# Anyon Superconductivity and Off-Diagonal Long-Range Order

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## § 1. Introduction to fractional statistics

Particles obeying fractional statistics[1)~3)] can, in two space dimensions form an exotic quantum fluid which can condense into a new type of superconductor.[4)] The simplest realization of a fractional statistics particle or 'anyon' consists of an ordinary boson (or fermion) bound to a magnetic flux tube (oriented perpendicular to the 2D plane) as shown in Fig. 1.

If one anyon is dragged adiabatically around another in a closed counterclockwise path, the quantum state acquires an extra (Berry's) phase due to the Aharanov-Bohm effect. If we call this phase factor $\exp(i2\theta)$ then adiabatic counterclockwise exchange of two particles will yield an extra phase $\exp(i\theta)$. Hence $\theta$ (whose value is determined by the particle charge and the strength of the flux tube) is called the statistics angle. If $\theta=\pi$ and the original particles were bosons, the composite particles are fermions since they yield a minus sign up exchange. There are no quantization conditions which constrain the strength of the flux tube and hence $\theta$ is continuously adjustable. Systems for which $\theta/\pi$ is fractional are said to obey fractional statistics. We shall be particularly concerned here with the special case of 'semions' which have $\theta/\pi=1/2$.

The Hamiltonian for a system of $N$ non-relativistic anyons is

$$H=\frac{1}{2M}\sum_{j=1}^{N}\left[\boldsymbol{p}_j+\frac{e}{c}(\boldsymbol{A}+\boldsymbol{a}_j)\right]^2+V\,, \tag{1}$$

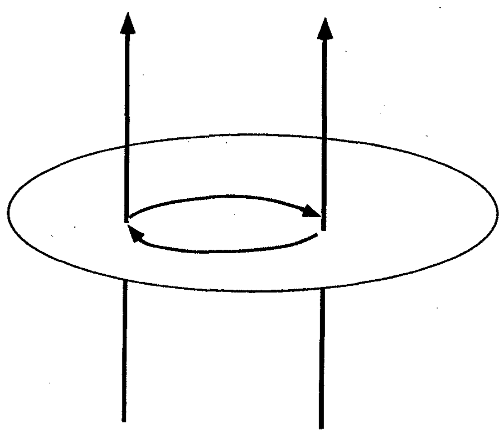

Fig. 1. A pair of particles with their attached flux tubes exchange positions in the 2D plane. The Aharanov-Bohm effect generates a phase factor $\exp(i\theta)$ during this process.

where $V$ is an arbitrary scalar potential (either external or interaction), $\boldsymbol{A}$ is the vector potential associated with the externally applied magnetic field

$$\nabla\times\boldsymbol{A}=\boldsymbol{B}_{\text{ext}}\,, \tag{2}$$

and the vector potential seen by the $j$th particle due to the other particles obeys

$$\nabla_j\times\boldsymbol{a}_j=\frac{\theta}{\pi}\Phi_0\sum_{k\neq j}\delta^2(\boldsymbol{r}_k-\boldsymbol{r}_j)\,, \tag{3}$$

where $\Phi_0\equiv hc/e$ is the flux quantum. The fact that $\boldsymbol{a}_j$ is curl-free "almost everywhere" means that we can remove it by means of a singular gauge transformation

$$\tilde{\psi}(z_1, \cdots, z_n) = \prod_{i<j}\left[\frac{(z_i - z_j)}{|z_i - z_j|}\right]^{\theta/\pi} \psi(z_1, \cdots, z_n), \tag{4}$$

where $z_j = x_j + iy_j$ is a complex number representing the 2D coordinate, $\psi$ is the original boson wave function and $\tilde{\psi}$ obeys (assuming $V=0$) the free-particle Schrödinger equation with Hamiltonian

$$\tilde{H} = \frac{1}{2M}\sum_{j=1}^{N}\left(\boldsymbol{p}_j + \frac{e}{c}\boldsymbol{A}\right)^2. \tag{5}$$

This seems like a great simplification until we realize that $\tilde{\psi}$ obeys very difficult boundary conditions due to the fact that it is multiple-valued. For the special case of converting bosons to fermions ($\theta = \pi$), $\tilde{\psi}$ is single-valued. However, to eliminate the bad effect of the singular phase factor we need a hard-core repulsion among the bosons to cause $\psi$ and hence $\tilde{\psi}$ to vanish when any two particles come together.

The surprising message here is that in two dimensions statistics are ambiguous. Statistics can be represented by an exchange symmetry of the wave function or by long-range vector potential interactions among the particles. Of course free bosons are distinguishable from free fermions because there is no Pauli exclusion but we can cure this by adding a hard-core repulsion. This hard-core repulsion also appears in the path-integral formulation of the problem where one requires that the braiding of the world-lines of the particles be well-defined.[3)]

In order to see that bosons with flux tubes really act like fermions let us examine the following highly-simplified model. Consider a rigid rotor with a boson fixed at each end. We ignore the flux tubes for the moment. We invoke the hard-core exclusion by keeping the particles a fixed distance apart. This neglect of the radial degree of freedom greatly simplifies the problem without affecting the essential physics which is found in the angular degree of freedom.

The Hamiltonian is

$$H = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\varphi}\right)^2, \tag{6}$$

where $I$ is the moment of inertia and $\varphi$ is the angular coordinate. The energy eigenfunctions are easily found

$$\psi_m(\varphi) = e^{im\varphi}. \tag{7}$$

The corresponding eigenvalues are

$$\epsilon_m = \frac{\hbar^2}{2I}m^2, \tag{8}$$

where $m = 0, \pm 1, \pm 2, \cdots$ are the angular momentum eigenvalues. We invoke the fact that the particles are bosons (that is, the wave function is exchange symmetric) by requiring that $\psi$ be symmetric under $\varphi \to \varphi + \pi$. Thus $m$ must be *even*. For fermions, $m$ would have to be odd. This is analogous to the symmetry requirements in ortho- and para-hydrogen.

If we now add flux tubes to each of the particles, and make a judicious choice of gauge, the Hamiltonian becomes

$$H=\frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\varphi}+\frac{\theta}{\pi}\right)^2, \tag{9}$$

where the "offset" in the mechanical angular momentum represents the extra Aharanov-Bohm phase acquired by the rotor as it turns.

It is now clear that for $\theta/\pi=1$ the boson spectrum maps into the fermi spectrum and vice-versa. One cannot tell whether a given spectrum is due to bosons with flux tubes or due to fermions. This is true, not just for the energy, but for all other observables as well. For example, the fermions have a non-zero current ("fermi velocity") in the ground state ($m=1$, $\theta/\pi=0$). The bosons have zero angular momentum ($m=0$) but a finite diamagnetic current due to the flux tubes ($\theta/\pi=1$). In either case the *mechanical* angular momentum (velocity) is the same.

This two-particle example makes the case that statistics are ambiguous in 2D but perhaps seems rather trivial. The many-body problem for anyons is highly non-trivial however. After all, the problem is equivalent to bosons with long-range (vector) interactions. One signature of the complications can be seen by considering the phase acquired by the system upon exchange of two of the $N$ particles *along some particular path*:

$$X=e^{i\theta}e^{2i\theta N_I}. \tag{10}$$

The first factor is from the exchange. The second factor counts the number of particles $N_I$ which are in the interior of the region bounded by the path. Thus we have the added complication that the phase $X$ depends in detail on the particular path. This complication makes it difficult to estimate amplitudes for path integrals, ring exchanges, and so forth. Of course for the particular cases $\theta=0$ and $\theta=\pi$ the exchange phase is independent of $N_I$ as expected for ordinary bosons and fermions.

While the $N$-body problem has so far proved largely insoluble, it is generally believed that systems of anyons have a very rich phase diagram with special condensed states forming at rational fractional values of $\theta/\pi$ (in a manner reminiscent of the dependence of the fractional quantum Hall state on filling factor $\nu$). In particular, Laughlin has proposed that for the case of semions ($\theta=\pi/2$) *pairs* of particles condense to form a new type of superconductor.[4] The essence of the argument is that while a semion is in a sense, 'half a fermion ', a pair of semions actually form a boson (which can then condense). Exchange of two semions gives a phase $e^{i\theta}=i$, while exchange of one pair of semions with another gives, not $e^{2i\theta}=-1$ but rather $e^{4i\theta}=+1$. The reason for this is that each semion in one pair sees two flux tubes in the other. The phase contributed by one is thus doubled, but so is the phase contributed by the *other* member of the pair. Hence the total phase is quadrupled.

Since pairs of semions are bosons, they can form a charge $2e$ condensate. The proper description of ordering within this condensate is non-trivial and will be discussed in detail shortly. One immediate observation is that the condensate may, depending on the particular model, break time-reversal symmetry. This is because counter-clockwise exchange of two semions gives $\exp(i\theta)=i$ while a clockwise exchange gives $\exp(-i\theta)=-i$. Such $P$ and $T$ breaking could in principle be observed in scattering experiments with circularly polarized light. Two groups[5),6)] have reported non-trivial signatures of $P$ and/or $T$ breaking, but the experiment which appears to be the "cleanest" has found a null result.[7)]

The question naturally arises as to how anyons and anyon superconductors could occur in nature since we have been talking about unphysical particles with infinitesimal flux tubes attached to them. It turns out that in certain quantum condensates of ordinary particles, the elementary excitations or quasiparticles can obey fractional statistics. Anyons clearly have special topological significance associated with their interactions. Moving one anyon in the presence of another gives no effect unless the path of the one winds around the other. Thus we are lead to consider condensates whose elementary excitations possess some nontrivial topological feature. This gives us really only one possible candidate —— the vortex. A vortex is a topological defect around which the phase of the condensate has a non-zero winding. A simple variational ansatz for such an object is

$$\Psi=\prod_{j=1}^{N} e^{in_W\varphi_j}\Psi_0\,, \tag{11}$$

where $\varphi_j$ is the azimuthal angle of the $j$th particle relative to the vortex position, $\Psi_0$ is the ground state wave function and $n_W$ is the topological winding number, which must be an integer to preserve the single-valuedness of the wave function. We see that the vortex does indeed act like a flux tube in the sense that there is a topologically non-trivial Berry's phase for moving particles around closed loops in the presence of the vortex. Unfortunately, the vortex acts like a flux tube containing an integer number of flux quanta since Berry's phase, $\pm 2\pi n_W$, for dragging a particle around the vortex, is a multiple of $2\pi$. Recall that for the previous construction of an anyon we used an integer charge and a fractional flux tube to obtain a fractional statistics angle $\theta/\pi$. Mother nature uses a different recipe: an integer flux tube (a vortex) and a *fractional charge*.

The prototypical example occurs in the fractional quantum Hall effect (FQHE), where the condensate has vortex excitations which happen for dynamical (energetic) reasons to carry fractional charge.[8)] These objects have non-trivial exchange phases[9)] and for the fractional state at filling factor $\nu=1/m$, carry charge $e^*=\pm e/m$ and statistics angle $\theta/\pi=1/m$. The sign of the charge is associated with the handedness of the vortex and reflects the underlying broken $T$ and $P$ symmetry due to the externally applied magnetic field. In anyon superconductors one expects that the anyon physics arises from *spontaneously* broken $P$ and $T$ symmetry at low temperatures.

There is a deep connection between the fact that vortices carry charge and the existence of a quantized Hall coefficient. Consider the following gedanken experiment.[10)] We have a system on the Hall plateau characterized by transport coefficients

$$\sigma_{xx}=0\,, \tag{12a}$$

$$\sigma_{xy}=\nu\frac{e^2}{h}\,. \tag{12b}$$

Since we have zero dissipation, we may adiabatically (reversibly) increase the magnetic field in some region $\Gamma$. Faraday's law gives us the line integral of the electric field around the boundary of this region

$$\oint_{\partial\Gamma} \boldsymbol{E}\cdot d\boldsymbol{r} = -\frac{1}{c}\frac{d\Phi}{dt}, \tag{13}$$

where $\Phi(t)$ is the total added flux. From the transport coefficients we see that there is a well-defined current flowing at right angles to this electric field and hence into or out of the region $\Gamma$. The rate of change of the charge obtained from the continuity equation is

$$\frac{dQ}{dt} = \sigma_{xy}\frac{1}{\Phi_0}\frac{d\Phi}{dt}. \tag{14}$$

For each quantum of flux that we add to the system an additional vortex is induced in the state. But we also see that charge $e^* = \nu e$ also appears in the region $\Gamma$. This charge is bound to the vortices and can be seen explicitly in the Laughlin variational wave function for the quasi-particles.

This binding of charge to vorticity is the key to understanding the ordering in the FQHE.. It implies that the Ginzburg-Landau description of the FQHE must be a Chern-Simons topological field theory.[11)~15)] Once we understand the nature of the ordering in the FQHE, it is a relatively small step to describe the ordering in anyon superconductors.

One difficulty with the 'anyons as vortices' model is that the votices exist within a background condensate of particles which cause the vortex to see a uniform background magnetic field.[9)] This is just what is needed for the FQHE but must be neglected to achieve anyonic superconductivity, a point which has been largely ignored in the literature. It may be possible that lattice effects (such as species doubling[16)]) can be used to eliminate the background magnetic field, but this has not as yet been clearly demonstrated.

## § 2. Soluble model of semion superconductivity

We turn now to a discussion of a soluble model of semion superconductivity.[17)] The model differs from the usual one with hard-core repulsion in that it has an *attractive* short-range interaction. The model is soluble only for a special value of the strength of the attraction and hence unfortunately is *not* adiabatically deformable into the model we really want to solve, in which the world lines of the particles can never interact. Nevertheless it is of interest because there exist very few soluble many-body problems in two dimensions and because the nature of the solutions suggests a good variational wave function for the problem with repulsive interactions. The present model also teaches us a great deal about the mean-field theory of statistics and was inspired by notions of supersymmetry and the Atiyah-Singer index theorem which has been applied to the problem of a single 2D electron in an arbitrary position-dependent magnetic field.[18)] Jackiw and Pi[19)] have recently investigated soliton solutions at the classical level in this same model. Interesting related work has also been carried out by Greiter and Wilczek.[20)]

Let us modify the Hamiltonian in Eq. (1) by giving the flux tube a finite diameter so that Eq. (3) becomes

$$b_j \equiv \nabla_j \times \boldsymbol{a}_j = \frac{\theta}{\pi}\Phi_0 \sum_{k\neq j} F(\boldsymbol{r}_k - \boldsymbol{r}_j)\,, \tag{15}$$

where $F$ is the flux tube 'form factor' which is a smooth, circularly symmetric function obeying

$$\int d^2 r F(\boldsymbol{r}) = 1\,. \tag{16}$$

Now let us invoke a special scalar interaction among the particles of the form

$$V_\pm = \mp \sum_i g\mu_B (b_i + B_{\rm ext})\,, \tag{17}$$

where $g=2$ and $\mu_B \equiv e\hbar/2Mc$ is the Bohr magneton. This potential corresponds to the Zeeman energy of a Dirac particle in the local magnetic field. [The sign choice in Eq. (17) will ultimately determine whether we are considering attractive or repulsive scalar interactions among the anyons and is not intended to represent the Zeeman energy.] With this special choice of interaction, the Hamiltonian factorizes into the 'supersymmetric' (SUSY) form

$$H^\pm = \sum_j (Q_j^\pm)^\dagger (Q_j^\pm)\,, \tag{18}$$

where

$$Q_j^\pm \equiv \Pi_j^x \mp i\Pi_j^y\,, \tag{19}$$

and

$$\boldsymbol{\Pi}_j \equiv \boldsymbol{p}_j + \frac{e}{c}(\boldsymbol{A} + \boldsymbol{a}_j)\,. \tag{20}$$

If we can find a normalizable function which obeys, for every $j$, the first-order differential equation

$$Q_j^\pm \Psi^\pm[\boldsymbol{r}] = 0\,, \tag{21}$$

where $[\boldsymbol{r}] \equiv (\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$, then we have an exact, zero-energy eigenfunction of the Hamiltonian. This result is a generalization of the supersymmetry of a single electron in an arbitrary magnetic field[18)] to the non-trivial $N$-body case in which the flux is *not fixed in time but is carried on the particles themselves.* In order to find the appropriate solution, it is useful to define the function $S$ by

$$\boldsymbol{A} + \boldsymbol{a}_j = -\hat{z} \times \nabla_j S[\boldsymbol{r}]\,, \tag{22}$$

and impose the gauge choice $\nabla \cdot (\boldsymbol{A} + \boldsymbol{a}) = 0$. Equation (15) now yields

$$\nabla_j^2 S[\boldsymbol{r}] = -\frac{2\pi}{\Phi_0}(B_{\rm ext} + b_j)\,. \tag{23}$$

We may interpret this as Poisson's equation for the electrostatic potential of a plasma with charge density

$$\rho(\boldsymbol{r})=\frac{1}{\Phi_0}(B_{\text{ext}}+\sum_j b_j)\,. \tag{24}$$

A little algebra shows that the state

$$\Psi^+=f[\bar{z}]e^{+S}\,, \tag{25}$$

where $f$ is any entire function of the $\bar{z}$'s (with $\bar{z}_j$ being the 2D complex coordinate $\bar{z}_j \equiv x_j - iy_j$) is annihilated by $Q_j^+$ for every $j$. Likewise the state

$$\Psi^-=f[z]e^{-S} \tag{26}$$

is annihilated by $Q_j^-$ for every $j$. The solution of Eq. (23) is readily obtained from the 2D Coulomb Green's function

$$\nabla^2(-\ln|\boldsymbol{r}|)=-2\pi\delta^2(\boldsymbol{r})\,. \tag{27}$$

Using this we obtain

$$S[\boldsymbol{r}]=-2\pi\frac{B_{\text{ext}}}{\Phi_0}\sum_j\frac{1}{4}|z_j|^2+\sum_{i<j}v(\boldsymbol{r}_i-\boldsymbol{r}_j)\,, \tag{28}$$

where

$$v(\boldsymbol{r}_i-\boldsymbol{r}_j)\equiv-\frac{\theta}{\pi}\int d^2r F(\boldsymbol{r}_i-\boldsymbol{r})\ln|\boldsymbol{r}-\boldsymbol{r}_j|\,. \tag{29}$$

While the above results are valid for arbitrary form factors it is convenient for the moment to consider the limit of point charges

$$F(\boldsymbol{r})\to\delta^2(\boldsymbol{r})\,, \tag{30}$$

although one has to be careful with questions of the self-adjointness of $H$ in this limit.[21] Equation (28) then becomes

$$S[\boldsymbol{r}]=-\frac{1}{4l^2}\sum_j|z_j|^2-\frac{\theta}{\pi}\sum_{i<j}\ln|\boldsymbol{r}_i-\boldsymbol{r}_j|\,, \tag{31}$$

where

$$\frac{1}{l^2}=\frac{2\pi B_{\text{ext}}}{\Phi_0}\,. \tag{32}$$

The ground state of $H^+$ may now be written

$$\Psi^+=f[\bar{z}]\exp\left(-\frac{1}{4l^2}\sum_j|z_j|^2\right)\prod_{i<j}|z_i-z_j|^{-\theta/\pi}\,, \tag{33}$$

and that of $H^-$ is

$$\Psi^-=f[z]\exp\left(+\frac{1}{4l^2}\sum_j|z_j|^2\right)\prod_{i<j}|z_i-z_j|^{+\theta/\pi}\,. \tag{34}$$

In the limit of Eq. (30), $H^-$ corresponds to hard-core repulsively interacting anyons and $\Psi^-$ is normalizable only in the presence of an external magnetic field ($l^2$, $B_{\rm ext}<0$). These solutions were known previously.[22]

We see from Eq. (33) that $\Psi^+$ has a cusp (possibly regularized by the form factor $F$) when two particles approach each other. This is imposed by the short-range scalar attraction in $H^+$. In the absence of an external field, the degree of homogeneity of the polynomial $F$ cannot be made large enough to give zeros whenever any two particles approach. Hence the cusp is unavoidable. If $F$ is appropriately chosen however, $\Psi^+$ is normalizable even in the absence of an external field.

We are interested here in a specific extension of $\Psi^+$ to the case of semions ($\theta/\pi=1/2$) which carry a spin quantum number, or a flavor quantum number related to species doubling on the lattice.[17),16),23),24] We choose (at first) a singular gauge in which every particle carries the same gauge charge and a flux tube corresponding to statistics angle $\theta/\pi=1/2$. The Hamiltonian is then spin-independent and we seek the spin-singlet-pairing superconducting ground state. In the fermion representation ($\Phi$ antisymmetric) we have

$$\Phi=\frac{1}{\sqrt{(2N)!}}\mathcal{A}\Psi[z](\alpha,\ \alpha,\cdots,\ \alpha,\ \beta,\ \beta,\cdots,\ \beta)\,, \tag{35}$$

where $\alpha$, $\beta$ are the 'up' and 'down' spinors respectively and $\mathcal{A}$ is the antisymmetrizer. Let us take the spatial part of the wave function to be

$$\Psi[z]=\prod_{i<j}(\bar{z}_i-\bar{z}_j)(\bar{z}_{[i]}-\bar{z}_{[j]})e^S\,, \tag{36}$$

where $i=1, 2, \cdots, N$ and $[j]=N+1, N+2, \cdots, 2N$ refer to spin up and down respectively. Using Eq. (33) the plasma factor is

$$e^S=\prod_{i<j}|z_i-z_j|^{-1/2}|z_{[i]}-z_{[j]}|^{-1/2}\prod_{k,l}|z_k-z_{[l]}|^{-1/2}\,. \tag{37}$$

Notice that the polynomial factor $F$ in Eq. (36) is precisely the one that appears in the mean-field approximation in which the statistical flux tubes are replaced by a uniform magnetic field and for $\theta/\pi=1/2$, both spin states of the lowest Landau level are fully occupied. This guarantees that in the present case $\Phi$ obeys the Fock cyclic condition[25] and is a spin singlet. From the generalized index theorem, $\Phi$ is an exact zero-energy eigenfunction of the semion Hamiltonian with appropriate attractive scalar interactions among the particles. [We will comment on the normalizability of $\Phi$ further below.]

The antisymmetrizer in Eq. (35) is rather inconvenient to deal with. Fortunately, provided that we are interested only in matrix elements of operators which do not flip spins, we can ignore the antisymmetrization and deal only with $\Psi$, the spatial part of the wave function. [This is because nontrivial permutations of the spinors yield orthogonal states.] Combining Eqs. (36) and (37) we have the remarkable result that

$$|\Psi|^2=\exp(-\beta\sum_{i<j}^{2N}(q_iq_j)\ln|z_i-z_j|)\,, \tag{38}$$

where the sum runs over all $2N$ particles, $\beta=1$, and $q_i=\pm 1$ for up and down spins, respectively. As for Laughlin's FQHE state[26] we have a plasma analogy but in this case it is to the two-dimensional two-component (neutral) Coulomb gas, with spin playing the role of Coulomb charge. The statistical mechanics here is quite different from that of Laughlin's one-component incompressible plasma. The two-component neutral Coulomb gas undergoes the Kosterlitz-Thouless transition at coupling constant $\Gamma=4$ (where $\Gamma\equiv\beta q^2$). For $\Gamma>4$ the charges (here spins) are bound together in (real-space) pairs and one has a spin 'insulator'. For $\Gamma<4$ there is spin 'deconfinement' into a 'metallic' phase. The present model has $\Gamma=1$ and so the quantum ground state is in the high-temperature metallic phase. The spins are not bound together and hence spin currents are free to flow. As in an ordinary metal, however, there is perfect metallic screening of isolated charges (spins) with a screening wave vector given in the Debye approximation by[27]

$$K^2=2\pi n_0\beta q^2\,, \tag{39}$$

where $n_0$ is the mean particle density. The implications of this screening will be discussed further below.

We return for a moment to the question of the normalizability of $\Psi$. Clearly there is a cusp in $|\Psi|^2$ of the form $|\boldsymbol{r}_i-\boldsymbol{r}_j|^{-1}$ when opposite spin particles approach. This is integrable in two dimensions so there are no 'ultraviolet' problems with the wave function. The infra-red behavior is more subtle.[17] Clearly the norm of the wave function is equivalent to the partition function of the corresponding plasma. This is ill-defined unless we impose a characteristic scale through periodic boundary conditions. To see this assume the plasma is self-bound and move particle $j$ off towards infinity. $|\Psi|^2$ vanishes only like $|\boldsymbol{r}_j|^{-1}$ which gives a divergent infra-red contribution. This is readily cured by the imposition of periodic boundary conditions which effectively places the system on a torus. [There is a technical subtlety in dealing with semion flux tubes on compact surfaces which requires the introduction of a global two-state degree of freedom.[28]]

The perfect screening of spin discussed above implies that this state is not merely a singlet but a *local singlet* in a precise sense which we now discuss. Let us start with what a local singlet is *not*. Imagine a magnet divided into two halves each cotaining $N$ spins ($s=1/2$). Couple the spins in each half ferromagnetically to a state of total spin $S=N/2$. Now couple these two large spins into a singlet. This is a *global* singlet but not a *local* singlet since the spins in any small neighborhood tend to be parallel. A local singlet is defined as a state in which the coarse-grained spin density tends to zero. We define this coarse-graining by

$$\boldsymbol{s}_i=\sum_{j=1}^{N}w_{ij}\boldsymbol{S}_j\,, \tag{40}$$

where $\boldsymbol{S}_j$ is the spin operator at lattice site $j$ and $w_{ij}$ is a non-negative weight factor which falls of smoothly with the separation of sites $i$ and $j$, and obeys the normalization condition

$$\sum_j w_{ij}=1\,. \tag{41}$$

The mean square value of the coarse-grained spin is measured by

$$\langle \boldsymbol{s}_i\cdot\boldsymbol{s}_j\rangle=\sum_{j,k=1}^{N} w_{ij}w_{ik}G_{jk}\,, \tag{42}$$

where $G$ is the spin correlation function

$$G_{jk}\equiv\langle \boldsymbol{S}_j\cdot\boldsymbol{S}_k\rangle\,. \tag{43}$$

Thus the coarse-grained spin will tend to zero only if the spin-spin correlation function has certain properties. Any liquid (i.e., translationally invariant) singlet state will automatically satisfy

$$\sum_j G_{ij}=0\,. \tag{44}$$

However local singlets obey the more stringent condition that the coarse-grained spin decays rapidly to zero as the length scale of the smoothing is increased. This requires that we be able to define a characteristic (finite) size of the spin screening cloud by

$$\sum_j G_{ij}|\boldsymbol{r}_i-\boldsymbol{r}_j|^2=R^2\sim K^{-2}<\infty\,. \tag{45}$$

We see here in Eqs. (44) and (45) the analogs for the spin problem of the 'charge neutrality' and 'perfect screening' sum rules for the Coulomb plasma.[29)~31)] The two-component neutral plasma obeys these sum rules and hence the quantum state we have found is a *local* spin singlet.

The classical Coulomb plasma exhibits a finite frequency plasmon oscillation mode due to the long-range Coulomb interaction. The quantum semion model considered here has an analogous spin vave excitation gap. Within the single-mode approximation[32)] (SMA) one has a triplet of spin wave excited states defined by

$$\Psi_k^{\mu}=\rho_k^{\mu}\Phi\,, \tag{46}$$

where $\Phi$ is the ground state, $\mu=x,\ y,\ z$ are the three spin components and

$$\rho_k^{\mu}\equiv\sum_{j=1}^{2N} S_j^{\mu}e^{i\boldsymbol{k}\cdot\boldsymbol{r}_j} \tag{47}$$

is the Fourier transform of the $\mu$th component of the spin density. Taking advantage of the spin rotational invariance, the variational energy estimate for these three degenerate modes may be written as

$$\Delta(k)=\frac{f(k)}{s(k)}\,, \tag{48}$$

where $f(k)\equiv(3/4)\hbar^2k^2/2M$ is the oscillator strength and

$$s(k)=\frac{1}{N}\sum_{\mu}\langle\Phi|\rho_{-k}^{\mu}\rho_k^{\mu}|\Phi\rangle \tag{49}$$

is the spin structure factor which is the Fourier transform of the correlation function $G$ given in Eq. (43). Within the Debye screening approximation

$$s(k)=(3/4)\frac{k^2}{k^2+K^2}, \tag{50}$$

we see that the spin structure function vanishes at small wave vectors due to the long-range Coulomb forces and from Eq. (48) we have a finite 'spin-plasmon' gap given by

$$\Delta(k)=\frac{\hbar^2}{2M}(k^2+K^2). \tag{51}$$

The spin wave mode consists of out-of-phase density oscillations of the up and down spin fluids. The in-phase or density mode produces no long-range 'electric fields' and hence is gapless.[33] This is the Goldstone mode associated with the superconductivity. Superconductivity ordinarily manifests itself as (algebraic in 2D) off-diagonal long-range order (ODLRO) in the two-body density matrix. In second quantized language this means that there is a significant amplitude to destroy a single pair at one location and create a pair far away without perturbing the state of the system

$$\rho(\boldsymbol{r},\ \boldsymbol{r}')=\langle c_\uparrow^\dagger(\boldsymbol{r}')c_\downarrow^\dagger(\boldsymbol{r}')c_\downarrow(\boldsymbol{r})c_\uparrow(\boldsymbol{r})\rangle\sim|\boldsymbol{r}-\boldsymbol{r}'|^{-\eta}, \tag{52}$$

where the exponent $\eta$ vanishes in the limit of zero temperature. In the language of first quantization using the ground state wave function we may write

$$\begin{aligned}\rho(\boldsymbol{r},\ \boldsymbol{r}')=N^2\int d^2r_2\cdots\int d^2r_N\int d^2r_{[2]}\cdots\int d^2r_{[N]}\\ \times\Psi^*(\boldsymbol{r},\ \boldsymbol{r}_2,\cdots,\ \boldsymbol{r}_N;\ \boldsymbol{r},\ \boldsymbol{r}_{[2]},\cdots,\ \boldsymbol{r}_{[N]})\\ \times\Psi(\boldsymbol{r}',\ \boldsymbol{r}_2\cdots,\ \boldsymbol{r}_N;\ \boldsymbol{r}',\ \boldsymbol{r}_{[2]},\cdots,\ \boldsymbol{r}_{[N]}).\end{aligned} \tag{53}$$

The analogous objects for anyons are slightly more complicated. When we create an anyon we create both a fermion (say) and an associated flux tube. We have learned from the theory of ODLRO in the FQHE[12),34] that the analog of Eq. (53) for anyons is

$$\begin{aligned}\rho(\boldsymbol{r},\ \boldsymbol{r}')=N^2\int d^2r_2\cdots\int d^2r_N\int d^2r_{[2]}\cdots\int d^2r_{[N]}\\ \times\Psi^*(\boldsymbol{r},\ \boldsymbol{r}_2,\cdots\boldsymbol{r}_N;\ \boldsymbol{r},\ \boldsymbol{r}_{[2]},\cdots,\ \boldsymbol{r}_{[N]})\\ \times\exp\left[+i\frac{e}{\hbar c}\int_r^{r'}d\boldsymbol{R}\cdot(\boldsymbol{a}_1(\boldsymbol{R})+\boldsymbol{a}_{[1]}(\boldsymbol{R}))\right]\\ \times\Psi(\boldsymbol{r}',\ \boldsymbol{r}_2,\cdots,\ \boldsymbol{r}_N;\ \boldsymbol{r}',\ \boldsymbol{r}_{[2]},\cdots,\ \boldsymbol{r}_{[N]}).\end{aligned} \tag{54}$$

It is straightforward to show that, in contrast to the FQHE, this correlation function exhibits *true* (not just algebraic) ODLRO. To see this consider what happens to the wave function in Eq. (36) when particle 1 and particle [1] are brought close together[35] as shown in Fig. 2. The phase factor does not depend on the relative orientation of the pair so they form an $S$-wave singlet as expected. Furthermore, the pair is 'charge neutral' according to our plasma analogy. Hence *the amplitude of $\Psi$ is independent of the location of the pair relative to the other particles.* Note however that as any

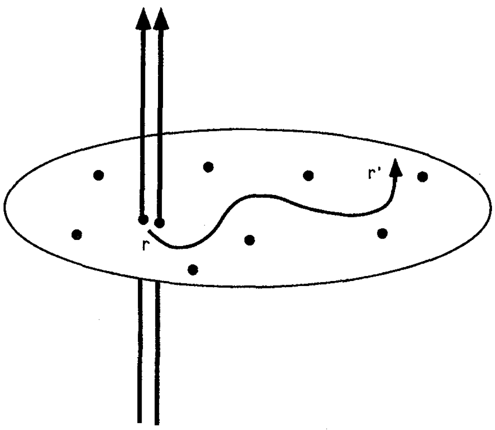


Fig. 2. Off-diagonal long-range order of singlet pairs in the superconductor. Particles 1 and [1] are brought close together and then dragged from position $\boldsymbol{r}$ to $\boldsymbol{r}'$. The amplitude of the wave function is independent of the position of the pair. The phase fluctuates wildly because the pair looks like a vortex to the other particles. However this is precisely cancelled by the vector potential phase factor.

other particle (either spin up or spin down) circles the pair counterclockwise there is a phase winding of $-2\pi$. Thus each pair looks like an antivortex to the other particles. Conversely, the pair sees antivortices at the position of the other particles as it moves among them. Normally this would cause wild phase fluctuations which would destroy the ODLRO. However, as we show below, these are precisely compensated by the vector potential term in Eq. (54), in a manner similar to that which occurs in the FQHE.[12),34)]

Notice that the curl of $\boldsymbol{a}_1$ and $\boldsymbol{a}_{[1]}$ vanish 'almost everywhere' but nevertheless the delta-function fluxes make the line integral in Eq. (54) path-dependent and hence ill-defined. Note however that if the path of the pair circles a third particle, the vector potential line integral contributes

$$\oint d\boldsymbol{R}\cdot(\boldsymbol{a}_1(\boldsymbol{R})+\boldsymbol{a}_2(\boldsymbol{R}))=4\theta=+2\pi\,. \tag{55}$$

Thus while the line integral in Eq. (54) is path-dependent, the exponential is well-defined and unambiguous. Furthermore, the $+2\pi$ in Eq. (55) exactly cancels the $-2\pi$ that would be obtained from the gradient of the phase along the same path

$$\oint d\boldsymbol{R}\cdot\nabla \mathrm{Im}\ \ln\ \Psi=-2\pi\,. \tag{56}$$

Hence the vector potential term in Eq. (54) exactly cancels the effects of the antivortices.

The upshot of all this is that a pair of semions is completely 'gauge neutral' and the amplitude and phase of the integral in Eq. (54) is independent of $\boldsymbol{r}$ and $\boldsymbol{r}'$. Hence

$$\rho(\boldsymbol{r},\ \boldsymbol{r}')\to \text{constant} \tag{57}$$

and this state exhibits true $S$-wave singlet-pair ODLRO.

This result is in contrast to similar manipulations in the FQHE which yield only algebraic ODLRO because of the incompressibility of the one-component Laughlin plasma. Further note that in the FQHE, it is not the electrons which condense, but 'artificial bosons' constructed by a singular gauge transformation which attaches an odd number of flux quanta to each electron.[12),34)] *Here it is the semions themselves which condense and no singular gauge changes need be made.*[36)] The point has been emphasized by Fisher, Lee and Kane[37)] and by Jain and Read.[38)] The latter authors considered spinless semions at the mean-field level which gives fermions filling the two-lowest Landau levels. They demonstrated an algebraic ODLRO for this integer QHE state using an expression equivalent to Eq. (54) using a gauge in which the statistical vector potential is zero but $\Psi$ is multiple-valued. The Stanford group has done an RPA calculation which demonstrates that the spinless model has a gapless, linearly dispersing collective mode and exhibits the Meissner effect.[39] Curiously, these standard indicators of superconductivity appeared in calculations without any need to invoke ODLRO. The present results[17] and those of Jain and Read[38] indicate that semions exhibit 'reasonably standard' superconducting pairing ODLRO.

The ideas presented here can be readily extended to study fluctuations beyond the mean-field theory of statistics and suggest a good variational wave function for the spinless case. Lack of space prevents discussion of these points here and the reader interested in these points is directed to Ref. 17).

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