Anyon Field Theory and Fractional Quantum Hall States

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We analyze the fractional quantum Hall effect by using an anyon field theory defined perturbatively in terms of a boson field. It is shown explicitly that there exist fractional quantum Hall (FQH) states which are condensed phases of bosonized electrons at zero momentum. When the Coulomb interaction is switched off, these states are degenerate with other states which are obtained by solving a certain self-dual equation. Taking into account the electron spin degrees of freedom we also show the existence of spin-singlet FQH states at the half-filling ($\nu=1/2$) and find vortex soliton excitations carrying the electric charge $e/4$. Furthermore, by using singular gauge transformations, we construct field theories of vortex solitons and derive the hierarchy of the FQH states.

§ 1. Introduction

The discovery of the fractional quantum Hall effect (FQHE) revealed that the two-dimensional electron system has rich ground state structures in a strong magnetic field.[1] Depending on filling factor ν the system exhibits a beautiful hierarchy of new quantum states called fractional quantum Hall (FQH) states. A microscopic theory has been presented to account for the FQHE and its hierarchy by using Laughlin's wave functions[2] and by exact numerical calculations on system of a few particles.[3] Landau-Ginsburg models have also been proposed.[4)~8]

However, all these approaches are rather phenomenological and unsatisfactory from a purely theoretical point of view. The FQHE and its hierarchy indicate the existence of new phases of the two-dimensional electron system in an external magnetic field, but their existence and their precise nature are not manifest in these approaches.

In this paper we present a microscopic formulation of the FQHE and its hierarchy where the structure of these new phases are shown explicitly.

§ 2. Anyons and planar electrons

We start with a field theory of anyons in a magnetic field. The theory is formulated by using a bosonic field. Later, we regard electrons as anyons, and use the field theory of anyons for describing electrons. It is well known that such a field theory of anyons is given by the Hamiltonian

$$H_0=\frac{1}{2m}\int d^2x|(i\partial_j+a_j-eA_j)\psi|^2\,, \tag{1}$$

together with a constraint equation on a_i,

$$\frac{1}{2\alpha}\varepsilon_{ij}\partial_i a_j=\psi^\dagger\psi\,, \tag{2}$$

where ψ is the charged boson field, a_i is the Chern-Simons (CS) gauge field, α is a statistics parameter of the anyon*) and A_i is an external electromagnetic potential,

$$A_i=-\frac{B}{2}\varepsilon_{ij}x_j \tag{3}$$

with a magnetic field B.

By representing the anyon by a boson field, there arises a possibility to analyze the anyon physics from a new point of view, in particular, by using the semiclassical approximation. In this approach, we quantize small fluctuations around a classical solution of the system, where the classical solution is a bosonic object. (In this paper we only consider the classical solution.) In such a case, in order to reproduce quantum spectrum of anyons,[9),10)] a modification of the Hamiltonian is necessary.[11)] The point of the argument is the following. On one hand, anyon wave functions vanish with a fractional power of r as

$$\psi_{\text{anyon}}(r)\to r^{\alpha/\pi}\,, \tag{4}$$

when two anyons come close. On the other hand, unperturbed boson states ($\alpha=0$) do no have this fractional power of r; their wave functions vanish like r^l with l being angular momentum. Hence, we cannot expand anyon states by using the unperturbed boson states. However, we may expand $r^{-\alpha/\pi}\psi_{\text{anyon}}$ by using the standard partial waves. Schematically the expansion looks like

$$\psi_{\text{anyon}}=r^{\alpha/\pi}(\sum \text{partial waves})\,. \tag{5}$$

This expansion leads to a modification of the perturbation series of α/π. Especially, in the first order of α/π, we have to add to the naive Hamiltonian a δ-function type repulsive force with the strength $2|\alpha|/M$.

The δ-function type interaction term with the coupling constant g reads

$$H_g=\frac{g}{2}\int d^2x\!:(\psi^\dagger\psi)^2\!:\,, \tag{6}$$

where normal ordering has been taken. Therefore, we expect that the second-quantized Hamiltonian is given by

$$H=\int dx\left[\frac{1}{2m}|(i\partial_j+a_j-eA_j)\psi|^2+\frac{g}{2}\!:(\psi^\dagger\psi)^2\!:\right], \tag{7}$$

where

*) The parameter α is defined such that a wave function of two anyons changes its phase by $e^{i\alpha}$ for the exchange of the anyons.

$$g=\frac{2|\alpha|}{M} \tag{8}$$

in the first order of perturbation in α/π.

We have shown that anyons are described by the Hamiltonian (7) with (8) for small parameter α/π. Now, electrons may be regarded as anyons with α/π being an odd integer. Although α/π is not a small quantity in this case, we assume that the Hamiltonian (7) with (8) describes electrons in the external magnetic field. The success of this description will justify this assumption. The boson field ϕ is identified with the electron bound to statistical flux, which we call *bosonized* electrons. The Coulomb interaction between electrons reads

$$\mathcal{V}[\phi]=\frac{e^2}{2\varepsilon}\int d^2xd^2y\colon\{\phi^\dagger\phi(x)-\rho\}\frac{1}{|x-y|}\{\phi^\dagger\phi(y)-\rho\}\colon, \tag{9}$$

where ε is the dielectric constant. Here, we have added a uniform background charge $e\rho$ to the Coulomb term $\mathcal{V}$ for charge neutrality. Thus, our Hamiltonian is

$$H^{\text{electron}}=H+\mathcal{V} \tag{10}$$

with the constraint equation (2).

It is convenient to rewrite (10) as

$$H^{\text{electron}}=\int d^2x\left[\frac{1}{2m}|(D_1-iD_2)\phi|^2+\frac{1}{2}\omega_c|\phi|^2+\mathcal{V}\right] \tag{11}$$

with $iD_j=i\partial_j+a_j-eA_j$ and $\omega_c=eB/m$ being the cyclotron frequency. This is easily derived by using the Bogomol'nyi decomposition:

$$|D_k\phi|^2=|(D_1-iD_2)\phi|^2-\varepsilon_{jk}\partial_j(\phi^\dagger iD_k\phi)-\varepsilon_{jk}\colon\partial_j(a_k-eA_k)|\phi|^2\colon. \tag{12}$$

It should be noted that the term $g|\phi|^4$ in the Hamiltonian (7) is cancelled with the third term in (12) because of our choice of $g=2|\alpha|/m$.

§ 3. FQH states at odd-denominator filling

We consider a system of N electrons in the presence of an external uniform magnetic field B perpendicular to the plane. We assume that the spin degrees of freedom can be ignored. In the next section we take account of the spin degrees of freedom.

In order to find the ground state of our system, we make a mean field approximation. First, we expand

$$\phi=\frac{1}{\sqrt{V}}[a_0+\sum_{p\neq0}a_pe^{ipx}] \tag{13}$$

with V being the volume of the system, and

$$[a_p,a_q^\dagger]=\delta_{pq}\,,\quad [a_p,a_q]=[a_p^\dagger,a_q^\dagger]=0\,. \tag{14}$$

We define the state

$$|N\rangle=\frac{1}{\sqrt{N!}}(a_0^\dagger)^N|0\rangle \tag{15}$$

with N being the number of the electrons in the system, which is assumed to be sufficiently large; $|0\rangle$ is the vacuum such that $a_{\boldsymbol{p}}|0\rangle=0$ for all $\boldsymbol{p}$. When the filling factor $\nu=2\pi\rho/eB$ is given by π/α, we can solve the constraint equation (2) as

$$a_i-eA_i=-\frac{\alpha}{\pi}\varepsilon_{ij}\int d^2y\frac{x^j-y^j}{(x-y)^2}(\psi^\dagger\psi-\rho)\,. \tag{16}$$

Taking the expectation value with $|N\rangle$, we find that $\langle N|(a_i-eA_i)|N\rangle=0$, which implies that in this approximation

$$\langle N|H^{\text{electron}}|N\rangle=\frac{1}{2}\omega_c N\,. \tag{17}$$

Therefore, the state $|N\rangle$ has the same energy as that of a state in which all of N electrons occupy the lowest Landau level. Furthermore, it follows that this state has the uniform number density:

$$\langle N|\psi^\dagger\psi|N\rangle=\frac{N}{V}=\rho\,. \tag{18}$$

This is the ground state in this approximation.

Let us see this problem from a slightly different way. We introduce a coherent state $|f\rangle$, which is defined by

$$|f\rangle=e^{-N/2}e^{\int d^2x\psi^\dagger(x)f(x)}|0\rangle\,, \tag{19}$$

where $\int d^2x|f(x)|^2=N$. This implies that the average number of electrons of the state $|f\rangle$ is equal to N. Note that

$$\psi|f\rangle=f|f\rangle\,,\quad \langle f|f\rangle=1\,. \tag{20}$$

Then, we find that

$$\begin{aligned}E(f)&\equiv\langle f|H^{\text{electron}}|f\rangle\\&=\int d^2x\left[\frac{1}{2m}|(D_1-iD_2)f|^2+\frac{1}{2}\omega_c N+\mathcal{V}(f)\right],\end{aligned} \tag{21}$$

where a_k in D_k is a c-number function given by (16) with $\psi=f$. It follows that

$$E(f)\geq\frac{1}{2}\omega_c N\,, \tag{22}$$

because the Coulomb interaction contributes a positive energy.

Let us neglect the Coulomb term $\mathcal{V}$ temporally. Then, any states $|f\rangle$ satisfying the self-dual equation

$$(D_j-i\varepsilon_{jk}D_k)f=0 \tag{23}$$

are degenerate with each other since all these states have the same energy $(\omega_c/2)N$. In particular, we can see that only at the filling factor $\nu=\pi/\alpha$ there exists the constant solution, $f=\sqrt{\rho}$, in this self-dual equation. (This constant solution remains to be a solution with the minimum energy even if the Coulomb interaction is switched on.) This solution describes a condensed state of the bosonized electron with a uniform density

$$\langle f=\sqrt{\rho}|\phi^\dagger\phi|f=\sqrt{\rho}\rangle=\rho\,, \tag{24}$$

while all other solutions describe the states with non-uniform densities. Obviously, these non-uniform states are degenerate with the uniform state only if the Coulomb interaction is switched off. Once the interaction operates among electrons, the degeneracy is removed, and the uniform state $|f=\sqrt{\rho}\rangle$ becomes the real ground state of the system. This is because the state possesses the smallest Coulomb energy due to its uniform electric charge distribution. Furthermore, it is straightforward to show that the Hall conductance in the state is given by $\sigma_{xy}=\nu(e^2/2\pi)$. Thus, this state is a FQH state.

In the condensed phase of the bosonized electrons there are vortex solitons carrying the statistical flux.[12] Such soliton solutions are obtained numerically by solving the self-dual equation (23). A soliton solution behaves as

$$\phi\rightarrow\sqrt{\rho}\,e^{-i\theta}\,, \tag{25}$$

at large distances, where θ is the azimuthal angle. It is shown that the electric charge and the mass of the soliton are $(\pi/\alpha)e$ and $(\pi/\alpha)m$, respectively. They are Laughlin's quasiholes. States $|f\rangle$ satisfying the condition $\int d^2x|f(x)|^2=N$ and containing some vortices are excited states above the ground state $|f=\sqrt{\rho}\rangle$. As an example we have estimated[13] an excitation energy of the state containing three vortex solitons at $\nu=1/3$,

$$\Delta E\simeq 0.13e^2/\varepsilon l_B\,, \tag{26}$$

where l_B is the magnetic length, $1/\sqrt{eB}$. Obviously, this is not a state with the minimum gap energy.

We emphasize that the state $|f=\sqrt{\rho}\rangle$ we have found at $\nu=\pi/\alpha$ represents a new phase of the planar electron system in the magnetic field. As we have seen, the existence and the nature of such a new phase is manifest in our microscopic formalism.

§ 4. FQH states at even-denominator filling

So far we have taken into account only one spin component of electrons. It is easy to generalize our scheme to include the two spin degrees of freedom of the electron.[14] The relevant bosonized Hamiltonian reads

$$H_2{}^{\text{electron}}=\frac{1}{2m}\int d^2x|(D_1{}^\uparrow-iD_2{}^\uparrow)\phi^\uparrow|^2+\frac{1}{2}\omega_c N^\uparrow$$
$$+\frac{1}{2m}\int d^2x|(D_1{}^\downarrow-iD_2{}^\downarrow)\phi^\downarrow|^2+\frac{1}{2}\omega_c N^\downarrow$$

$$+\text{Coulomb interactions} \tag{27}$$

with $iD_j^{\uparrow\downarrow}=i\partial_j+a_j^{\uparrow\downarrow}-eA_j$, where the indices $\uparrow$ and $\downarrow$ denote quantities associated with spin up and down components, respectively. CS gauge fields $a_j^{\uparrow\downarrow}$ satisfy the constraint equations

$$\varepsilon_{ij}\partial_i a_j^{\uparrow}=2\alpha|\phi^{\uparrow}|^2+2\gamma|\phi^{\downarrow}|^2\,,$$
$$\varepsilon_{ij}\partial_i a_j^{\downarrow}=2\alpha|\phi^{\downarrow}|^2+2\gamma|\phi^{\uparrow}|^2\,, \tag{28}$$

where $\alpha=\pi\times$odd integer, and $\gamma=\pi\times$ integer; the statistics parameter α describes commutativity between electrons with the same spin component, while γ describes the one between electrons with the different spin component.

We examine diagonal matrix elements of the Hamiltonian (27) by introducing coherent states $|f^{\uparrow},f^{\downarrow}\rangle$,

$$\phi^{\uparrow\downarrow}|f^{\uparrow},f^{\downarrow}\rangle=f^{\uparrow\downarrow}|f^{\uparrow},f^{\downarrow}\rangle\,,\quad \int d^2x|f^{\uparrow\downarrow}(x)|^2=N^{\uparrow\downarrow}\,. \tag{29}$$

When the Coulomb interaction is neglected, the ground state configurations are found by solving the self-dual equations

$$(D_1^{\uparrow\downarrow}-iD_2^{\uparrow\downarrow})f^{\uparrow\downarrow}=0\,, \tag{30}$$

together with the constraint equations (28) by setting $\phi^{\uparrow\downarrow}=f^{\uparrow\downarrow}$. When the Coulomb interaction is included, the ground state becomes unique with constant $f^{\uparrow\downarrow}=\sqrt{\rho^{\uparrow\downarrow}}$ at the filling factor

$$\nu\equiv\frac{2\pi(\rho^{\uparrow}+\rho^{\downarrow})}{eB}=\frac{2\pi}{\alpha+\gamma}\,, \tag{31}$$

where $\rho^{\uparrow\downarrow}$ are the densities of each spin components. The ground state describes uniform condensations of both up and down spin components. It follows from the constraint equations (28) that $\rho^{\uparrow}=\rho^{\downarrow}$, which indicates that the ground state is spin singlet. When $\alpha=3\pi n$ and $\gamma=\pi n$, n=integer, the filling factor becomes $\nu=1/2n$. Namely, these states are FQH states at the filling factors of even denominator.

Associated with these FQH states there are also vortex solitons. For instance, we may consider soliton solutions characterized by the asymptotic behaviors such as

$$\phi^{\uparrow}\to\sqrt{\rho^{\uparrow}}\,e^{-i\theta}\,,\quad \phi^{\downarrow}\to\sqrt{\rho^{\downarrow}} \tag{32a}$$

or

$$\phi^{\uparrow}\to\sqrt{\rho^{\uparrow}}\,,\quad \phi^{\downarrow}\to\sqrt{\rho^{\downarrow}}\,e^{-i\theta}\,, \tag{32b}$$

at large distances. Their electric charges are found to be

$$-eQ=\frac{\pi}{\alpha+\gamma}e=\frac{\nu}{2}e \tag{33}$$

for both cases (32a) and (32b), where

$$Q\equiv\int d^2x(|\phi^\uparrow|^2+|\phi^\downarrow|^2-\rho)\,. \tag{34}$$

This is the half of the electric charge of the soliton in the previous case of the odd-denominator filling factor. Thus, the result (33) shows a clear distinction of the singlet half-filling states.

§ 5. Field theory of vortex solitons and hierarchy

We now wish to construct a local field theory of vortex solitons and derive a hierarchy of the FQH states. We consider local (point-like) vortices by regarding them as point-particles. Such vortices are easily introduced by considering a singular gauge transformation such that[15)]

$$\phi\to e^{if}\phi\,,\quad a_\mu\to a_\mu+\partial_\mu f\,, \tag{35}$$

where $f(x)=\sum_{r=1}^{N_v}\theta(x-z_r)$, and $\theta(x-z_r)$ is the azimuthal angle. Local vortices are assumed to be located at $x^k=z_r^k(t)$ in the 2-dimensional space at time t, where $r=1, 2, \cdots, N_v$, N_v=total number of vortices.

Let us recall that the Lagrangian density which leads to the Hamiltonian (7) together with the constraint equation (2) is given by

$$\mathcal{L}=\phi^\dagger iD_0\phi-\frac{1}{2m}|D_k\phi|^2-\frac{g}{2}(\phi^\dagger\phi)^2-\frac{1}{4\alpha}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda\,. \tag{36}$$

Performing this singular gauge transformation to the Lagrangian density, we obtain

$$\mathcal{L}\to\mathcal{L}+\varDelta\mathcal{L}^{\text{vortex}} \tag{37}$$

with

$$\varDelta\mathcal{L}^{\text{vortex}}=-\frac{\pi}{\alpha}a_\mu K^\mu+\tilde{\alpha}_\phi G\,, \tag{38}$$

where $\tilde{\alpha}_\phi$ is given by the reciprocal relation in terms of α,

$$\tilde{\alpha}_\phi=-\frac{\pi^2}{\alpha}\,. \tag{39}$$

Here,

$$\begin{aligned}K^\mu&=(1/2\pi)\sum_{r=1}^{N_v}\varepsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta(x-z_r)\\&=\sum_{r=1}^{N_v}\dot{z}_r{}^\mu\delta^2(x-z_r)\,,\end{aligned} \tag{40}$$

which represents world-lines of the local vortices; $z^\mu=(t, z^k)$. On the other hand,

$$\begin{aligned}G&=(1/4\pi^2)\sum_{r,s}\varepsilon^{\mu\nu\lambda}\partial_\mu\theta(x-z_r)\partial_\nu\partial_\lambda\theta(x-z_s)\\&=(1/2\pi)\sum_s K^\mu\partial_\mu\theta(x-z_s)\,.\end{aligned} \tag{41}$$

This quantity in not well defined when two vortices coincide, i.e., for $r=s$. However, with a careful analysis it is shown[13] that this coincidence can be neglected, and that the quantity $\widehat{\alpha}_\phi$ is only defined mod 2π. Therefore, we can replace $\widehat{\alpha}_\phi$ in formula (40) with α_ϕ which obeys the generalized reciplocal relation[15]

$$\alpha_\phi=-\frac{\pi^2}{\alpha}+2\pi p \tag{42}$$

with p being an integer.

Integrating over the 2-dimensional space we get

$$\int d^2x\Delta\mathcal{L}=-\frac{\pi}{\alpha}\sum_{r=1}^{N_v}a_\mu\frac{dz_r{}^\mu}{dt}+\frac{\alpha_\phi}{\pi}\sum_{r<s}\frac{d}{dt}\theta(z_r-z_s)\,, \tag{43}$$

which describes how local vortices interact with the background CS field a_k and among themselves. It turns out that after quantization the statistic parameter of the vortices is given by α_ϕ.

So far we have treated the vortex solitons as point particles, and have analyzed their interaction terms, i.e., potential terms. In order to find their kinetic terms we have to carefully analyze so-called collective coordinates of the solitons.[16] But, it is natural to guess that the terms are given by $(1/2)M\dot{z}^2$ in our nonrelativistic formulation, where M is the mass of the soliton; $M=(\pi/\alpha)m$. Hereafter, we assume this kinetic term. Then, the classical mechanics of vortex solitons is described by

$$\mathcal{L}^{\text{vortex}}=\sum_r\left[\frac{M}{2}\left(\frac{dz_r{}^k}{dt}\right)^2-\frac{\pi}{\alpha}a_\mu\frac{dz_r{}^\mu}{dt}\right]+\frac{\alpha_\phi}{\pi}\sum_{r<s}\frac{d}{dt}\theta(z_r-z_s)\,. \tag{44}$$

It is straightforward to second-quantize this system by introducing the vortex field ϕ. Taking into account the contact terms similar to (6), we get the second-quantized vortex Hamiltonian H^{vortex}. The Hamiltonian which describes the combined system of the electrons and the vortices is given by $H^{(1)}_{\text{FQHE}}=H^{\text{electron}}+H^{\text{vortex}}$, i.e.,

$$H^{(1)}_{\text{FQHE}}=\int d^2x\left[\frac{1}{2m}|(D_1-iD_2)\psi|^2+\frac{eB}{2m}N+\mathcal{V}\right]$$
$$+\int d^2x\left[\frac{1}{2M}|(D_1{}^{(1)}-iD_2{}^{(1)})\phi|^2\right] \tag{45}$$

with $iD_k{}^{(1)}=i\partial_k+c_k-(\pi/\alpha)a_k$. Here, a_k and c_k are determined by the constraint equations

$$\psi^\dagger\psi-\frac{\pi}{\alpha}\phi^\dagger\phi=\frac{1}{2\alpha}\varepsilon_{ij}\partial_i a_j\,, \tag{46a}$$

$$\phi^\dagger\phi=\frac{1}{2\alpha_\phi}\varepsilon_{ij}\partial_i c_j\,. \tag{46b}$$

In order to obtain the ground state of the new Hamiltonian (45), we may repeat the same procedure as we did for $H^{(0)}_{\text{FQHE}}\equiv H^{\text{electron}}$. When the Coulomb interaction is neglected, the ground state configurations are obtained by solving the self-dual equations

$$(D_1-iD_2)\psi=0\,,\quad (D_1{}^{(1)}-iD_2{}^{(1)})\phi=0\,. \tag{47}$$

When the Coulomb interaction is included, there are two ground states described by two classical solutions with constant ψ and ϕ at $\nu=\nu^{(0)}$ and $\nu=\nu^{(1)}$, where $\nu^{(0)}=\pi/\alpha$ and

$$\nu^{(1)}=\frac{\pi}{\alpha}\left(1+\frac{\pi^2}{\alpha\alpha_\phi}\right)=\frac{1}{k+\frac{-1}{2p}}\,. \tag{48}$$

Here, $k\equiv\pi/\alpha$, and p is the parameter which appears is the generalized reciprocal relation (42). The both energies of these two states are given by $E=(1/2)\omega_c N$, as should be the case.

The solution at $\nu=\nu^{(0)}$ corresponds to the condensed phase of only bosonized electrons, while the solution at $\nu=\nu^{(1)}$ corresponds to the condensed phase of both bosonized electrons and vortices, and given by

$$\tilde{\psi}=\sqrt{\rho}\,,\quad \tilde{\phi}=\sqrt{\frac{\rho}{2p}}\,,$$

$$\tilde{a}_k=\frac{\alpha}{\pi}\tilde{c}_k=eA_k\,. \tag{49}$$

They represent the FQH states of the 0th stage and the 1st stage in the hierarchy of the FQHE, respectively. There exist again vortex solitons in this FQH state, which can be second-quantized.

Precisely in the same manner we can construct the Hamiltonian $H^{(n)}_{\rm FQHE}$ at the nth stage, by second-quantizing the local vortices at the $(n-1)$th stage; it involves the field operators $\phi^{(i)}$, the CS gauge fields $c_k{}^{(i)}$ together with the statistics parameters $\alpha^{(i)}$ for $i=1,\cdots,n$ in addition to ψ, a_k and α. The statistics parameters are determined by the reciplocal relations similar to (42), i.e.,

$$\alpha^{(i)}=-\frac{\pi^2}{\alpha^{(i-1)}}+2\pi p^{(i)} \tag{50}$$

with $p^{(i)}$ integers. The mass $M^{(i)}$ and the electric charge $e^{(i)}$ of the vortex $\phi^{(i)}$ are given by $M^{(i)}=m|Q^{(i)}|$ and $e^{(i)}=eQ^{(i)}$, respectively, with

$$Q^{(i)}=\left(\prod_{k=0}^{i-1}\frac{\pi}{\alpha^{(k)}}\right) \tag{51}$$

with $\alpha^{(0)}=\alpha$. It is found that the ground state of the Hamiltonian $H^{(n)}_{\rm FQHE}$ describing a condensed phase of all ψ and $\phi^{(i)}$ exists uniquely if and only if the filling factor ν takes a particular value, i.e., $\nu=\nu^{(n)}$ with

$$\nu^{(n)}=\frac{\pi}{\alpha}\left[1+\frac{\pi^2}{\alpha\alpha^{(1)}}\left[1+\frac{\pi^2}{\alpha^{(1)}\alpha^{(2)}}\left[1+\cdots\frac{\pi^2}{\alpha^{(n-1)}\alpha^{(n)}}\right]\cdots\right]\right.$$

$$=\cfrac{1}{k+\cfrac{-1}{2p^{(1)}+\cfrac{-1}{2p^{(2)}+\cfrac{-1}{\ddots\cfrac{}{\cfrac{-1}{2p^{(n)}}}}}}}. \tag{52}$$

The ground state is the FQH state at the nth stage of the hierarchy. The Hall conductance is found to be $\sigma_{xy}=\nu^{(n)}(e^2/2\pi)$.

§ 6. Discussion

We have shown in our microscopic formulation that the two-dimensional electron system possesses distinct phases of the FQHE, which are characterized by condensations of the bosonized electrons and vortices. Our formulation consistently reproduces the energy and the degeneracy of the electron system when the Coulomb interaction is switched off. Using our formulation we can analyze various aspect of the FQHE (e.g., effects of spatial variation of external magnetic field, etc.) which are beyond the scope of the standard approach based on the variational wave functions.

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