Quantized Theta-Functions

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Introduction

0.1. *Quantized linear group*

The quantum special linear group $SL_q(n)$ over a field k where $q\in k^*$ is a quantization parameter is represented by the Hopf algebra

$$F[SL_q(n)]=k\langle z_i^j\rangle/R;\quad i,j=1,\cdots,n\geq 2\,,$$

where R denotes the ideal in the free association algebra $k\langle z_i^j\rangle$, $z_i^j=z_{i,q}^j$, generated by the following relations: For all pairs $i<j$, $k<l$ put $a=z_i^k$; $b=z_i^l$; $c=z_j^k$; $d=z_j^l$. Then

$$ab=q^{-1}ba;\quad ac=q^{-1}ca;\quad cd=q^{-1}dc;\quad bd=q^{-1}db;$$

$$bc=cb;\quad ad-da=(q^{-1}-q)bc;$$

by

$$\mathrm{DET}_q(z_i^j)\equiv\sum_{s\in S_n}(-q)^{-l(s)}z_1^{s(1)}\cdots z_n^{s(n)}=1\,.$$

The comultiplication is represented by the usual formula $\Delta(z_i^k)=\sum_{j=1}^n z_i^j\otimes z_j^k$ and the antipode is given by a quantized version of Kramer's rule. For $q=1$ we obtain the polynomial function ring of the usual $SL(n)$.

A very large part of the theory of classical Lie and algebraic groups can be extended in this way by deforming it into a growing domain of non-commutative geometry: cf., Refs. 1) and 2) for basic constructions and results.

We want to remind also that the theory acquires some specific properties when the quantization parameter q is a root of unity. In particular, if $q^l=1$, $l\equiv 1$ mod 2, q primitive, there is a non-commutative Frobenius morphism, defined over $\boldsymbol{Z}[q,q^{-1}]$(!):

$$\Phi_l\colon SL_q(n)\to SL(n)\colon\ \Phi_l^*(z_{i,1}^j)=(z_{i,q}^j)^l\,.$$

The ring $A[SL_q(n)]$ becomes finite over its center, the category of representations ceases to be semi-simple and acquires some properties akin to those of the finite characteristic case.

For some reason, precisely these values of q are important in two-dimensional conformal field theory.

0.2. *Problem of quantization of abelian varieties*

It is natural to expect that not only linear algebraic groups but also projective ones, i.e., abelian varieties, can be quantized in a similar manner. One obstacle, however, is that even in the classical case there are no natural Hopf algebras of functions on an abelian variety A. Of course, one can consider the graded algebra $F = F(A, L) = \oplus_{n=0}^{\infty} \Gamma(A, L^n)$, where L is an ample invertible sheaf on A. However, it has no natural comultiplication. In fact, if $m: A\times A \to A$ denotes the addition map, it defines $m^*: F(A, L) \to F(A\times A, m^*(L))$, but $m^*(L) \not\simeq p_1^*(L)\otimes p_2^*(L)$ so that $F(A\times A, m^*(L)) \not\simeq F(A, L)\otimes F(A, L)$.

Mumford in Ref. 3) found a clever remedy. Namely, if $i^*(L)\simeq L$ where $i: A\to A$, $i(x)=-x$, then instead of $m: A\times A\to A$ one should consider $M: A\times A\to A\times A$, $M(x, y)=(x+y, x-y)$, and try to describe $M^*: F(A\times A, p_1^*L\otimes p_2^*L)\to F(A\times A, M^*(p_1^*L\otimes p_2^*L))\simeq F(A\times A, p_1^*L^2\otimes p_2^*L^2)$. Mumford succeeded to do this and obtained a very detailed picture (at least for appropriate L's). Unfortunately, to my knowledge, there is no axiomatization of the algebraic structure he studied, which would be similar to that of the Hopf algebras. Therefore, paradoxically, we do not understand what properties of the ring $F(A, L)$ should be conserved (or lost) after a quantum deformation.

0.3. *Quantized theta-functions*

The main idea of this note is to construct some quantum deformations not of the ring $F(A, L)$ itself but of its separate homogeneous components $F(A, L)_n = \Gamma(A, L^n)$, consisting classically of theta-functions of various types. We make non-commuting the basic Fourier harmonics $e^{2\pi i n t z}$ (instead of matrix coefficients z_i^j in the $SL_q(n)$-case), and the degree of the non-commutativity is measured by an appropriate quantization parameter.

To be more precise, in the classical theory we start (for $K=\mathbb{C}$) with the universal covering map $\pi: \mathbb{C}^n\to A$, trivialize $\pi^*(L)$ and identify $\Gamma(A, L)$ with a certain subspace of functions on $\mathbb{C}^n$ behaving in a quasi-periodic way with respect to the lattice Ker π. If this lattice is of the form $\mathbb{Z}^n\oplus\Omega$, in $\mathbb{C}^n$, we can instead consider an intermediate covering ρ of A by an algebraic torus $\mathbb{C}^{*n}$:

$$\mathbb{C}^n \xrightarrow{\exp(2\pi\cdot)} \mathbb{C}^{*n} \xrightarrow{\rho} A$$

and consider $\rho^*(\Gamma(X, L))$ as a subspace of an appropriate space of entire functions on $(\mathbb{C}^*)^n$.

In order to define the quantized theta-functions, we suggest to replace $(\mathbb{C}^*)^n$ by Connes' quantum torus T_q[4),5)] whose polynomial function ring is $\mathbb{C}\langle e_1^{\pm1}, \cdots, e_n^{\pm1}\rangle/(e_ie_j - q_{ij}^{-1}e_je_i | i<j)$, $q=(q_{ij})$ the quantization parameter. Although generally such a ring is not a Hopf algebra, it is acted upon by a usual torus (with $q_{ij}\equiv1$) so that the period lattice corresponding to A can be used to describe the functional equations for the deformed theta-functions on T_q.

These functions possess many properties similar to the classical ones and constitute a welcome addition to the growing family of quantized special functions (cf., in particular Wess's and Zumino's calculus on quantum plane). One major problem is that we generally cannot multiply quantized theta-functions since the exponential factors are not central.

0.4. *Plan*

The first section of this paper is devoted to the category of non-commutative (or quantum) tori. We describe several types of morphisms in that category. The point is that classical relations between theta-functions involve various argument changes which traditionally are expressed via the addition law. Since, however, quantum tori are not even quantum groups, we must be prepared to replace these argument changes by morphisms.

The second section introduces the basic functional equation for our quantized theta-functions. We construct and investigate the respective linear spaces $\Gamma(X, L)_q$.

In the third section, we consider the "root of unity" case, its relation to the Brauer group and commutative geometry.

Our presentation of theta-functions is very much motivated by the p-adic theory, and we hope that the root of unity case may have number-theoretical applications. For this reason we consider tori over arbitrary complete normed fields, non-necessarily archimedean ones.

§ 1. Category of quantum tori

1.1. *Notation*

We fix once for all a base field K. Consider the class of pairs (H, α), where H is a finitely generated abelian group and α: $H\times H\to K^*$ an alternating pairing:

$$\alpha(\chi, \eta)=\alpha(\eta, \chi)^{-1}; \quad \alpha(\chi_1\chi_2, \eta)=\alpha(\chi_1, \eta)\alpha(\chi_2, \eta)$$

for all χ, $\eta\in H$. They form objects of *category of quantum character groups,* whose morphisms f: $(H_1, \alpha_1)\to(H_2, \alpha_2)$ consist of group homomorphisms f: $H_1\to H_2$ consistent with α_i^2:

$$\alpha_2^2(f(\chi), f(\eta))=\alpha_1^2(\chi, \eta) \tag{1·1}$$

for all χ, $\eta\in H_1$. The form $\varepsilon(\chi,\eta)=\alpha_1(\chi, \eta)\alpha_2^{-1}(f(\chi), f(\eta))$ with values in $\{\pm1\}$ is called the *characteristic* of H. The *quantum toroid* $T(H, \alpha)$ is defined by its polynomial function ring $A(H, \alpha)$ which as a linear space is freely generated over K by symbols $e_{\eta,\alpha}(\chi)=e(\chi)$, $\chi\in H$, with multiplication law

$$e(\chi)e(\eta)=\alpha(\chi, \eta)e(\chi+\eta)\,. \tag{1·2}$$

We write H additively; $e(\chi)$ should be viewed as "quantum Fourier harmonics", α is the quantization parameter.

If H is free, $T(H, \alpha)$ is called a *quantum torus*, toroids also occur naturally as kernels, cokernels, quantum Tate groups, etc. For $\alpha=1$, $A(H, \alpha)$ is the ring of a usual commutative group scheme Spec $A(H, \alpha)$. In general, we define a morphism F: $T(H_2, \alpha_2)\to T(H_1,\alpha_1)$ as the inverse morphism of function rings F^*: $A(H_1, \alpha_1)\to A(H_2, \alpha_2)$. The following result describes all morphisms of quantum tori.

1.2. *Proposition*

a) If H is free, then all invertible elements in $A(H, \alpha)$ are of the form $ae(\chi)$, a

$\in K^*$, $\chi\in H$. Therefore, any morphism of quantum tori F: $T(H_2, \alpha_2)\to T(H_1, \alpha_1)$ defines an induced morphism $[F]=f$: $(H_1, \alpha_1)\to(H_2, \alpha_2)$: $F^*(e(\chi))=a_\chi e(f(\chi))$ for some $a_\chi\in K^*$.

b) The set of all morphisms F: $T(H_2, \alpha_2)\to T(H_1, \alpha_1)$ with a fixed $[F]$ is either empty or has a natural structure of the principal homogeneous space over the group of K-points of the commutative torus $T(H_1, l)(K)=\mathrm{Hom}(H_1, K^*)$.

Proof Let H be free. Consider a linear form l: $H\otimes R\to R$ such that any equation $l(\chi)=a\in R$ has no more than one solution $\chi\in H$. The relation $\chi\leq\eta\Leftrightarrow l(\chi)\leq l(\eta)$ defines on H a structure of a well-ordered group. In particular, the highest order term of a product pq, p, $q\in A(H, \alpha)$ is the product of the highest order terms, and the same is true for the lowest order terms. It follows that if p is invertible, then $p=ae(\chi)$. Hence F: $T(H_2, \alpha_2)\to T(H_1, \alpha_1)$ induces a map f: $H_1\to H_2$ as in the Proposition. Since F^* is a ring homomorphism we have

$$\begin{array}{ccc} F^*(e(\chi)e(\eta)) & = & F^*(e(\chi))F^*(e(\eta)) \\ \| & & \backslash\backslash \\ F^*(\alpha_1(\chi,\eta)e(\chi+\eta)) & & a_\chi e(f(\chi))a_\eta e(f(\eta)) \\ \| & & \backslash\backslash \\ \alpha_1(\chi,\eta)a_{\chi+\eta}e(f(\chi+\eta)) & & a_\chi a_\eta \alpha_2(f(\chi),f(\eta))e(f(\chi)+f(\eta)) . \end{array}$$

It follows that f: $H_1\to H_2$ is a group homomorphism. Furthermore, in the equality

$$\alpha_1(\chi, \eta)\alpha_2^{-1}(f(\chi), f(\eta))=a_\chi a_\eta a_{\chi+\eta}^{-1}: =\varepsilon(\chi, \eta) \tag{1·3}$$

the left-hand side is alternate while the right one is symmetric in χ, η. Hence $\varepsilon(\chi, \eta)=\pm1$, and $\alpha_1^2(\chi, \eta)=\alpha_2^2(f(\chi), f(\eta))$. It follows that f is a morphism $(H_1, \alpha_1)\to(H_2, \alpha_2)$ with characteristic $\varepsilon(\chi, \eta)$. If a system $\{a_\chi\}$ verifying (1·3) exists at all, any other such system is of the form $a_\chi c(\chi)$ where c: $H_1\to K^*$ is a homomorphism. ∎

Notice that if f: $(H_1, \alpha_1)\to(H_2, \alpha_2)$ has characteristic $\varepsilon\equiv1$, it defines a canonical morphism F: $T(H_2, \alpha_2)\to T(H_1, \alpha_1)$ with

$$F^*(e(\chi))=e(f(\chi)), \tag{1·4}$$

which we shall sometimes denote also f.

1.3. *Analytic functions on quantum tori*

Our theta-functions will be certain infinite linear combinations of the formal exponents $e(\chi)$. In order to define them we shall furthermore assume that K is a complete normed field and that α is *unitary,* that it takes values in the subgroup $K_1^*=\{a\in K|\ |a|=1\}$. In particular, if all values of α are roots of unity, it is unitary.

A formal series $\sum_{\chi\in H}a_\chi e(\chi)$, $a_\chi\in K$, is called an *analytic function* on $T(H, \alpha)$ if for any $N>0$ there exists $c>0$ such that

$$|a_\chi|<c(\|\chi\|+1)^{-N},$$

where $\|\cdot\|$ is an Euclidean norm on $H\otimes R$.

A standard computation shows the following fact.

1.4. *Lemma*

If α is unitary, the space of analytic functions $An(H, \alpha)$ on $T(H, \alpha)$ is a ring with respect to the usual product of formal series. ■

Now we shall list some morphisms of quantum tori.

1.5. *Multiplication by n*

The morphism

$$[n]:\ T(H,\alpha)\to T(H,\alpha^{n^2})$$

is defined by

$$[n]^*(e(\chi))=e((n\chi)).$$

It is an endomorphism of $T(H,\alpha)$ in two important cases:

a) $n=-1$.

b) α takes values in roots of unity of degree d, and $n^2\equiv 1 \bmod d$.

1.6. *Direct products and multiplications*

In non-commutative geometry the tensor product of function rings plays the same role as in commutative geometry: it morally corresponds to the direct product of quantum spaces, although is not the direct product in the usual categorical sense (in dual category).

Following this convention, we put

$$T(H_1,\alpha_1)\times T(H_2,\alpha_2)=T(H_1\oplus H_2,\alpha_1\oplus\alpha_2)$$

by

$$e_{H_1,\alpha_1}(\chi)\otimes e_{H_2,\alpha_2}(\eta)=e_{H_1\oplus H_2,\alpha_1\oplus\alpha_2}((\chi,\eta)).$$

Now, we can define the multiplication maps

$$m_{\alpha,\beta}:\ T(H,\alpha)\times T(H,\beta)\to T(H,\alpha\beta)$$

by

$$m^*_{\alpha,\beta}(e_{H,\alpha\beta}(\chi))=e_{H,\alpha}(\chi)\otimes e_{H,\beta}(\chi).$$

In particular, $m_{1,\beta}$ is the action of the usual torus $T(H,1)$ upon its quantized form $T(H,\beta)$ which was implicitly used in 1.2, and which will be used in the definition of the period lattices of quantum theta-functions.

This construction can be generalized as follows. First consider n tori $T_i=T(H_i,\alpha_i)$, $i=1,\cdots,n$. For each $1\le i<j\le n$ choose a scalar product γ_{ij}: $H_i\times H_j\to K^*$ and define the skew product of T_i w.r.t. $\gamma=(\gamma_{ij})$ by

$$\prod_{(\gamma)}T_i=T(H_1\oplus\cdots\oplus H_n,\alpha),$$

where

$$\alpha((\chi_1,\cdots,\chi_n),(\eta_1,\cdots,\eta_n))=\prod_{i=1}^{n}\alpha_i(\chi_i,\eta_i)\prod_{i<j}\gamma_{ij}(\chi_i,\eta_j)\prod_{i>j}\gamma_{ji}^{-1}(\eta_j,\chi_i).$$

As above, we can define a multiplication morphism

$$m:\ \prod_{(\gamma)} T_i \to T(H, \alpha_1\cdots\alpha_n)\ ;\ m^*(e(\chi))=e((\chi,\cdots,\chi))\,,$$

if the following condition is fulfilled: $\prod_{i<j}\gamma_{ij}(\chi,\eta)$ is symmetric in χ, η. In fact:

$$\alpha_1\cdots\alpha_n(\chi,\eta)=\alpha_1(\chi,\eta)\cdots\alpha_n(\chi,\eta)\,,$$

$$\alpha((\chi,\cdots,\chi),(\eta,\cdots,\eta))=\prod_{i=1}^{n}\alpha_i(\chi,\eta)\prod_{i<j}\gamma_{ij}(\chi,\eta)\prod_{j<i}\gamma_{ji}^{-1}(\eta,\chi)\,.$$

1.7. *Opposite torus*

Let $T(H,\alpha)^{\mathrm{opp}}$ be defined by the ring $A(H,\alpha)^{\mathrm{opp}}$ with multiplication reversed with respect to that in $A(H,\alpha)$. We have a canonical isomorphism

$$T(H,\alpha)^{\mathrm{opp}} \xrightarrow{\sim} T(H,\alpha^{-1}):\ e_{H,\alpha}(\chi)\mapsto e_{H,\alpha^{-1}}(\chi)\,.$$

1.8. *Mumford's morphism*

By definition, it is

$$M:\ T(H\oplus H,\alpha\oplus\alpha)\to T(H\oplus H,\alpha^2\oplus\alpha^2)\,,$$

$$M^*(e(\chi,\eta))=e(\chi+\eta,\chi-\eta)\,.$$

It is well defined because

$$(\alpha\oplus\alpha)[(\chi+\eta,\chi-\eta),(\chi'+\eta',\chi'-\eta')]$$
$$=\alpha(\chi+\eta,\chi'+\eta')\alpha(\chi-\eta,\chi'-\eta')=\alpha^2(\chi,\chi')\alpha^2(\eta,\eta')\,.$$

§ 2. Quantized theta-functions

2.1. *Periods and formal thetas*

For some time, we can work with formal series in $e(\chi)\in A(H,\alpha)$. Consider a subgroup $B\subset T(H,1)(K)=\mathrm{Hom}(H,K^*)$. It acts upon $A(H,\alpha)$ by

$$b^*(e(\chi))=\chi(b)e(\chi)\,. \tag{2·1}$$

We shall call a formal series $\theta=\sum a_\chi e(\chi)$ a *formal left quantized theta-function* with respect to B if there exist two maps $B\to K^*$: $b\mapsto\lambda_b$, and $B\to H$: $b\mapsto\chi_b$ such that for all $b\in B$ we have

$$b^*(\theta)=\lambda_b e(\chi_b)\theta\,. \tag{2·2}$$

Similarly one can introduce right thetas by putting $e(\chi_b)$ to the right-hand side of θ in (2·2). They are reduced to left thetas on the opposite torus (cf. 1.7).

2.2. *Lemma*

If $\theta\neq 0$ verifies (2·2), then

a) $b\mapsto\chi_b$ is a homomorphism $B\to H$;

b) $\chi_{b_1}(b_2)$ bimultiplicatively depends on b_1, b_2 and can be represented in the form

$$\chi_{b_2}(b_1)=(b_1,b_2)[b_1,b_2]\,, \tag{2·3}$$

where $(\cdot,\cdot)$ is symmetric, $[\cdot,\cdot]$ is alternate, and moreover

$$[b_1, b_2]=\alpha(\chi_{b_1}, \chi_{b_2})\,. \tag{2·4}$$

Proof We have

$$(b_1b_2)^*(\theta)=\lambda_{b_1b_2}e(\chi_{b_1b_2})\theta\,. \tag{2·5}$$

On the other hand, it equals $b_1^*[b_2^*(\theta)]$, that is,

$$(b_1)^*[\lambda_{b_2}e(\chi_{b_2})\theta]=\lambda_{b_1}\lambda_{b_2}\chi_{b_2}(b_1)\alpha(\chi_{b_2}, \chi_{b_1})e(\chi_{b_1}+\chi_{b_2})\theta\,. \tag{2·6}$$

It follows, that $\chi_{b_1b_2}=\chi_{b_1}+\chi_{b_2}$ (compare (2·5) and (2·6)). Moreover, (2·6) should be symmetric in b_1, b_2, which gives

$$\chi_{b_1}(b_2)\chi_{b_2}(b_1)=\alpha^{-2}(\chi_{b_1}, \chi_{b_2})\,.$$

On the other hand, (2·5) and (2·6) show that

$$\chi_{b_2}(b_1)\alpha(\chi_{b_2}, \chi_{b_1})=\frac{\lambda_{b_1b_2}}{\lambda_{b_1}\lambda_{b_2}}:=(b_1, b_2) \tag{2·7}$$

so that the left-hand side is a symmetric pairing. This shows (2·3) and (2·4).

2.3. *Theta-types*

Let now (b_1, b_2) be the symmetric pairing from (2·3). Assume that there is a symmetric pairing $B\times B\to K^*$: $(b_1, b_2)^{1/2}$, whose square is (b_1, b_2). It certainly exists if $K=C$; otherwise it exists in a finite extension of K (if B is finitely generated). Two such roots differ by a pairing $B\times B\to\mu_2=\{\pm 1\}$.

Choose $(b_1, b_2)^{1/2}$, and put $\psi(b)=\lambda_b\cdot(b, b)^{-1/2}$. From (2·7) it follows that $\psi(b_1b_2)=\psi(b_1)\psi(b_2)$. A change in the choice of $(b_1, b_2)^{1/2}$ can be compensated by the corresponding change of ψ without influencing λ_b.

This justifies the following definitions:

A (left) formal theta-type for $T(H, \alpha)$ w.r.t.: the period subgroup $B\subset T(H, 1)$ is a triple $L=(\varphi, \psi, (\cdot,\cdot)^{1/2})$ consisting of two group homomorphisms and one symmetric pairing:

$$\varphi: B\to H,\ \varphi(b)=\chi_b\,;\ \psi: B\to K^*\,;\ (\cdot,\cdot)^{1/2}: B\times B\to K^* \tag{2·8}$$

such that

$$\forall\, b_i\in B,\quad (b_1, b_2)=\chi_{b_2}(b_1)\alpha(\chi_{b_2}, \chi_{b_1})\,. \tag{2·9}$$

A (left) formal theta-function on the torus $T(H, \alpha)$ of the type L is a formal series θ verifying the functional equations

$$b^*(\theta)=\psi(b)(b, b)^{1/2}e(\chi_b)\theta \tag{2·10}$$

for all $b\in B$.

Clearly, all formal theta-functions of given type L form a linear space which we denote $\Gamma(L)$.

We turn now to the situation of 1.3, assuming K complete normal and α unitary. Then we have the following result which classically leads to the introduction of

Siegel's upper half-space, parametrizing abelian varieties.

2.4. *Theorem*

a) We have always (even without unitarity assumption)

$$\dim\Gamma(L)=[H:\varphi(B)].$$

b) The space $\Gamma(L)$ consists only of analytic functions iff $[H:\varphi(B)]<\infty$ and $\log|(b_1, b_2)|$ is a positively defined bilinear form on B. In particular, if B is free, we have $rkB=rkH$ and B is a discrete subgroup of Hom (H, K^*).

Proof For $\theta=\sum a_\chi e(\chi)$, we can rewrite (2·10) as follows:

$$b^*(\theta)=\sum_{\chi\in H} a_\chi\chi(b)e(\chi)=\sum_{\chi\in H} a_{\chi+\chi_b}(\chi+\chi_b)(b)e(\chi+\chi_b)$$
$$=\psi(b)(b, b)^{1/2}e(\chi_b)(\theta)=\sum_{\chi\in H} a_\chi\chi(b)\psi(b)(b, b)^{1/2}\alpha(\chi_b, \chi)e(\chi+\chi_b).$$

This means that for all $b\in B$ we have

$$a_{\chi+\chi_b}=a_\chi\psi(b)(b, b)^{1/2}\chi_b(b)^{-1}\alpha(\chi_b, \chi)$$
$$=a_\chi\psi(b)(b, b)^{-1/2}\alpha(\chi_b, \chi). \tag{2·11}$$

Hence one can arbitrarily choose values a_χ for all χ in a system of representatives of $H/\varphi(B)$ and then uniquely reconstruct θ. This shows the first part of the theorem.

In particular, if $[H:\varphi(B)]=\infty$, there are always non-analytic elements in $\Gamma(L)$.

On the other hand, if $[H:\varphi(B)]<\infty$, we have for a fixed χ and varying b:

$$\log|a_{\chi+\chi_b}|=\log|(b, b)^{-1/2}|+\log|a_\chi\psi(b)\alpha(\chi_b, \chi)|.$$

The first summand is quadratic in b while the second is linear. Hence analyticity of all $\theta\in\Gamma(L)$ is ensured precisely when $\log|(b, b)^{1/2}|$ is positively defined. ∎

2.5. *Definition*

A theta-type L is called a polarization if $[H:\varphi(B)]<\infty$ and $\log|(b, b)|>0$. It is called a principal polarization if $H=\varphi(B)$. ∎

Let now f: $T(H_2, \alpha_2)\to T(H_1, \alpha_1)$ be a morphism of tori, such that $F^*(e(\chi))=a_\chi e(f(\chi))$ for some f: $H_1\to H_2$ (cf. 1.1~1.2). Let θ be a formal theta-function on $T(H_1, \alpha_1)$ of the type (φ_1: $B_1\to H_1$; ψ_1: $B_1\to K^*$; $(\cdot, \cdot)_1^{1/2}$: $B_1\times B_1\to K^*$). Consider a period subgroup $B_2\subset T(H_2, 1)(K^*)$ such that $f^{-1}(B_2)\subset B_1$.

2.6. *Lemma*

If F has characteristic 1, that is, $a_{\chi+\eta}=a_\chi a_\eta$ (cf. (1.3)), then $F^*(\theta)$ is a formal theta-function w.r.t. B_2 of the type $(\varphi_2, \psi_2, (\cdot, \cdot)_2^{1/2})$, where (writing $a(\chi)$ instead a_χ):

$$\psi_2(b)=\psi(f^*(b))a(\chi_b), \qquad b\in B_2\subset\mathrm{Hom}(H_2, K^*),$$

$$(b, b)_2^{1/2}=(f^*(b), f^*(b))_1^{1/2},$$

φ_2 is defined by commutativity of the diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\varphi_1} & H_1 \\ g\uparrow & & \downarrow f \\ B_2 & \overset{\varphi_2}{\cdots\cdots} & H_2 \end{array} , \qquad \begin{array}{c} g\colon B_2 \subset \mathrm{Hom}(H_2, K^*) \\ \| \\ \text{restriction of } f^* \quad H_1 \xrightarrow{f} H_2 \to K^* . \end{array}$$

Proof is a formal application of definitions.

§ 3. Tori and thetas at roots of unity

3.1. *Cross-products*

Let A be a commutative ring, G a commutative group acting upon A: $G\times A\to A$, $(\sigma, f)\mapsto {}^{\sigma}f$. Consider a cocycle $A=\{a(\sigma, \tau)\}: G\times G\to A^*$. The cross product $A[\sigma; \sigma]$ is a free A-module $\oplus_{\sigma\in G}Ae_{\sigma}$ with multiplication

$$\begin{cases} e_{\sigma}f={}^{\sigma}fe_{\sigma} , & (3\cdot1) \\ e_{\sigma}e_{\tau}=a_{\sigma,\tau}e_{\sigma+\tau} . & (3\cdot2) \end{cases}$$

Its center contains the ring of G-invariants of A.

3.2. *Function ring on a non-commutative torus as a cross product*

Consider a torus $T(H, \alpha)$ for which α takes values in roots of unity. Assume that there is a filtration $H'\subset I\subset H$ such that $H'=\mathrm{Ker}\ \alpha^2$, I is α^2 isotropic and α^2 induces a perfect duality α: $H/I\times I/H'\to K^*$.

Clearly, $A=A(I, 1)$ is a commutative subring of $A(H, \alpha)$, and $A(H', 1)\subset A$ is the center of $A(H, \alpha)$. The group

$$G=\mathrm{Hom}\ (I/H', K^*)\simeq H/I$$

acts upon A by

$$\sigma[e(\chi)]=\sigma(\chi)e(\chi)=\alpha^2(\eta, \chi)e(\chi) ,$$

where $\sigma\in G$, $\chi\in I$, $\eta\in H$, σ corresponds to η mod I.

Choose now a system of representatives $\bar{G}\subset H$ for H mod I and define the application β: $\bar{G}\xrightarrow{\sim}G$ as reduction mod I. For $\eta\in H$, let $\bar{\eta}\in\bar{G}$ be its representative. Put for $\eta, \chi\in H$:

$$a_{\beta(\eta),\beta(\chi)}=\alpha(\eta, \chi)\alpha(\overline{\eta+\chi}, \eta+\chi)e(\eta+\chi-(\overline{\eta+\chi})) . \qquad (3\cdot3)$$

3.3. *Proposition*

a) $(3\cdot3)$ is a cocycle $G\times G\to A^*$.

b) There is an isomorphism

$$F\colon A(H, \alpha)\xrightarrow{\sim} A(I, 1)[G, a] \qquad (3\cdot4)$$

such that

$$F(e(\chi))=e(\chi) \quad \text{for} \quad \chi\in I ,$$

$$F(e(\eta))=e_{\beta(\eta)} \quad \text{for} \quad \eta\in\bar{G} .$$

Proof Clearly, $A(H, \alpha)=\oplus_{\eta\in\bar{G}}A(I, 1)e(\eta)$ as $A(I, 1)$-module. It remains to check the multiplication table. For $\chi\in I$ we have

$$e(\eta)e(\chi)=\alpha^2(\eta, \chi)e(\chi)e(\eta),$$

which after applying F becomes (3·1):

$$e_{\beta(\eta)}f={}^{\beta(\eta)}fe(\eta)$$

for $f=e(\chi)$. Moreover, for $\chi, \eta\in\bar{G}$ we have

$$e(\eta)e(\chi)=\alpha(\eta, \chi)e(\eta+\chi)=\alpha(\eta, \chi)\alpha(\overline{\eta+\chi}, \eta+\chi)e(\eta+\chi-(\overline{\eta+\chi}))e(\overline{\eta+\chi}),$$

which after applying F becomes (3·2):

$$e_{\beta(\eta)}e_{\beta(\chi)}=a_{\beta(\eta),\beta(\chi)}e_{\beta(\eta)+\beta(\chi)}. \qquad \blacksquare$$

3.4. *Central types and sheaves*

With the previous notation, call a theta-type $L'=(\varphi', \psi', (\cdot, \cdot)_1{}^{1/2})$ for $T(H, \alpha)$ central, if $\psi'(B)\subset H'$.

Any type $L=(\varphi, \psi, (\cdot, \cdot)^{1/2})$ can be multiplied on the right by a central type:

$$LL'=(\varphi+\varphi', \psi\psi', (\cdot, \cdot)^{1/2}(\cdot, \cdot)_1{}^{1/2})$$

and

$$\Gamma(L)\Gamma(L')\subset\Gamma(LL')$$

in the ring of analytic functions of $T(H, \alpha)$.

In particular, let L_0 be a central polarization. Then it defines an invertible sheaf on the abelian variety $\mathcal{A}=T(H', 1)/B$ which we shall denote again by L_0. The space of sections $\Gamma(\mathcal{A}, L_0{}^n)$ is a subspace of $\Gamma(L_0^n)$ in the sense of 2.3.

For any formal type L for (H, α) put

$$M=\oplus_{n\geq 0}\Gamma(LL_0^n).$$

It is a Z-graded module over $\oplus_{n\geq 0}\Gamma(\mathcal{A}, L_0^n)$. Therefore it defines a coherent sheaf of $\mathcal{A}$. In this way theta-functions at roots of unity become sections of sheaves on usual algebraic varieties.

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