, the Boussinesq equation, the nonlinear Schrödinger (NLS) equation and so on. It has also been shown that there is a class of soliton equations with singular integral terms such as the Benjamin-Ono (BO) equation, the intermediate long wave (ILW) equation and so on. Besides such continuous systems, there also exist nonlinear discrete systems with soliton solutions such as the Toda lattice.

There are several ways to get soliton solutions for these equations. For example in the method of the inverse scattering transformation, soliton solutions are associated with the discrete spectra of the scattering operator and may be written in determinant form. In the direct method, they are obtained by a simple perturbational technique and represented as polynomials in exponentials. The Bäcklund transformation (BT) gives a way of constructing $N+1$-soliton solution from N-soliton solution.

A simple and useful expression of these solutions may be the one in terms of Wronskian. The Wronskian representation of the solutions was first given for the KdV and modified KdV equations.[1] This technique has been developed by Freeman and Nimmo[2)~5)] for other soliton equations such as the KP, Boussinesq, NLS and Davey-Stewartson (DS) equations. In this expression of the solutions, these nonlinear equations are derived from the Laplace expansion of the determinants which are equal to zero.

The general theory of τ function has been developed by Sato[6] and it has been shown that a hierarchy of equations (KP hierarchy), all of which have common solutions, can be obtained by introducing infinitely many independent variables. In the Sato theory, the equations of the KP hierarchy are nothing but the Plücker relations (PRs) and their solutions are expressed with τ functions. The simplest example of the equation of the KP hierarchy is the KP equation itself. The two-dimensional Toda lattice (2DTL) equation also belongs to the extension of the KP hierarchy.[7),8)]

In this paper the Wronskian technique for the KP hierarchy is explained in the case of the KP equation as an example and the N-soliton solution is given in terms of the Wronskian. We also present solutions of the 2DTL equation and the two-dimensional Toda molecule (2DTM) equation in the form of two-directional Wronskians which are determinants having Wronskian properties into two directions.[9)] We shall see that the structures of the solutions for these two types of the Toda equations are quite different. It is shown that the solution of the 2DTM equation can also be represented in terms of the double Wronskian. Then we discuss about the correspondence between the two-directional Wronskian and double Wronskian solutions of the 2DTM equation.

It is possible to construct coupled systems of these equations satisfied by the same Wronskian using the fact that the solutions of the KP equation (or other equations of the KP hierarchy) and those of the Toda equations have common Wronskian structure. Various nonlinear evolution equations can be derived from these systems of the KP and Toda equations by suitable variable transformations. As such examples the DS and NLS equations are briefly discussed.

§ 2. Equations of the KP hierarchy and their Wronskian solutions

As an example of equation of the KP hierarchy, we consider the KP equation,

$$\left(u_t-\frac{3}{2}uu_x-\frac{1}{4}u_{xxx}\right)_x-\frac{3}{4}u_{yy}=0\,, \tag{2·1}$$

where the subscripts indicate the partial differentiations with respect to the indicated variables. Equation (2·1) may be written in a bilinear form,

$$(D_x^4-4D_xD_t+3D_y^2)\tau\cdot\tau=0\,, \tag{2·2}$$

where $u=(2\log\tau)_{xx}$ and D is the bilinear differential operator defined by

$$D_x^4f\cdot g=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^4 f(x)g(x')\Big|_{x'=x}\,,$$

$$D_xD_tf\cdot g=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)f(x,t)g(x',t')\Big|_{x'=x,t'=t}$$

and so on.

Using the Wronskian technique developed by Freeman and Nimmo,[2)] we can show that solutions of Eq. (2·2) are written in a Wronskian form,

$$\tau=W(\varphi_1,\varphi_2,\cdots,\varphi_N)\,, \tag{2·3}$$

where

$$W(\varphi_1, \varphi_2, \cdots, \varphi_N)=\begin{vmatrix} \varphi_1 & \dfrac{\partial}{\partial x}\varphi_1 & \cdots & \left(\dfrac{\partial}{\partial x}\right)^{N-1}\varphi_1 \\ \varphi_2 & \dfrac{\partial}{\partial x}\varphi_2 & \cdots & \left(\dfrac{\partial}{\partial x}\right)^{N-1}\varphi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_N & \dfrac{\partial}{\partial x}\varphi_N & \cdots & \left(\dfrac{\partial}{\partial x}\right)^{N-1}\varphi_N \end{vmatrix}, \tag{2·4}$$

and φ_i satisfies

$$\frac{\partial}{\partial y}\varphi_i=\left(\frac{\partial}{\partial x}\right)^2\varphi_i\,, \tag{2·5a}$$

$$\frac{\partial}{\partial t}\varphi_i=\left(\frac{\partial}{\partial x}\right)^3\varphi_i\,. \tag{2·5b}$$

It is convenient to write $x=x_1$, $y=x_2$ and $t=x_3$. Suppressing the columns and the reference to the function φ, we shall write as

$$\begin{vmatrix} \varphi_1 & \cdots\cdot & \left(\dfrac{\partial}{\partial x}\right)^{N-1}\varphi_1 \\ \vdots & \ddots & \vdots \\ \varphi_N & \cdots\cdot & \left(\dfrac{\partial}{\partial x}\right)^{N-1}\varphi_N \end{vmatrix}=|0, \cdots\cdot, N-1|\,. \tag{2·6}$$

Then we observe that

$$\tau=|0, \cdots\cdot, N-1|\,, \tag{2·7a}$$

$$\frac{\partial}{\partial x_1}\tau=|0, \cdots, N-2, N|\,, \tag{2·7b}$$

$$\frac{1}{2}\left\{\left(\frac{\partial}{\partial x_1}\right)^2+\frac{\partial}{\partial x_2}\right\}\tau=|0, \cdots, N-2, N+1| \tag{2·7c}$$

and so on. These are expressed by using Sato's Maya diagram as follows:

$$\tau=\begin{array}{|c|c|c|c|c|c|c|c|c|c}\hline \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & \bigcirc & & & & \cdots \\ \hline \end{array}\,, \atop \begin{array}{cccccccccc} 0 & 1 & & N-3 & N-2 & N-1 & N & N+1 & N+2 & \end{array} \tag{2·8a}$$

$$\frac{\partial}{\partial x_1}\tau=\begin{array}{|c|c|c|c|c|c|c|c|c|c}\hline \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & & \bigcirc & & & \cdots \\ \hline \end{array}\,, \atop \begin{array}{cccccccccc} 0 & 1 & & N-3 & N-2 & N-1 & N & N+1 & N+2 & \end{array} \tag{2·8b}$$

$$\frac{1}{2}\left\{\left(\frac{\partial}{\partial x_1}\right)^2+\frac{\partial}{\partial x_2}\right\}\tau=\begin{array}{|c|c|c|c|c|c|c|c|c|c}\hline \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & & & \bigcirc & & \cdots \\ \hline \end{array}\,, \atop \begin{array}{cccccccccc} 0 & 1 & & N-3 & N-2 & N-1 & N & N+1 & N+2 & \end{array} \tag{2·8c}$$

and are also expressed in terms of the Young diagram,

$$\tau=\tau_{\phi}\,, \tag{2·9a}$$

$$\frac{\partial}{\partial x_1}\tau=\tau_{\ydiagram{1}}\,, \tag{2·9b}$$

$$\frac{1}{2}\left\{\left(\frac{\partial}{\partial x_1}\right)^2+\frac{\partial}{\partial x_2}\right\}\tau=\tau_{\ydiagram{2}}\,, \tag{2·9c}$$

and so on. In these notations, Eq. (2·2) is rewritten as

$$\tau_{\phi}\tau_{\ydiagram{2,2}}-\tau_{\ydiagram{1}}\tau_{\ydiagram{2,1}}+\tau_{\ydiagram{2}}\tau_{\ydiagram{1,1}}=0\,, \tag{2·10}$$

which is nothing but the PR. Especially Eq. (2·3) gives N-soliton solution by taking

$$\varphi_i=\exp\,\eta_i+\exp\,\xi_i \tag{2·11}$$

with

$$\eta_i=p_ix_1+p_i^2x_2+p_i^3x_3+\eta_{i0}\,, \tag{2·12a}$$

$$\xi_i=q_ix_1+q_i^2x_2+q_i^3x_3+\xi_{i0}\,. \tag{2·12b}$$

The other equations of the KP hierarchy are also reduced to the PRs and common solutions of all the equations are written in the form of Wronskian,

$$\tau=W(\varphi_1,\varphi_2,\cdots,\varphi_N)\,, \tag{2·13}$$

where φ_i depends on infinitely many independent variables $x_1, x_2, x_3, \cdots$, and satisfies the linear differential equations,

$$\frac{\partial}{\partial x_k}\varphi_i=\left(\frac{\partial}{\partial x_1}\right)^k\varphi_i\,. \qquad (k=1,2,3,\cdots) \tag{2·14}$$

For example, the bilinear equation,

$$(D_1^2+D_2)\tau\cdot\tau'=0\,, \tag{2·15}$$

which is often referred as a BT of the KP equation, is satisfied by the Wronskian,

$$\tau=W(\varphi_1,\varphi_2,\cdots,\varphi_N)=|0,1,\cdots,N-1|\,, \tag{2·16a}$$

$$\tau'=W\left(\frac{\partial}{\partial x_1}\varphi_1,\frac{\partial}{\partial x_1}\varphi_2,\cdots,\frac{\partial}{\partial x_1}\varphi_N\right)=|1,2,\cdots,N|\,. \tag{2·16b}$$

In Eq. (2·15) and hereafter, D_k means the bilinear differential operator with respect to x_k. In addition to Eqs. (2·16a) and (2·16b) there is another Wronskian solution of Eq. (2·15),

$$\tau=W(\varphi_1,\varphi_2,\cdots,\varphi_N)\,, \tag{2·17a}$$

$$\tau'=W(\varphi_1,\varphi_2,\cdots,\varphi_N,\varphi_{N+1})\,. \tag{2·17b}$$

The fact that there are these two types of solutions for Eq. (2·15) is important to construct the so-called dark and bright soliton solutions for equations of the NLS class.

§ 3. The two-dimensional Toda lattice equation

The 2DTL equation,

$$\frac{\partial^2}{\partial x\partial y}\log(1+V_n)=V_{n+1}-2V_n+V_{n-1}\,, \tag{3·1}$$

describes a nonlinear coupled oscillator with exponential type potential. By writing

$$V_n=\frac{\partial^2}{\partial x\partial y}\log\tau_n\,, \tag{3·2}$$

Eq. (3·1) reduces to the bilinear form,

$$D_xD_y\tau_n\cdot\tau_n-2(\tau_{n+1}\tau_{n-1}-\tau_n{}^2)=0\,. \tag{3·3}$$

We show that solutions of Eq. (3·3) are given by

$$\tau_n=W\left(\left(\frac{\partial}{\partial x}\right)^n\varphi_1,\left(\frac{\partial}{\partial x}\right)^n\varphi_2,\cdots,\left(\frac{\partial}{\partial x}\right)^n\varphi_N\right), \tag{3·4}$$

where φ_i satisfies

$$\frac{\partial^2}{\partial x\partial y}\varphi_i=-\varphi_i\,. \tag{3·5}$$

Equation (3·4) is a Wronskian on x, that is, each column is the derivative with respect to x of the next left column. It is also a Wronskian on $-y$ in the inverse direction, that is, each column is the derivative with respect to $-y$ of the next right column. Thus we may call it a two-directional Wronskian. It is natural to write $x=x_1$ and $-y=x_{-1}$.

We observe that τ_n, its derivatives and $\tau_{n\pm1}$ are given by

$$\tau_n=|n,\cdots\cdot,n+N-1|\,, \tag{3·6a}$$

$$\frac{\partial\tau_n}{\partial x_1}=|n,\cdots,n+N-2,n+N|\,, \tag{3·6b}$$

$$\frac{\partial\tau_n}{\partial x_{-1}}=|n-1,n+1,\cdots,n+N-1|\,, \tag{3·6c}$$

$$\frac{\partial^2\tau_n}{\partial x_1\partial x_{-1}}=|n-1,n+1,\cdot\cdot,n+N-2,n+N|+|n,\cdots\cdot,n+N-1|\,, \tag{3·6d}$$

$$\tau_{n+1}=|n+1,\cdots\cdot,n+N|\,, \tag{3·6e}$$

$$\tau_{n-1}=|n-1,\cdots\cdot,n+N-2|\,, \tag{3·6f}$$

which are denoted in the Maya diagram as

$$\tau_n = \cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & \\ \hline \multicolumn{1}{c}{n-2} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-2} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{n+N} \end{array}\ \underset{n+N+1}{\boxed{}} \cdots, \tag{3·7a}$$

$$\frac{\partial\tau_n}{\partial x_1} = \cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & \bigcirc & \bigcirc & \cdots & \bigcirc & & \bigcirc \\ \hline \multicolumn{1}{c}{n-2} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-2} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{n+N} \end{array}\ \underset{n+N+1}{\boxed{}} \cdots, \tag{3·7b}$$

$$\frac{\partial\tau_n}{\partial x_{-1}} = \cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline & \bigcirc & & \bigcirc & \cdots & \bigcirc & \bigcirc & \\ \hline \multicolumn{1}{c}{n-2} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-2} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{n+N} \end{array}\ \underset{n+N+1}{\boxed{}} \cdots, \tag{3·7c}$$

$$\begin{aligned}\frac{\partial^2\tau_n}{\partial x_1\partial x_{-1}} = &\cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline & \bigcirc & & \bigcirc & \cdots & \bigcirc & & \bigcirc \\ \hline \multicolumn{1}{c}{n-2} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-2} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{n+N} \end{array}\ \underset{n+N+1}{\boxed{}} \cdots \\ &+ \cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & \\ \hline \multicolumn{1}{c}{n-2} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-2} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{n+N} \end{array}\ \underset{n+N+1}{\boxed{}} \cdots,\end{aligned} \tag{3·7d}$$

$$\tau_{n+1} = \cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & \bigcirc & \cdots & \bigcirc & \bigcirc & \bigcirc \\ \hline \multicolumn{1}{c}{n-2} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-2} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{n+N} \end{array}\ \underset{n+N+1}{\boxed{}} \cdots, \tag{3·7e}$$

$$\tau_{n-1} = \cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline & \bigcirc & \bigcirc & \bigcirc & \cdots & \bigcirc & & \\ \hline \multicolumn{1}{c}{n-2} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-2} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{n+N} \end{array}\ \underset{n+N+1}{\boxed{}} \cdots. \tag{3·7f}$$

If we rearrange the columns of the determinants in the alternate order such as $\cdots$, $n+1$, $n+N-1$, n, $n+N$, $n-1$, $n+N+1$, $\cdots$, then the Maya diagram for Eqs. (3·7a)~(3·7f) may be written as follows:

$$\tau_n = \begin{array}{|c|c|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{} \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & \bigcirc & & & \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N+1} \end{array} \cdots, \tag{3·8a}$$

$$\frac{\partial\tau_n}{\partial x_1} = -\begin{array}{|c|c|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{} \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & & \bigcirc & \bigcirc & & \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N+1} \end{array} \cdots, \tag{3·8b}$$

$$\frac{\partial\tau_n}{\partial x_{-1}} = \begin{array}{|c|c|c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n-1} & \multicolumn{1}{c}{} \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & & & \bigcirc & \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N-1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{n+N+1} \end{array} \cdots, \tag{3·8c}$$

$$\frac{\partial^2 \tau_n}{\partial x_1 \partial x_{-1}} = \begin{array}{cccccccccc} & & & n+1 & & n & & n-1 & & \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & & & \bigcirc & \bigcirc & & \cdots \\ \hline & & & & n+N-1 & & n+N & & n+N+1 & \end{array}$$

$$+\begin{array}{cccccccccc} & & & n+1 & & n & & n-1 & & \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & \bigcirc & & & & \cdots \\ \hline & & & & n+N-1 & & n+N & & n+N+1 & \end{array}, \tag{3·8d}$$

$$\tau_{n+1} = (-)^{N-1} \begin{array}{cccccccccc} & & & n+1 & & n & & n-1 & & \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & \bigcirc & & \bigcirc & & & \cdots \\ \hline & & & & n+N-1 & & n+N & & n+N+1 & \end{array}, \tag{3·8e}$$

$$\tau_{n-1} = (-)^{N} \begin{array}{cccccccccc} & & & n+1 & & n & & n-1 & & \\ \hline \bigcirc & \bigcirc & \cdots & \bigcirc & & \bigcirc & & \bigcirc & & \cdots \\ \hline & & & & n+N-1 & & n+N & & n+N+1 & \end{array}, \tag{3·8f}$$

where the sign is changed by the rearrangement. In Eqs. (3·8a)~(3·8f), the quantities are normalized by multiplying them by $+1$ or -1 in a manner that the sign of τ_n is $+$. We can also express them in terms of the Young diagram as follows:

$$\tau_n = \tau_\phi , \tag{3·9a}$$

$$\frac{\partial \tau_n}{\partial x_1} = -\tau_{\yng(1,1)}, \tag{3·9b}$$

$$\frac{\partial \tau_n}{\partial x_{-1}} = \tau_{\yng(2)}, \tag{3·9c}$$

$$\frac{\partial^2 \tau_n}{\partial x_1 \partial x_{-1}} = \tau_{\yng(2,2)} + \tau_\phi , \tag{3·9d}$$

$$\tau_{n+1} = (-)^{N-1} \tau_{\yng(1)}, \tag{3·9e}$$

$$\tau_{n-1} = (-)^{N} \tau_{\yng(2,1)}. \tag{3·9f}$$

In this notation Eq. (3·3) is written as

$$\tau_\phi \tau_{\yng(2,2)} - \tau_{\yng(1)} \tau_{\yng(2,1)} + \tau_{\yng(2)} \tau_{\yng(1,1)} = 0 , \tag{3·10}$$

which is again the PR. Therefore it is proved that the Wronskian (3·4) actually satisfies the 2DTL equation (3·3). If φ_i is chosen such that

$$\varphi_i = \exp \eta_i + \exp \xi_i \tag{3·11}$$

with

$$\eta_i = p_i^{-1} x_{-1} + p_i x_1 + \eta_{i0} , \tag{3·12a}$$

$$\xi_i = q_i^{-1} x_{-1} + q_i x_1 + \xi_{i0} , \tag{3·12b}$$

then Eq. (3·4) gives an N-soliton solution of the 2DTL equation.

If we introduce independent variables $x_2, x_3, \cdots, x_{-2}, x_{-3}, \cdots$, the 2DTL equation can be coupled with any equation of the KP hierarchy and its solution is also given in two-directional Wronskian. For example a coupled system of the 2DTL equation and the BT of the KP equation,

$$\begin{cases} D_1D_{-1}\tau_n\cdot\tau_n+2(\tau_{n+1}\tau_{n-1}-\tau_n^2)=0\,, & (3\cdot13\text{a})\\ (D_1^2+D_2)\tau_n\cdot\tau_{n+1}=0\,, & (3\cdot13\text{b})\\ (D_{-1}^2-D_{-2})\tau_n\cdot\tau_{n+1}=0 & (3\cdot13\text{c})\end{cases}$$

is satisfied by the solution (3·4) with

$$\frac{\partial}{\partial x_k}\varphi_i=\left(\frac{\partial}{\partial x_1}\right)^k\varphi_i\,. \qquad (k=-2,\,-1,\,1,\,2) \tag{3·14}$$

Here the solution of the type of Eqs. (2·16a) and (2·16b) is used for the BTs (3·13b) and (3·13c).

§ 4. The two-dimensional Toda molecule equation

We have the 2DTM equation,

$$\frac{\partial^2}{\partial x\partial y}\log V_n=V_{n+1}-2V_n+V_{n-1}\,, \tag{4·1}$$

which is rewritten in the bilinear form,

$$D_xD_y\tau_n\cdot\tau_n-2\tau_{n+1}\tau_{n-1}=0\,, \tag{4·2}$$

through the variable transformation (3·2). We impose boundary conditions,

$$V_0=V_M=0\,, \tag{4·3}$$

which are satisfied with

$$\tau_0=\Phi_1(x)\Psi_1(y)\,, \tag{4·4a}$$

$$\tau_M=\Phi_2(x)\Psi_2(y)\,, \tag{4·4b}$$

where $\Phi_l(x)$ and $\Psi_l(y)$ $(l=1, 2)$ are arbitrary functions of x and y, respectively. The main difference between the TL and TM equations is that the former describes an infinite or periodic lattice system and the latter a finite or semi-infinite lattice system, respectively.

Leznov and Saveliev[10] have obtained solutions of Eq. (4·2) with the boundary conditions (4·4a) and (4·4b). The solutions are expressed as

$$\tau_n=\begin{vmatrix} \varphi & \frac{\partial}{\partial x}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-1}\varphi \\ \frac{\partial}{\partial y}\varphi & \frac{\partial}{\partial x}\frac{\partial}{\partial y}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-1}\frac{\partial}{\partial y}\varphi \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-1}\left(\frac{\partial}{\partial y}\right)^{n-1}\varphi \end{vmatrix} \tag{4·5}$$

with

$$\varphi=\sum_{j=1}^{M}\phi_j(x)\psi_j(y)\,, \tag{4·6}$$

where $\phi_j(x)$ and $\psi_j(y)$ are arbitrary functions of x and y, respectively. The solution (4·5) is a Wronskian with respect to x in the horizontal direction and with respect to y in the vertical direction, respectively. This type of two-directional Wronskian satisfies the 2DTM equation (4·2) with the boundary condition (4·4a), which is proved by using the Laplace expansion of a determinant.

Consider the following identity for $(2n+2)\times(2n+2)$ determinant:

$$\left|\begin{array}{ccc:c:ccc:cc}
\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-1}\varphi & \left(\frac{\partial}{\partial x}\right)^{n}\varphi & & & & 0 & 0 \\
\vdots & & \vdots & \vdots & & & & \vdots & \vdots \\
\left(\frac{\partial}{\partial y}\right)^{n-2}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-1}\left(\frac{\partial}{\partial y}\right)^{n-2}\varphi & \left(\frac{\partial}{\partial x}\right)^{n}\left(\frac{\partial}{\partial y}\right)^{n-2}\varphi & & 0 & & 0 & 0 \\
\left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-1}\left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & \left(\frac{\partial}{\partial x}\right)^{n}\left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & & & & 1 & 0 \\
\left(\frac{\partial}{\partial y}\right)^{n}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-1}\left(\frac{\partial}{\partial y}\right)^{n}\varphi & \left(\frac{\partial}{\partial x}\right)^{n}\left(\frac{\partial}{\partial y}\right)^{n}\varphi & & & & & 1 \\
\hdashline
 & & & \left(\frac{\partial}{\partial x}\right)^{n}\varphi & \varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-2}\varphi & 0 & 0 \\
 & & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & 0 & & \left(\frac{\partial}{\partial x}\right)^{n}\left(\frac{\partial}{\partial y}\right)^{n-2}\varphi & \left(\frac{\partial}{\partial y}\right)^{n-2}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-2}\left(\frac{\partial}{\partial y}\right)^{n-2}\varphi & 0 & 0 \\
 & & & \left(\frac{\partial}{\partial x}\right)^{n}\left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & \left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-2}\left(\frac{\partial}{\partial y}\right)^{n-1}\varphi & 1 & 0 \\
 & & & \left(\frac{\partial}{\partial x}\right)^{n}\left(\frac{\partial}{\partial y}\right)^{n}\varphi & \left(\frac{\partial}{\partial y}\right)^{n}\varphi & \cdots & \left(\frac{\partial}{\partial x}\right)^{n-2}\left(\frac{\partial}{\partial y}\right)^{n}\varphi & 0 & 1
\end{array}\right| = 0. \tag{4·7}$$

Applying a Laplace expansion in $(n+1)\times(n+1)$ minors to the left-hand side of Eq. (4·7), we get

$$\tau_{n+1}\tau_{n-1}+\frac{\partial\tau_n}{\partial y}\frac{\partial\tau_n}{\partial x}-\tau_n\frac{\partial^2\tau_n}{\partial x\partial y}=0\,, \tag{4·8}$$

where τ_n is given by Eq. (4·5). Equation (4·8) coincides with Eq. (4·2). Therefore we have proved that the Wronskian (4·5) satisfies Eq. (4·2). It is also possible to prove this by using the Jacobi formula for the determinant.[11] The boundary condition (4·4b) is obeyed by the solution (4·5) if φ is taken to be the form of Eq. (4·6).

If we introduce extra structure with an infinite number of independent variables $x_2, x_3, \cdots, y_2, y_3, \cdots$, such that

$$\frac{\partial}{\partial x_k}\varphi=\left(\frac{\partial}{\partial x_1}\right)^k\varphi\,, \tag{4·9a}$$

$$\frac{\partial}{\partial y_k}\varphi=\left(\frac{\partial}{\partial y_1}\right)^k\varphi\,, \tag{4·9b}$$

where $x_1=x$ and $y_1=y$, then it is possible to make τ_n satisfy the 2DTM equation and all equations of the KP hierarchy simultaneously. For example we consider a coupled system of the 2DTM equation and the BT of the KP equation,

$$\begin{cases} D_1D_1'\tau_n\cdot\tau_n-2\tau_{n+1}\tau_{n-1}=0\,, & (4\cdot10a)\\ (D_1^2+D_2)\tau_n\cdot\tau_{n+1}=0\,, & (4\cdot10b)\\ (D_1'^2+D_2')\tau_n\cdot\tau_{n+1}=0\,, & (4\cdot10c)\end{cases}$$

where D_k' means the bilinear differential operator with respect to y_k. The Wronskian solution of Eqs. (4·10a)~(4·10c) is given by Eq. (4·5) with the conditions (4·9a) and (4·9b) for $k=2$. In this case the solution for the BTs (4·10b) and (4·10c) is the one in the form of Eqs. (2·17a) and (2·17b).

We here remark about a transformation between the 2DTL and 2DTM equations. Let τ_n be a solution of the 2DTL equation (3·3), then, for instance, $\tau_n'=e^{xy}\tau_n$ satisfies the 2DTM equation (4·2). However if the boundary conditions (4·4a) and (4·4b) are imposed on the 2DTM equation, their solutions are not always simply transformed. Actually the solutions (3·4) and (4·5) for the 2DTL and 2DTM equations do not correspond each other. We note that the size of Wronskian relates to the number of solitons for Eq. (3·3) and to the lattice site number for Eq. (4·2). The boundary condition on the Toda equation makes a substantial difference in the structure of the solutions.

§ 5. Double Wronskian representation of the solution for the 2DTM equation

Solutions for the 2DTM equation (4·2) with the boundary conditions (4·4a) and (4·4b) are also expressed in terms of double Wronskian,

$$\tau_n=\left|\begin{array}{cccc:cccc} \varphi_1 & \frac{d}{dx}\varphi_1 & \cdot\cdot & \left(\frac{d}{dx}\right)^{n-1}\varphi_1 & \psi_1 & \frac{d}{dy}\psi_1 & \cdot\cdot & \left(\frac{d}{dy}\right)^{M-n-1}\psi_1\\ \varphi_2 & \frac{d}{dx}\varphi_2 & \cdot\cdot & \left(\frac{d}{dx}\right)^{n-1}\varphi_2 & \psi_2 & \frac{d}{dy}\psi_2 & \cdot\cdot & \left(\frac{d}{dy}\right)^{M-n-1}\psi_2\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \varphi_M & \frac{d}{dx}\varphi_M & \cdot\cdot & \left(\frac{d}{dx}\right)^{n-1}\varphi_M & \psi_M & \frac{d}{dy}\psi_M & \cdot\cdot & \left(\frac{d}{dy}\right)^{M-n-1}\psi_M \end{array}\right|, \tag{5·1}$$

where $\varphi_i(x)$ and $\psi_i(y)$ are arbitrary functions of x and y, respectively. After Freeman we shall use the notation for the above double Wronskian as

$$\tau_n=|0,1,\cdot\cdot,n-1;0,1,\cdot\cdot,M-n-1|\,. \tag{5·2}$$

We see that the derivatives of τ_n and $\tau_{n\pm1}$ are given by

$$\frac{\partial\tau_n}{\partial x}=|0,\cdot\cdot,n-2,n;0,\cdots,M-n-1|\,, \tag{5·3a}$$

$$\frac{\partial\tau_n}{\partial y}=|0,\cdots,n-1;0,\cdot\cdot,M-n-2,M-n|\,, \tag{5·3b}$$

$$\frac{\partial^2\tau_n}{\partial x\partial y}=|0,\cdot\cdot,n-2,n;0,\cdot\cdot,M-n-2,M-n|\,, \tag{5·3c}$$

$$\tau_{n+1}=|0,\cdot\cdot\cdot\cdot,n;0,\cdot\cdot,M-n-2|\,, \tag{5·3d}$$

$$\tau_{n-1}=|0,\cdot\cdot,n-2;0,\cdot\cdot\cdot\cdot,M-n|\,. \tag{5·3e}$$

These are also written by using the double Maya diagram as follows:

$$\tau_n=\begin{array}{c|c|c|c|c|cc} & n-2 & n-1 & n & n+1 & & \\ \hline \cdots & \bigcirc & \bigcirc & & & \cdots & x \\ \hline \cdots & \bigcirc & \bigcirc & & & \cdots & y \\ \hline & M-n-2 & M-n-1 & M-n & M-n+1 & & \end{array}\,, \tag{5·4a}$$

$$\frac{\partial\tau_n}{\partial x}=\begin{array}{c|c|c|c|c|cc} & n-2 & n-1 & n & n+1 & & \\ \hline \cdots & \bigcirc & & \bigcirc & & \cdots & x \\ \hline \cdots & \bigcirc & \bigcirc & & & \cdots & y \\ \hline & M-n-2 & M-n-1 & M-n & M-n+1 & & \end{array}\,, \tag{5·4b}$$

$$\frac{\partial\tau_n}{\partial y}=\begin{array}{c|c|c|c|c|cc} & n-2 & n-1 & n & n+1 & & \\ \hline \cdots & \bigcirc & \bigcirc & & & \cdots & x \\ \hline \cdots & \bigcirc & & \bigcirc & & \cdots & y \\ \hline & M-n-2 & M-n-1 & M-n & M-n+1 & & \end{array}\,, \tag{5·4c}$$

$$\frac{\partial^2\tau_n}{\partial x\partial y}=\begin{array}{c|c|c|c|c|cc} & n-2 & n-1 & n & n+1 & & \\ \hline \cdots & \bigcirc & & \bigcirc & & \cdots & x \\ \hline \cdots & \bigcirc & & \bigcirc & & \cdots & y \\ \hline & M-n-2 & M-n-1 & M-n & M-n+1 & & \end{array}\,, \tag{5·4d}$$

$$\tau_{n+1}=\begin{array}{c|c|c|c|c|cc} & n-2 & n-1 & n & n+1 & & \\ \hline \cdots & \bigcirc & \bigcirc & \bigcirc & & \cdots & x \\ \hline \cdots & \bigcirc & & & & \cdots & y \\ \hline & M-n-2 & M-n-1 & M-n & M-n+1 & & \end{array}\,, \tag{5·4e}$$

$$\tau_{n-1}=\begin{array}{c|c|c|c|c|cc} & n-2 & n-1 & n & n+1 & & \\ \hline \cdots & \bigcirc & & & & \cdots & x \\ \hline \cdots & \bigcirc & \bigcirc & \bigcirc & & \cdots & y \\ \hline & M-n-2 & M-n-1 & M-n & M-n+1 & & \end{array}\,. \tag{5·4f}$$

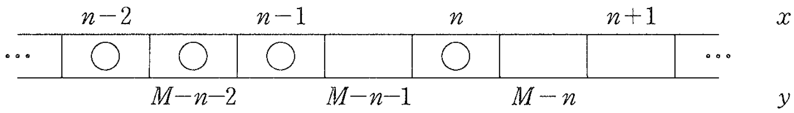
Rearranging the columns of the determinants in the alternate order of $\varphi(x)$ and $\psi(y)$, we can express them in terms of the single Maya diagram as

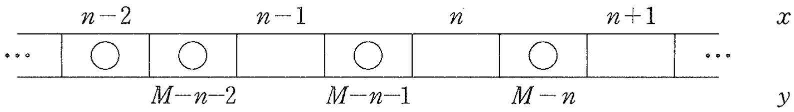
$$\tau_n=\begin{array}{c|c|c|c|c|c|c|c|c|cc} & n-2 & & n-1 & & n & & n+1 & & & x \\ \hline \cdots & \bigcirc & \bigcirc & \bigcirc & \bigcirc & & & & & \cdots & \\ \hline & & M-n-2 & & M-n-1 & & M-n & & & & y \end{array}\,, \tag{5·5a}$$

$$\frac{\partial\tau_n}{\partial x}=-\begin{array}{c|c|c|c|c|c|c|c|c|cc} & n-2 & & n-1 & & n & & n+1 & & & x \\ \hline \cdots & \bigcirc & \bigcirc & & \bigcirc & \bigcirc & & & & \cdots & \\ \hline & & M-n-2 & & M-n-1 & & M-n & & & & y \end{array}\,, \tag{5·5b}$$

$$\frac{\partial\tau_n}{\partial y}=\begin{array}{c|c|c|c|c|c|c|c|c|cc} & n-2 & & n-1 & & n & & n+1 & & & x \\ \hline \cdots & \bigcirc & \bigcirc & \bigcirc & & & \bigcirc & & & \cdots & \\ \hline & & M-n-2 & & M-n-1 & & M-n & & & & y \end{array}\,, \tag{5·5c}$$

$$\frac{\partial^2\tau_n}{\partial x\partial y}=\begin{array}{c|c|c|c|c|c|c|c|c|cc} & n-2 & & n-1 & & n & & n+1 & & & x \\ \hline \cdots & \bigcirc & \bigcirc & & & \bigcirc & \bigcirc & & & \cdots & \\ \hline & & M-n-2 & & M-n-1 & & M-n & & & & y \end{array}\,, \tag{5·5d}$$

$\tau_{n+1}=(-)^{M-n-1}$  , (5·5e)

$\tau_{n-1}=(-)^{M-n-1}$  , (5·5f)

where the sign of τ_n is normalized to be $+$ by multiplication of these quantities by $+1$ or -1. Using the Young diagram we rewrite Eqs. (5·5a)~(5·5f) as follows:

$$\tau_n=\tau_\phi\,, \tag{5·6a}$$

$$\frac{\partial\tau_n}{\partial x}=-\tau_{\substack{\square\\\square}}\,, \tag{5·6b}$$

$$\frac{\partial\tau_n}{\partial y}=\tau_{\square\square}\,, \tag{5·6c}$$

$$\frac{\partial^2\tau_n}{\partial x\partial y}=\tau_{\substack{\square\square\\\square\square}}\,, \tag{5·6d}$$

$$\tau_{n+1}=(-)^{M-n-1}\tau_{\square}\,, \tag{5·6e}$$

$$\tau_{n-1}=(-)^{M-n-1}\tau_{\substack{\square\square\\\square}}\,. \tag{5·6f}$$

Hence the 2DTM equation (4·2) is again reduced to the PR,

$$\tau_\phi\tau_{\substack{\square\square\\\square\square}}-\tau_{\square}\tau_{\substack{\square\square\\\square}}+\tau_{\square\square}\tau_{\substack{\square\\\square}}=0\,, \tag{5·7}$$

and we have proved that the double Wronskian (5·1) satisfies Eq. (4·2).

Comparing τ_1/τ_0 in the forms of two-directional Wronskian and double Wronskian, we can easily see the correspondence between these two representations. Namely, φ in Eq. (4·5) can be expressed in terms of φ_i and ψ_i in Eq. (5·1) as

$$\varphi=\frac{\begin{vmatrix} \varphi_1 & \psi_1 & \dfrac{d}{dy}\psi_1 & \cdots & \left(\dfrac{d}{dy}\right)^{M-2}\psi_1 \\ \varphi_2 & \psi_2 & \dfrac{d}{dy}\psi_2 & \cdots & \left(\dfrac{d}{dy}\right)^{M-2}\psi_2 \\ \vdots & \vdots & \vdots & & \vdots \\ \varphi_M & \psi_M & \dfrac{d}{dy}\psi_M & \cdots & \left(\dfrac{d}{dy}\right)^{M-2}\psi_M \end{vmatrix}}{\begin{vmatrix} \psi_1 & \dfrac{d}{dy}\psi_1 & \cdots & \left(\dfrac{d}{dy}\right)^{M-1}\psi_1 \\ \psi_2 & \dfrac{d}{dy}\psi_2 & \cdots & \left(\dfrac{d}{dy}\right)^{M-1}\psi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_M & \dfrac{d}{dy}\psi_M & \cdots & \left(\dfrac{d}{dy}\right)^{M-1}\psi_M \end{vmatrix}}\,, \tag{5·8}$$

where we use the arbitrariness of Φ_l and Ψ_l in the boundary conditions (4·4a) and (4·4b).

It is possible to introduce variables $x_2, x_3, \cdots, y_2, y_3, \cdots$ so that the solutions of various coupled systems of the 2DTM equation and the equations of the KP hierarchy can be constructed.

§ 6. Conclusion

The equations of the KP hierarchy have common solutions in the form of Wronskian (2·13) whose elements satisfy the linear differential equations (2·14) with respect to infinitely many independent variables. These equations are reduced to the bilinear identities of the Wronskian which are nothing but the PRs. The 2DTL equation in the bilinear form (3·3) is also reduced to the PR (3·10) and has the two-directional Wronskian solution which is the forward and backward directional one. The solution of the 2DTM equation (4·2) is written in terms of the two-directional Wronskian which is the horizontal and vertical directional one and also written in terms of the double Wronskian. An important point is that the structures of these Wronskian, multi-directional Wronskian and multiple Wronskian are quite similar. Using this fact, we can construct coupled systems of the equations of the KP hierarchy and the Toda equations and express their solutions in the form of Wronskian with multi-structure.

From these coupled systems it is possible to derive various nonlinear evolution equations. For example the DS equation can be obtained from Eqs. (3·13a)~(3·13c) or from Eqs. (4·10a)~(4·10c) and their soliton solutions can be expressed in the form of Wronskian with multi-structure. In the case of Eqs. (3·13a)~(3·13c), letting

$$Q=(2\log\tau_0)_{xx}\,, \tag{6·1}$$

$$A=\frac{\tau_{-1}}{\tau_0}e^{i(kx+ly-\omega t)} \tag{6·2}$$

with

$$\tau_0:\text{real}\,, \tag{6·3}$$

$$\tau_1=\tau_{-1}{}^*\,, \tag{6·4}$$

where * indicates complex conjugate, and changing independent variables suitably, we get the DS equation with the dark soliton solution,[5]

$$\begin{cases} iA_t-A_{xx}+A_{yy}=A|A|^2+2QA\,, & (6·5\text{a})\\ Q_{xx}+Q_{yy}=-(|A|^2)_{xx}\,. & (6·5\text{b})\end{cases}$$

On the other hand in the case of Eqs. (4·10a)~(4·10c), choosing M even number, taking

$$\tau_N:\ \text{real}\,, \tag{6·6}$$

$$\tau_{N+1}=-\tau_{N-1}^*\,, \tag{6·7}$$

and writing

$$Q=-(2\log\tau_N)_{xx}\,, \tag{6·8}$$

$$A=\frac{\tau_{N-1}}{\tau_N}\,, \tag{6·9}$$

where $N=M/2$, we obtain the DS equation with the bright soliton solution,

$$\begin{cases} iA_t-A_{xx}+A_{yy}=-(A|A|^2+2QA)\,, & (6·10a) \\ Q_{xx}+Q_{yy}=-(|A|^2)_{xx}\,, & (6·10b) \end{cases}$$

through a suitable independent variable transformation. The NLS equation,

$$iu_t+u_{xx}\pm 2|u|^2u=0\,, \tag{6·11}$$

can be derived from Eqs. (6·5a) and (6·5b) or from Eqs. (6·10a) and (6·10b) by using a suitable reduction. Many other equations such as the BO, ILW and Mel'nikov equations can be obtained in the similar way.

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