# Conformal Field Theory Approach to One-Dimensional Quantum Liquids

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We review the critical behavior of 1D highly correlated electron systems and 1D quantum many-body systems with long-range interactions. These models exhibit the common universal long-distance properties which are characterized as Luttinger liquids. The microscopic foundation of Luttinger liquids is formulated on the basis of conformal field theory and Bethe-ansatz solutions.

## § 1. Introduction

Conformal field theories describe the macroscopic fluctuations in two-dimensional (2D) critical phenomena based on the representation theory of underlying infinite dimensional symmetry.[1] The most fundamental conformal symmetry is generated by the celebrated Virasoro algebra. By virtue of the rapid development in conformal field theory (CFT) our knowledge of the 2D classical critical phenomena has become very rich and precise in recent years.[2),3)]

In view of physics of critical phenomena the current interest is much focused on quantum critical phenomena. For one-dimensional (1D) quantum systems with short-range interactions there is no phase transition at finite temperatures, and hence the correlation functions decay exponentially at long distance. Critical phenomena can take place only at absolute zero, $T=0$. The low-energy gapless excitations then follow the linear dispersion relation. Consequently the long-distance behavior of correlation functions is characterized by the power-law decay. It is clear that 1D quantum critical phenomena provide us with the active area for CFT to play a significant role.[4)]

In the study of phase transitions finite-size scaling has been recognized as an important tool. In CFT the finite-size scaling method turns out to be so powerful that we are able to evaluate the correlation exponents from the finite-geometry information.[5)] The application of this method to a variety of quantum spin chains has achieved remarkable success in both numerical and analytical approaches. We also found an important application to 1D highly correlated electron systems such as the Hubbard model[6)~8)] and the $t$-$J$ model.[9)] The correlation exponents have been calculated exactly starting with the microscopic models. We have thus obtained an answer to the long-standing problem in condensed matter physics since the discovery of the Bethe-ansatz wave functions.

Our purpose in this article is to give a review of our recent works on 1D highly

correlated systems[7),9)] and 1D quantum many-body systems with long-range interactions.[10),11)] In § 2 we would like to describe the basic aspects of CFT putting much emphasis on the points which play a vital role in our application. In § 3 applying the CFT technique to 1D correlated electron systems we derive the exact critical exponents for various correlation functions. In § 4 we discuss the conformal invariance properties of 1D critical systems with long-range interactions of $1/r^2$ type.

## § 2. Brief review of CFT

We consider a 2D critical system defined on the euclidean plane with coordinate $(x^1, x^2)$. It is convenient to introduce complex combinations

$$z=x^1+ix^2\,,\quad \bar{z}=x^1-ix^2\,. \tag{2·1}$$

The conformal transformation is to make a replacement

$$z\to w(z)\,,\quad \bar{z}\to\bar{w}(\bar{z})\,, \tag{2·2}$$

where $w(z)$ and $\bar{w}(\bar{z})$ are arbitrary analytic functions. Thus 2D conformal invariance is infinite dimensional symmetry. In quantum field theory the generators of the coordinate transformation are defined from a symmetric energy-momentum tensor $T_{\mu\nu}(x^1, x^2)$. Scale invariance at criticality implies that the trace of the energy-momentum tensor vanishes. The continuity equation $\partial_\mu T_{\mu\nu}=0$ is then reduced to

$$\partial_{\bar{z}}T(z)=0\,,\quad \partial_z\bar{T}(\bar{z})=0\,, \tag{2·3}$$

where we have defined

$$T(z)=\frac{1}{2}(T_{11}-iT_{12})\,,\quad \bar{T}(\bar{z})=\frac{1}{2}(T_{11}+iT_{12})\,. \tag{2·4}$$

Expanding $T(z)$ as well as $\bar{T}(\bar{z})$ in Laurent series

$$T(z)=\sum_{n\in\boldsymbol{Z}}z^{-n-2}L_n\,,\quad \bar{T}(\bar{z})=\sum_{n\in\boldsymbol{Z}}\bar{z}^{-n-2}\bar{L}_n\,, \tag{2·5}$$

we obtain the infinite set of symmetry generators $L_n$ and $\bar{L}_n$.

The energy-momentum tensor $T(z)$ transforms under the conformal transformation (2·2) as[1)]

$$T(z)\to\left(\frac{dw}{dz}\right)^2T(w)+\frac{c}{12}\{w, z\}\,, \tag{2·6}$$

where $\{w, z\}$ stands for the Schwartzian

$$\{w, z\}=\frac{d^3w/dz^3}{dw/dz}-\frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2. \tag{2·7}$$

The transformation property (2·6) is equivalent to the commutation relations for $L_n$

$$[L_m, L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0}\,, \tag{2·8}$$

which is called the Virasoro algebra. For $\bar{L}_n$ we have the same algebra, and $L_n$ and $\bar{L}_n$ are commuting. A number $c$, called the central charge, is an important parameter of the theory. The central charge labels each universality class of the critical systems.

In critical lattice statistical systems microscopic lattice variables renormalize to an infinite set of scaling fields in the continuum limit, each of which is characterized by its scaling dimension. What is remarkable in 2D phenomena is that this infinite set of scaling fields is under control. In CFT these fields are classified into subsets on top of that there exists a primary field $\Phi(z, \bar{z})$. Under the conformal transformation (2·2) a primary field behaves as an $(h^+, h^-)$ tensor

$$\Phi(z, \bar{z}) \to \left(\frac{dw}{dz}\right)^{h^+} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{h^-} \Phi(w, \bar{w}), \tag{2·9}$$

where $h^{\pm}$, called conformal weights, are another fundamental parameters in the theory. Consequently the two-point correlation function reads

$$\langle \Phi(z, \bar{z}) \Phi(z', \bar{z}') \rangle = \frac{1}{(z-z')^{2h^+} (\bar{z}-\bar{z}')^{2h^-}}. \tag{2·10}$$

It is now clear that the conformal weights $h^{\pm}$ determine the scaling dimension $x$ of $\Phi$ as $x=h^+ + h^-$. Starting with the primary field $\Phi(z, \bar{z})$ we see that each subset forms an infinite dimensional conformal tower whose elements are given by repeatedly acting $L_n$, $\bar{L}_n$ with $n>0$ on $\Phi(z, \bar{z})$.[1] Their conformal weights take $(h^+ + N^+, h^- + N^-)$ with $N^{\pm}=1, 2, 3, \cdots$.

To summarize the 2D universality class is completely specified by the value of the central charge and the spectrum of the primary fields $\{(h^+, h^-)\}$. Upon classifying CFT let us first assume unitarity, then $c>0$. CFT with $c \geq 1$ are qualitatively different from those with $0<c<1$. The unitarity condition selects discrete values of $c$ for $0<c<1$,

$$c = 1 - \frac{6}{m(m+1)}, \quad m=3, 4, 5, \cdots, \tag{2·11}$$

whereas unitarity is ensured for $c \geq 1$ theories.[12] For $0<c<1$ the number of primary fields is finite and their conformal weights are rational numbers which are determined by the Kac formula. In $c=1$ theory any non-negative coformal weight is allowed and there exist infinite number of primary fields. Symmetry is at most discrete for $0<c<1$, such as $Z_2$ symmetry for the Ising model, realizing $c=1/2$ CFT, and $S_3$ symmetry for the three-state Potts model, realizing $c=4/5$ CFT. At $c=1$ there can exist $h=1$ field, and hence the symmetry can be continuous.

The results for $0<c<1$ have been obtaind by making use of the powerful representation theory of the Virasoro algebra. In $c=1$ theory, on the other hand, the representation theory is not powerful enough to specify the theory. We know, however, that the $c=1$ theory is nothing but the gaussian field theory which we can directly deal with.

### 2.1. $c=1$ *CFT*

Since the $c=1$ theory is the most relevant CFT for our description of 1D quantum liquids we wish to summarize its main properties. Let us consider the gaussian theory on the cylinder geometry. The action reads

$$S=\frac{1}{2\pi}\int dt\int_0^{2\pi} d\sigma\partial_\mu\varphi\partial^\mu\varphi\,, \tag{2·12}$$

where $t$ is the temporal coordinate defined along the cylinder axis and $\sigma$ is the spatial coordinate across the cylinder with period $2\pi$. The gaussian field $\varphi(\sigma, t)$ is assumed to be periodic

$$\varphi(\sigma+2\pi, t)=\varphi(\sigma, t)+2\pi RN\,, \tag{2·13}$$

where $N\in\boldsymbol{Z}$ is the winding number and $R$ is the continuous real parameter in the theory. The cylinder coordinate $(\sigma, t)$ is related to that of the plane $(z, \bar{z})$ through the conformal mapping $z=e^{i(t+\sigma)}$, $\bar{z}=e^{i(t-\sigma)}$. Then the equation of motion implies the decomposition

$$\varphi(\sigma, t)=\frac{1}{2}[\phi(z)+\bar{\phi}(\bar{z})]\,. \tag{2·14}$$

The stress tensor is given by

$$T(z)=-\frac{1}{2}(\partial_z\phi(z))^2\,,\quad \bar{T}(\bar{z})=-\frac{1}{2}(\partial_{\bar{z}}\bar{\phi}(\bar{z}))^2\,. \tag{2·15}$$

We have primary fields with conformal weights $(h^+, h^-)=(1, 0)$ as well as $(0, 1)$

$$J(z)=i\partial_z\phi(z)\,,\quad \bar{J}(\bar{z})=i\partial_{\bar{z}}\bar{\phi}(\bar{z})\,, \tag{2·16}$$

which are responsible for the continuous $U(1)\times U(1)$ symmetry of the system. From these operators we may obtain the marginal operator $J\bar{J}(z, \bar{z})$ with conformal weight $(1, 1)$. Then the system realizes the continuonusly varying criticality.

The primary fields in the $c=1$ theory may be divided into two classes. One class consists of fields $\Phi_{M,N}(z, \bar{z})$ with continuously varying conformal weights

$$h^{\pm}=\frac{1}{2}\left(\frac{M}{2R}\pm RN\right)^2\,,\quad M, N\in\boldsymbol{Z}\,. \tag{2·17}$$

These fields are expressed in terms of the vertex operators

$$\begin{aligned}\Phi_{M,N}(z, \bar{z})&=e^{i[\alpha_{M,N}\phi(z)+\alpha_{M,-N}\bar{\phi}(\bar{z})]}\\&=e^{i[(M/R)\varphi(\sigma,t)+2RN\tilde{\varphi}(\sigma,t)]}\,,\end{aligned} \tag{2·18}$$

where $\alpha_{M,N}=(M/2R)+RN$ and $\tilde{\varphi}(\sigma, t)=[\phi(z)-\bar{\phi}(\bar{z})]/2$.*) The primary fields in the other class are given by the differential polynomials of $\partial_z\phi(z)$ and $\partial_{\bar{z}}\bar{\phi}(\bar{z})$. Their conformal weights are

*) The operators $\Phi_{M,N}(z, \bar{z})$ with $M, N\in\boldsymbol{Z}$ are mutually local. To describe the full operator content of the theory, however, we should also consider the parafermionic operators which generically do not respect the condition $M, N\in\boldsymbol{Z}$.

$$h=n^2/4\,,\quad n=0,1,2,\cdots\,, \tag{2·19}$$

which are obviously independent of $R$.

This system defines the gaussian critical line on which we observe many interesting symmetry enhancement points. We first note the duality symmetry under $R\leftrightarrow(1/2R)$ corresponding to the interchange $M\leftrightarrow N$. At the self-dual point $R=1/\sqrt{2}$ the $U(1)$ symmetry is enlarged to the $SU(2)$ symmetry since in addition to $J(z)$ two holomorphic $(1,0)$ primary fields $\Phi_{\pm1,\pm1}(z)=e^{\pm i\sqrt{2}\,\phi(z)}$ appear and these three currents provide the $SU(2)$ generators. It is well known that the spin-1/2 antiferromagnetic (AF) Heisenberg chain at $T=0$ renormalizes to this fixed point. The point $R=\sqrt{2}$ corresponds to the Kosterlitz-Thouless point in the $XY$ model. Other interesting points are $R=\sqrt{3}/2$, realizing $N=2$ supersymmetry, and $R=1$, realizing free Dirac fermion theory. In addition to this critical line we have the other $c=1$ critical line on which the $U(1)\times U(1)$ symmetry is broken. This line describes the critical behavior of the Baxter's eight-vertex model and the Ashkin-Teller model.[13)~15)] Many interesting physical as well as mathematical aspects of $c=1$ CFT are discussed in Ref. 16).

The gaussian universality class plays a fundamental role in our understanding of 2D critical phenomena and 1D quantum critical phenomena. Its ubiquitous nature was emphasized by several authors.[17)~20)] In the following we shall see several examples of quantum critical systems which are mapped onto the gaussian theory at long distance under renormalization. To complete this mapping we have to determine the parameter $R$ in terms of the microscopic parameters, such as interaction strength, particle density and so on. For the models we will discuss this calculation is carried out exactly by using the Bethe-ansatz results.

### 2.2. *Finite-size scaling in CFT*

We explain the finite-size scaling in CFT, following the work of Cardy.[5)] Let us consider a two-point function of primary field (2·10). We apply the conformal transformation

$$w=\frac{L}{2\pi}\ln z \tag{2·20}$$

which maps the entire $z$-plane onto the surface of the cylinder. Notice that in this finite geometry boundary conditions are periodic. The cylinder coordinate is $w=t+i\sigma$ just as used in § 2.1 except that our metric here is euclidean and $\sigma$ is the $L$-periodic coordinate. From (2·9) one can easily find the correlation function on the cylinder. Expanding this in Taylor series we get

$$\langle\Phi(w,\bar{w})\Phi(w',\bar{w}')\rangle_{\text{cylinder}}$$
$$\sim\sum_{N^{\pm}=0}^{\infty}A_{N^{\pm}}e^{-(2\pi/L)(x+N^{+}+N^{-})(t-t')}e^{(2\pi i/L)(s+N^{+}-N^{-})(\sigma-\sigma')}\,, \tag{2·21}$$

where $x=h^{+}+h^{-}$, $s=h^{+}-h^{-}$ and the factors $A_{N^{\pm}}$ are irrelevant in the present context. In terms of the transfer matrix formulation the left-hand side may be written as

$$\sum_{n,k}\langle 0|\Phi(\sigma)|n,k\rangle e^{-(E_n-E_0)(t-t')}\langle n,k|\Phi(\sigma')|0\rangle\,, \tag{2·22}$$

where $E_n$ and $k$ are the energy and momentum eigenvalues of the hamiltonian $\hat{H}$ and the momentum operator $\hat{P}$, respectively. Here the ground state $|0\rangle$ corresponds to $|n,k\rangle=|0,0\rangle$. Thus we see that for each primary state $(h^+, h^-)$ there correspond an infinite tower of the eigenstates of $\hat{H}$ and $\hat{P}$ with the eigenvalues $E=E_0+2\pi(x+N^++N^-)/L$ and $P=2\pi(s+N^+-N^-)/L$.

This result is understood in terms of the transformation property of the energy-momentum tensor. Substituting (2·20) into (2·6) we find that

$$\hat{H}=\frac{2\pi}{L}(L_0+\bar{L}_0)-\frac{\pi c}{6L} \tag{2·23}$$

up to a non-universal bulk term. The scaling law now follows from the fact that $\Phi(0,0)|0\rangle$ is an eigenstate of $L_0$ $(\bar{L}_0)$ with eigenvalue $h^+(h^-)$. In particular the ground state has zero eigenvalues $h^+=h^-=0$. Notice also the appearance of the central charge $c$. This leads to an important universal finite-size scaling behavior of the ground-state energy[21),22)]

$$E_0\sim\varepsilon_0 L-\frac{\pi c}{6L}\,, \tag{2·24}$$

where $\varepsilon_0$ is the ground-state energy density in the thermodynamic limit.

Therefore if we know the energy spectrum under periodic boundary conditions the $1/L$ corrections to the ground-state energy gives us the value of the central charge and the finite gap scaled as $1/L$ yields the conformal weights. Thus we can determine the universality class and scaling dimensions of the operators from the finite-size effects.

In 1D quantum critical phenomena we have to take into account the anisotropy factor in the finite-size scaling relations. The velocity $v$ of the elementary excitation fixes this factor. The scaling formulas are now written as

$$E_0\sim\varepsilon_0 L-\frac{\pi v c}{6L}\,,$$

$$E-E_0\sim\frac{2\pi v}{L}(x+N^++N^-)\,. \tag{2·25}$$

Replacing $1/L$ by $T$(=temperature) we realize that the specific heat $C$ vanishes linearly with respect to $T$[22)]

$$C\sim\frac{\pi c}{3v}T\,. \tag{2·26}$$

This is a useful relation to extract $c$ from thermodynamics.

The results offer an efficient way of evaluating the scaling dimensions in the numerical analysis of 1D quantum system. More crucial for our purpose is the fact that for Bethe-ansatz solvable models full information about the scaling dimensions is provided by the energy spectrum obtained exactly from the Bethe-ansatz equations.[23)~27)] In the Bethe-ansatz framework we are thus able to determine the long-distance behavior of correlation functions without directly dealing with them. When the system renormalizes to the gaussian fixed point we can also determine the

dependence of $R$ on the microscopic parameters as will be shown in the subsequent sections.

## § 3. Correlated electron systems

The universal role of Fermi liquids is well established to describe the low-energy properties of electron systems in higher dimensions. After the discovery of high-$T_c$ superconductivity, however, a fundamental issue in condensed matter physics has been to clarify if the normal state of low-dimensional highly correlated systems exhibits the non-Fermi liquid behavior.

Especially, in 1D electron systems the large quantum fluctuations give rise to the anomalous behavior from the point of view of conventional Fermi liquids. Such behavior was first discovered in the Tomonaga-Luttinger (TL) model many years ago.[28),29)] The TL model is recognized as the weakly correlated system in the sense that the dispersion law for bare electrons is completely linear. The model can be treated exactly by the bosonization method. The long-distance behavior of the equal-time correlation functions has been obtained as follows:[30)]

(a) charge density correlator

$$\langle n(r)n(0)\rangle \sim \text{const}+A_0 r^{-2}+A_2 r^{-\alpha_s}\cos 2k_F r+A_4 r^{-\alpha_c}\cos 4k_F r\,, \tag{3·1}$$

(b) spin density correlator

$$\langle S_z(r)S_z(0)\rangle \sim B_0 r^{-2}+B_2 r^{-\alpha'_s}\cos 2k_F r\,, \tag{3·2}$$

(c) electron correlator

$$G_\sigma \sim r^{-\eta}\cos k_F r\,,\quad \sigma=\uparrow,\downarrow\,, \tag{3·3}$$

(d) singlet and triplet pair correlators

$$P_s(r)=P_t(r)\sim r^{-\beta}\,, \tag{3·4}$$

where $k_F$ is the Fermi momentum and the logarithmic corrections have been suppressed.

The critical exponents $\alpha_s$, $\alpha_c$, $\alpha'_s$, $\eta$ and $\beta$ depend on the coupling constant of the forward scattering with zero-momentum transfer. Inspecting the explicit results for these exponents[30)] one finds the universal scaling relations among them

$$\begin{aligned}&\alpha_s=\alpha'_s\,,\\ &\alpha_s=1+\alpha_c/4\,,\\ &\eta=(\alpha_c+4)^2/16\alpha_c\,,\\ &\beta=1+4/\alpha_c\,.\end{aligned} \tag{3·5}$$

Taking the Fourier transform of the electron correlator we obtain the momentum distribution function near $k_F$

$$\langle n_k\rangle=\langle n_{k_F}\rangle-\text{const}|k-k_F|^\theta \text{sgn}(k-k_F)\,, \tag{3·6}$$

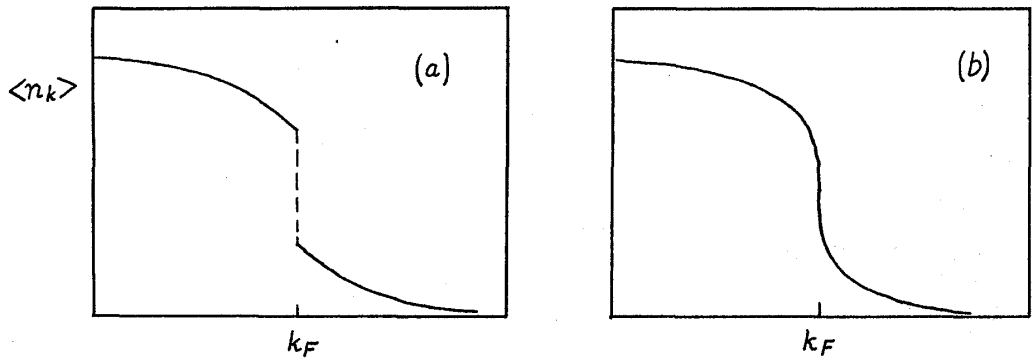


Fig. 1. Momentum distribution function for (a) Fermi liquid and (b) for Luttinger liquid.

where the exponent $\theta$ is given by

$$\theta=\eta-1=(a_c-4)^2/16a_c\,. \tag{3·7}$$

This power-law singularity in the momentum distribution features the non-Fermi liquid nature of the TL model since the Fermi liquid theory predicts the finite jump discontinuity at $k_F$. See Fig. 1.

This means that the quasiparticle picture, the heart of the Fermi liquid theory, breaks down. Instead the low-energy excitations in the TL model are realized as the collective modes. Furthermore the low-energy behavior is characterized by the scaling relations (3·5) and (3·7). The TL model thus possesses the distinctive features not observed in higher dimensions. Haldane pointed out that a large class of 1D quantum critical systems share in fact the common low-energy properties with the TL model. He expected this universality class to cover all 1D femion systems, and then proposed to call such systems Luttinger liquids.[19]

As for strongly correlated systems, however, the issue has not been settled until very recently. Let us consider two models of correlated electrons; the Hubbard model and the $t$-$J$ model. The Hubbard chain describes a system of itinerant electrons interacting through the on-site Coulomb repulsion $U$. The hamiltonian is

$$\mathcal{H}=-t\sum_{\langle ij\rangle,\sigma}c_{i\sigma}^{\dagger}c_{j\sigma}+U\sum_i n_{i\uparrow}n_{i\downarrow}\,,\quad U>0\,, \tag{3·8}$$

where the notation is standard. The hamiltonian of the $t$-$J$ model is given by[31]

$$\mathcal{H}=-t\sum_{\langle ij\rangle,\sigma}c_{i\sigma}^{\dagger}c_{j\sigma}+J\sum_{\langle ij\rangle}\left(\boldsymbol{S}_i\cdot\boldsymbol{S}_j-\frac{n_in_j}{4}\right) \tag{3·9}$$

with an antiferromagnetic coupling $J>0$, and it is assumed that every site is not doubly occupied. It is shown that the strong correlation limit ($U\gg t$) of the Hubbard model is effectively described by the hamiltonian (3·9) with $J\ll t$. The Bethe-ansatz diagonalization was performed for the Hubbard model in the whole parameter region[32] and for the $t$-$J$ model at the special point $t=J$.[33] We hereinafter set $t=J=1$ for convenience.

Let us define the filling factor $\nu$ by $\nu=$(density of electrons)/2. At half-filling $\nu=1/2$ there opens the Hubbard gap in the charge excitation, namely these systems are in the Mott insulating phase. In the long-distance limit, therefore, only the massless

spin excitation survives. The universality class is that of the spin-1/2 AF Heisenberg chain. Corresponding CFT is the $c=1$ $SU(2)$ Kac-Moody theory as mentioned in § 2.1. For $0<\nu<1/2$ both the charge and spin excitations are gapless. Our task is to specify the universality class of this metallic phase.

Recently several groups have carried out the numerical calculations to obtain the correlation functions in 1D highly correlated electron systems for $\nu<1/2$.[34)~37)] In particular the results in Refs. 36) and 37) suggest that the TL scaling relation, which is verified in the weakly correlated system, is also valid for the highly correlated Hubbard chain. Last year several authors have successfully obtained the correlation exponents in the Hubbard model[6)~8)] and the $t$-$J$ model.[9)] The results show explicitly that these models are characterized as the Luttinger liquid and their fixed point is of the TL type. In the following we summarize our main results derived from the Bethe-ansatz solution by using the finite-size scaling technique described in § 2.2.

### 3.1. *Finite-size corrections*

The corrections to the ground-state energy $E_0$ turn out to be [38),9)]

$$E_0 \sim \varepsilon_0 N_a - \frac{\pi v_c}{6N_a} - \frac{\pi v_s}{6N_a}, \tag{3·10}$$

where $N_a$ is the number of the lattice sites. Here $v_c$ and $v_s$ are the charge and spin velocities, explicit formulas of which are available in Refs. 6)~9). From this expression we identify the central charge to be $c=1$ both for charge and spin excitations.

The finite-size corrections to the excitation spectra are expressed in terms of the change of the number of electrons (or down spin electrons) denoted as $I_c$ (or $I_s$) and the number of particles moving from the left "fermi point" to the right one denoted as $D_c$ (or $D_s$) for the charge (or spin) excitations. The energy gap consists of two terms;[38),8),9)] one is proportional to the charge velocity and the other to the spin velocity

$$E-E_0 \sim \frac{2\pi v_c}{N_a}(h_c^+ + h_c^-) + \frac{2\pi v_s}{N_a}(h_s^+ + h_s^-), \tag{3·11}$$

where $h_a^{\pm}$ are the left and right conformal dimensions in the charge ($a=c$) and spin ($a=s$) sectors:

$$h_c^{\pm} = \frac{1}{2}\left(\frac{Z_c}{2}(2D_c+D_s) \pm \frac{1}{Z_c}\frac{I_c}{2}\right)^2 + N_c^{\pm},$$

$$h_s^{\pm} = \frac{1}{4}\left(I_s - \frac{I_c}{2} \pm D_s\right)^2 + N_s^{\pm}. \tag{3·12}$$

Here $Z_c$ is a non-universal function called the dressed charge in the Bethe-ansatz approach. The excitations carry the momentum

$$P = 4k_F D_c + 2k_F D_s + \frac{2\pi}{N_a}\sum_{a=c,s}(h_a^+ - h_a^-). \tag{3·13}$$

The number of low-energy particle-hole excitations is counted by the non-negative integers $N^{\pm}$ which also label the infinite conformal tower.

The dressed charge $Z_c$ is determined from the solution to the Bethe-ansatz integral equation

$$\eta_c(k)=1+\int_{-Q}^{Q}dk'\cos k'\,G(\sin k-\sin k')\eta_c(k') \tag{3·14}$$

for the Hubbard model, while for the $t$-$J$ model at $t=J$

$$\xi_c(\lambda)=1+\int_{-\Lambda}^{\Lambda}d\lambda' R(\lambda-\lambda')\xi_c(\lambda')\,, \tag{3·15}$$

where the kernel is given by $G(x)=K(x;\ U,\ t)$ and $R(x)=K(x;\ 2,\ 1)$ with

$$K(x;\ a,\ b)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\exp(-i\omega x)}{1+\exp(a|\omega|/(2b))}\,. \tag{3·16}$$

The Fermi level $Q$ as well as $\Lambda$ is fixed by the electron concentration. The dressed charge is then defined by $Z_c=\eta_c(Q)$ (or $\xi_c(\Lambda)$) for the Hubbard (or $t$-$J$) model. Therefore all the dependence of scaling dimensions on the microscopic parameters enters in the dressed charge. We also mention that there exists the restriction on the quantum numbers $I_\alpha$ and $D_\alpha$[38),9)]

$$D_c=\frac{I_c+I_s}{2}\bmod 1\,,\quad D_s=\frac{I_c}{2}\bmod 1\,, \tag{3·17}$$

which should be respected when making the identification between field operators and quantum numbers.

### 3.2. *Correlation functions*

The two-point correlation functions of the scaling fields $\Phi_{h^\pm}(r,\ t)$ with conformal dimensions $h^\pm$ are now written

$$\langle\Phi_{h^\pm}(r,\ t)\Phi_{h^\pm}(0,\ 0)\rangle=\frac{\exp(i4k_FD_cr)\exp(i2k_FD_sr)}{(r-iv_ct)^{2h_c^+}(r+iv_ct)^{2h_c^-}(r-iv_st)^{2h_s^+}(r+iv_st)^{2h_s^-}}\,. \tag{3·18}$$

Let us first discuss the charge density correlation function. The number operator $n_i$ will renormalize to a number of scaling fields at long distance. The scaling fields are determined by assigning the quantum numbers $(I_\alpha,\ D_\alpha,\ N_\alpha^\pm)$ to the field operators.[8)] The asymptotic behavior of the equal-time correlator then takes the same form as (3·1) in the TL model. The $4k_F$ piece arises from the excitation of $(I_c,\ I_s,\ D_c,\ D_s)=(0,\ 0,\ \pm1,\ 0)$, whereas the $2k_F$ piece from $(I_c,\ I_s,\ D_c,\ D_s)=(0,\ 0,\ \pm1,\ \mp1)$ and $(0,\ 0,\ 0,\ \pm1)$. The non-oscillating part is due to the lowest particle-hole excitation. We thus find

$$\alpha_c=2Z_c^2\,,\quad \alpha_s=1+\alpha_c/4\,. \tag{3·19}$$

The spin correlation function is also given by (3·2) as in the TL model. The critical exponent $\alpha_s'$ for the $2k_F$ part is equal to $\alpha_s$ of the charge density correlation

$$\alpha_s'=\alpha_s\,. \tag{3·20}$$

Turning to the electron correlator we again find the same form as (3·3) where the $k_F$ oscillation comes from $(I_c,\ I_s,\ D_c,\ D_s)=(1,\ 1,\ 0,\ 1/2)$. Therefore the momentum

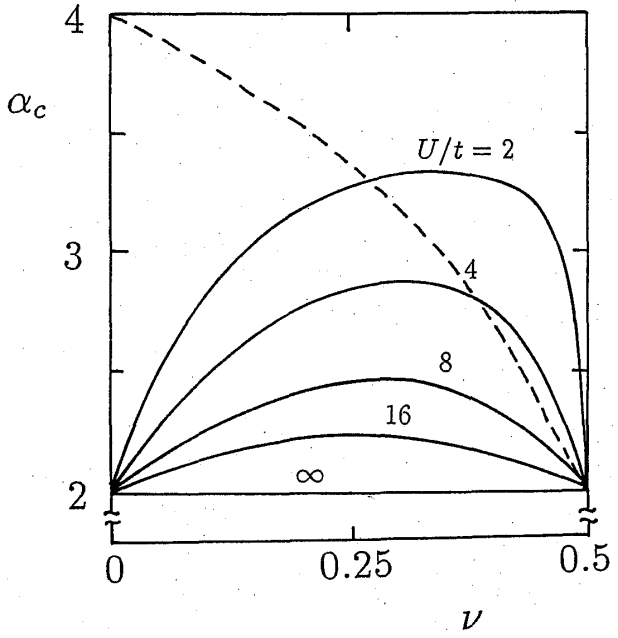


Fig. 2. Critical exponent $\alpha_c$ as functions of electron filling $\nu$: Solid curves for Hubbard model[7] and dashed curve for $t$-$J$ model at $t=J$.[9]

distribution function around $k_F$ exhibits the typical power-law singularity of Luttinger liquids with the exponent

$$\theta=\eta-1=(\alpha_c-4)^2/(16\alpha_c)\,. \qquad (3\cdot21)$$

Examine now the superconducting correlation functions. The singlet and triplet pair correlation functions have the leading uniform term

$$P_s(r,0)=P_t(r,0)\sim r^{-\beta}\,,$$
$$\beta=1+4/\alpha_c\,, \qquad (3\cdot22)$$

which correspond to $(I_c, I_s, D_c, D_s)=(2, 1, 1/2, -1)$ and $(I_c, I_s, D_c, D_s)=(2, 2, 0, 0)$ for $P_s$ and $P_t$, respectively. In Fig. 2 we depict the $4k_F$ CDW exponent $\alpha_c$ both for the Hubbard model and the $t$-$J$ model, from which one can readily read off other exponents using the relations (3·19), (3·21) and (3·22).

Comparing (3·12) with (2·17) one immediately recognizes a typical form for the $c=1$ CFT. In fact there exist two independent $c=1$ CFT's describing separately the charge and spin degrees of freedom. The action is given by

$$S=\frac{1}{2\pi}\sum_{\alpha=c,s}v_\alpha\int dt\int_0^{2\pi}d\sigma\partial_\mu\varphi_\alpha\partial^\mu\varphi_\alpha\,. \qquad (3\cdot23)$$

To fix the gaussian parameter $R_c$ in the charge sector we look at the $4k_F$ charge excitation $(0, 0, \pm1, 0)$. It may be appropriate to call this excitation holon. The corresponding primary field is represented by the vertex operator

$$\Phi_{\pm2,0}(z,\bar{z})=e^{\pm i2Z_c\varphi(\sigma,t)}\,. \qquad (3\cdot24)$$

Comparison with (2·18) gives $R_c=1/Z_c$.

For the spin sector we get from (3·12)

$$h_s{}^{\pm}=\begin{cases}0+\boldsymbol{Z}_{\geq0}\,,\\ \dfrac{1}{4}+\boldsymbol{Z}_{\geq0}\,.\end{cases} \qquad (3\cdot25)$$

This is exactly the spectrum of the $c=1$ $SU(2)$ Kac-Moody theory corresponding to the allowed values of spin 0 and 1/2 for the level 1 highest weight representations. Thus the spin sector stays in the $SU(2)$ Kac-Moody theory for arbitrary filling, $0<\nu\leq1/2$. Notice that the $2k_F$ spin excitation $(0, 0, 0, \pm1)$ is indeed identified as the spin 1/2 highest weight state with conformal weight 1/4. Let us refer to this spin excitation as spinon. It is then clear that the holon and spinon excitations are the elementary massless degrees of freedom in the metallic phase. As a result the electrons are

no longer regarded as elementary, but the composites of holon and spinon.

We see clearly that the TL scaling relations (3·5) and (3·7) hold even for highly correlated systems. Namely, in the metallic phase, the repulsive Hubbard model and the $t$-$J$ model at $t=J$ renormalize to the TL fixed point, and thus they belong to the universality class of Luttinger liquids. In CFT description the TL fixed point consists of two independent $c=1$ CFT's; one is associated with the charge sector which is characterized by continuously varying criticality under the $U(1)$ symmetry and the other represents the spin sector with the $SU(2)$ symmetry.

Finally we remark that in Luttinger liquids the bulk quantities are closely related to the correlation exponents.[19),39),6)~8)] This is a direct consequence of the $U(1)$ and $SU(2)$ Kac-Moody symmetry of the system, and hence the universal characterization of the TL fixed point. One can also study the transport properties by analyzing the finite-size effects under twisted boundary conditions.[40),41)] For further details we refer to original references.

## § 4. Critical systems with long-range interactions

In this section we wish to study the conformal properties of 1D quantum systems with long-range interactions. We first discuss the continuum many-body system and then turn to the anisotropic Heisenberg chain.

### 4.1. *Continuum many-body system*

We consider the $N$-body hamiltonian[42)]

$$\mathcal{H}=-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}+\sum_{j>l}V(x_j-x_l)+\omega^2\sum_{j=1}^{N}x_j^2\,, \tag{4·1}$$

where the potential is of the $1/r^2$ type

$$V(r)=g/r^2\,. \tag{4·2}$$

It is useful to define $\lambda=(1+\sqrt{1+2g})/2$ and we restrict ourselves to the region $\lambda\geq 1/2$. The thermodynamic limit is taken by letting $N\to\infty$, $\omega\to 0$ with $N\omega$ being kept fixed. Sutherland found the ground-state wave function in the Jastrow form[42)]

$$\psi_0=\prod_{i<j}|x_i-x_j|^{\lambda}\exp\left(-\frac{\omega}{2}\sum_i x_i^2\right) \tag{4·3}$$

with the energy eigenvalue

$$E_0=\omega N(1+\lambda(N-1))\,. \tag{4·4}$$

One may recognize that $\psi_0^2$ is identical to the probability distribution function for the eigenvalues of matrices from a gaussian ensemble. For $\lambda=1/2$, 1 and 2 the ensembles are orthogonal, unitary and symplectic types, respectively. On the basis of the results in the theory of random matrices Sutherland obtained the density correlation function whose long-distance behavior is[42)]

$$\langle\rho(r)\rho(0)\rangle\sim\text{const}+A_0r^{-2}+A_2r^{-a}\cos 2k_{\mathrm{F}}r\,, \tag{4·5}$$

where the exponent $a$ is equal to 4, 2 and 1 for $\lambda=1/2$, 1 and 2, respectively.

We now put the same system in the finite geometry with linear size $L$. To implement periodic boundary conditions we modify the form of the potential

$$V(r)\to V_L(r)=g\sum_{n=-\infty}^{\infty}(r+nL)^{-2}=\frac{g\pi^2}{L^2}\left[\sin\left(\frac{\pi r}{L}\right)\right]^{-2}. \tag{4·6}$$

The wave function for the hamiltonian with this finite-size potential again takes the Jastrow form

$$\phi_0=\prod_{i>j}\left|\sin\left(\frac{\pi(x_i-x_j)}{L}\right)\right|^{\lambda} \quad \text{for} \quad x_i>x_j\,, \tag{4·7}$$

which indicates that the two-body scattering is essential to describe the asymptotic form of the many-body scattering states. Moreover the integrability of the system has been proven at the quantum level.[43] These observations imply that we can adopt the Bethe-ansatz idea to describe the scattering states $\psi(x_1,\cdots,x_N)$ in the asymptotic region $x_1\ll\cdots\ll x_i\ll\cdots\ll x_N$. Sutherland called this approach the asymptotic Bethe-ansatz.[43]

This offers a way of systematic construction of the energy spectrum, which is summarized as the set of the algebraic equations á la Bethe ansatz

$$k_jL=2\pi J_j+\pi(\lambda-1)\sum_{l=1}^{N}\mathrm{sgn}(k_j-k_l) \quad \text{for} \quad j=1,2,\cdots,N\,, \tag{4·8}$$

where the quantum number $J_j$ is an integer (or half integer) for the Fermi (or Bose) statistics. The energy and the momentum are given by $E=\sum k_n{}^2$ and $P=\sum k_n$, respectively.

Since the system has no dimensionful parameter we may expect conformal invariance at low energies although the interaction is long range. Let us extract the information about the long-distance properties from (4·8).[10] First of all we evaluate the value of the central charge. We perform the low-temperature expansion of the free energy $F(T)$ since the analysis of the ground-state energy is rather subtle due to the potential form (4·6). The result is $F(T)\simeq F(T=0)-\pi T^2/(6v)$ with the velocity $v=2\pi\lambda N/L$. According to (2·26) the central charge is identified as $c=1$.

In order to obtain the conformal dimensions we compute the finite-size corrections in the energy spectrum. The change of the particle number is denoted as $I$ and the number of particles moving from the left to the right fermi point as $D$. The $1/L$ corrections to the low-energy excitations are

$$E-E_0\sim\frac{2\pi v}{L}\left(\frac{\lambda}{4}I^2+\frac{1}{\lambda}D^2+N^++N^-\right), \tag{4·9}$$

where the non-negative integers $N^{\pm}$ correspond to the simple particle-hole excitation. The associated momentum is

$$P=2\pi k_{\mathrm{F}}D+\frac{2\pi}{L}(ID+N^+-N^-)\,. \tag{4·10}$$

The selection rule for $D$ is $D=I/2$ mod 1 for fermions, while $D\in\boldsymbol{Z}$ for bosons. From

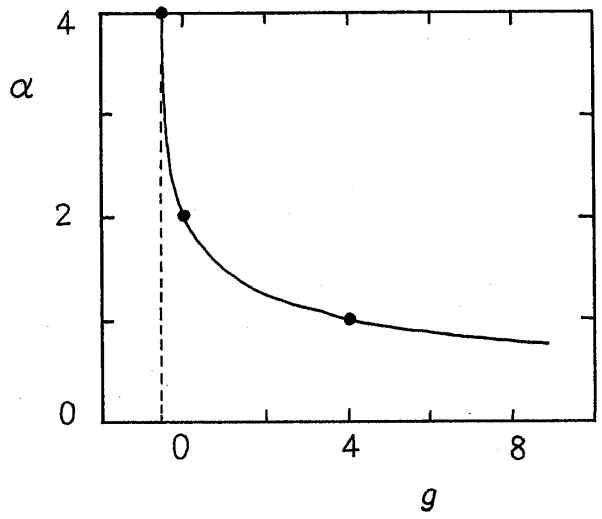


Fig. 3. Critical exponent $\alpha$ as a function of interaction strength $g$.[10] Sutherland's results[42] are represented by ●.

these expressions the conformal dimensions $h^{\pm}$ can be read off[10]

$$h^{\pm}(I;\, D;\, N^{\pm})=\frac{1}{2}\left(\frac{I}{2Z}\pm ZD\right)^2+N^{\pm}, \tag{4·11}$$

where the dressed charge function is $Z=1/\sqrt{\lambda}$. We thus obtain the typical formula for the $c=1$ CFT.

It is now easy to see that the asymptotic form of the denity correlation function is given by (4·5). The $2k_F$ oscillation term arises from the excitation $(I, D, N^{\pm})=(0, 1, 0)$, yielding

$$\alpha=2/\lambda\,, \tag{4·12}$$

which is plotted in Fig. 3. We see that the present result verified for any $\lambda\geq 1/2$ indeed agrees with Sutherland's result at $\lambda=1/2$, 1 and 2. For further discussion of correlation functions we refer to Ref. 10).

### 4.2. *Anisotropic Heisenberg chain*

Haldane[44] and Shastry[45] independently introduced a model of exactly solved spin-1/2 isotropic AF Heisenberg chain with long-range interactions. Let us investigate the anisotropy effect in the Haldane-Shastry model.[11] We take a periodic ring of $N$ sites and the position of the $i$-th site is represented by $x_i$. The hamiltonian is defined by

$$\mathcal{H}=\sum_{i<j}J_{ij}[S_i^xS_j^x+S_i^yS_j^y+\Delta S_i^zS_j^z]\,, \tag{4·13}$$

where the inverse-square exchange $J_{ij}$ is chosen to satisfy periodic boundary conditions. Following (4·6) we set

$$J_{ij}=\frac{\pi^2J}{N^2}\left[\sin\left(\frac{\pi(x_i-x_j)}{N}\right)\right]^{-2} \tag{4·14}$$

with the AF coupling $J>0$.

In the following we consider the case that there is a positive even integer $p$ such that[44]

$$p=\frac{1}{2}(1+\sqrt{1+8\Delta})\,. \tag{4·15}$$

For these particular values of $\Delta$ the ground-state wave function in the Jastrow form has been demonstrated to exist.[44] Applying the method of asymptotic Bethe ansatz one can express the excitation spectra in terms of the pseudomomenta $k_j$. Let $M$ be the number of down spins. The Bethe-ansatz-like algebraic equations for $k_j$ read[46]

$$k_jN=2\pi I_j+\pi(p-1)\sum_{l=1}^{M}\mathrm{sgn}(k_j-k_l) \quad \text{for} \quad j=1\sim M\,, \tag{4·16}$$

where $I_j$ is an integer or a half-odd integer and the second term on the right-hand side is the sum of two-body phase shifts. The total energy is given by

$$E(M,\Delta)=E_s(M,\Delta)+\frac{J}{4}\sum_{j=1}^{M}k_j^2 \tag{4·17}$$

with the constant energy shift

$$E_s=\frac{\pi^2 J}{24N^2}[(N^2-1)(N-4M)\Delta-2(N^2+2)M]\,. \tag{4·18}$$

It is clear that (4·16) does not produce all the eigenstates of (4·13). Since (4·16) describes only the scattering states it is not useful when the system is in the Ising-type ordered state. The scattering states, however, will describe certain massless disordered phase (liquid phase) which may be realized under applied magnetic fields.[11] In order to identify the liquid phase let us first look into (4·16). The $k_j$'s take the values in the region $[-\pi,\pi]$ due to the periodicity of the lattice system. For given $p$ we have $|I_j|<N/2-(p-1)(M-1)/2$. Thus one finds that $k_{j+1}=k_j+2\pi p/N$. As a consequence the allowed maximum number of down spins is $N/p$, which in turn gives the lower bound on the magnetization in the liquid phase, i.e., $\min s_z=1/2-1/p$. Notice that $\min s_z$ is finite for $p>2$. This is quite peculiar in view of the nearest-neighbor interaction models.

We now argue that the spin-liquid phase with the magnetization $s_z\geq 1/2-1/p$ exists in the presence of external magnetic fields. First of all the elementary excitation obtained from (4·16) has the linear dispersion at low energies for $s_z\geq 1/2-1/p$, and hence the system is in the massless phase. Let us next examine the energy shift due to the change $\Delta M$ of the number of down spins. Extracting the term linear in $\Delta M$ gives the magnetization curve as a function of magnetic field $H$

$$s_z=\frac{1}{2}-\frac{2}{\pi p}\sqrt{(H_{c2}-H)/J} \tag{4·19}$$

for $H_{c1}<H<H_{c2}$, where the upper and lower critical fields, $H_{c2}$ and $H_{c1}$, are given by

$$H_{c2}=\frac{\pi^2 J}{6}\left(\Delta(p)+\frac{1}{2}\right),\quad H_{c1}=\frac{\pi^2 J}{6}(\Delta(p)-1)\,. \tag{4·20}$$

At $H_{c2}$ all the spins are fully polarized ferromagnetically, while at $H_{c1}$ the magnetization takes its minimum value $s_z=1/2-1/p$ allowed in the liquid phase. From (4·19) we obtain the differential spin susceptibility

$$\chi_s=\frac{4}{\pi^2 p^2 J}(1-2s_z)^{-1}=\frac{1}{\pi p v}\,, \tag{4·21}$$

where the velocity $v$ of the spin wave excitation is $v=\pi p(1-2s_z)/(4J)$.

Let us consider the low-energy conformal properties of the spin-liquid phase.[11] The analysis is essentially the same as for the previous continuum model in § 4.1. The central charge is obtained from the $T$-linear coefficient of the specific heat. The

result is again $c=1$. The universal $1/N$ corrections to the excitation energy are expressed as

$$E(\Delta M;\,\Delta D)-E(0;\,0)\sim\frac{2\pi v}{N}\left(\frac{p}{4}\Delta M^2+\frac{1}{p}\Delta D^2+N^+ +N^-\right), \tag{4·22}$$

where the quantum number $\Delta D$ is related to the momentum change $2\pi M\Delta D/N$ and the non-negative integers $N^{\pm}$ represent the particle-hole excitations. The momentum carried by the excitation is

$$P=2k_{\mathrm{F}}\Delta D+\frac{2\pi}{N}(\Delta M\Delta D+N^+-N^-), \tag{4·23}$$

where $k_{\mathrm{F}}=\pi M/N$.

The scaling dimensions are read off from the energy spectra. One finds that the asymptotic form of the longitudinal spin correlation function is written

$$\langle S_r^z S_0^z\rangle\sim\mathrm{const}+A_0 r^{-2}+A_2 r^{-\alpha}\cos 2k_{\mathrm{F}}r\,, \tag{4·24}$$

where the $2k_{\mathrm{F}}$ exponent is $\alpha=2/p$ corresponding to the excitation $(\Delta M, \Delta D, N^{\pm})=(0, 1, 0)$. The transverse spin correlation, on the other hand, has the leading non-oscillation term

$$\langle S_r^+ S_0^-\rangle\sim B_0 r^{-\beta}\,, \tag{4·25}$$

where we get the correlation exponent $\beta=p/2$ which follows from $(\Delta M, \Delta D, N^{\pm})=(1, 0, 0)$. We see that the $2k_{\mathrm{F}}$ exponent $\alpha$ in $\langle S_r^z S_0^z\rangle$ becomes small as $\Delta$ increases, implying the tendency to stabilize the spin alignment with the period $\pi/k_{\mathrm{F}}=N/M$. Notice that this alignment will be stabilized as a long-range spin order under certain conditions if favored energetically.

One feature of interest is that there is no dependence of the critical exponents on magnetic fields but only on the anisotropy parameter $\Delta$. This is in marked contrast to the short-range interaction models. We also point out that the critical exponents $\alpha$ and $\beta$ obey the scaling relations inherent in the Luttinger liquid, $\beta=1/\alpha$, reflecting the $U(1)$ symmetry in the spin-liquid phase.[29),19)] Only in the isotropic limit $\Delta=1$ at $H=0$, the $SU(2)$ symmetry is recovered, then $\beta=\alpha=1$. This case has already been discussed in detail in Refs. 44)~46).

We have shown that the spin-liquid phase is realized under external magnetic fields. The low-energy behavior of this phase is described by the $c=1$ CFT, and critical exponents of spin correlation functions have been calculated. For low magnetic fields it is natural to expect the ordered state with a gap to appear. It will be an interesting issue to clarify the nature of this long-range ordered state.

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