Vector-Model Approach to Non-Perturbative Theory of Random Filaments

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It is explained how vector models can be used to describe the physics of random filaments with arbitrary random branches. The continuum limit is obtained in the large-N limit by taking a double scaling limit which is analogous to that in the matrix models. The models exhibit various interesting properties, such as, nonperturbative instability, connection with the KP hierarchy, Virasoro structure, among others, which can be compared with the corresponding behaviors of the matrix models. When the target space is a continuum, the critical theories in the scaling limit are equivalent with superrenormalizable local field theories and the fractal dimensions of the random filaments are given by $2k/(k-1)$ ($k=2, 3, \cdots$) at the k-th critical point.

§ 1. Introduction

The matrix model[1] is one of the subjects which have been the focus of intensive study of theoretical particle physicists in recent years. In this lecture, I would like to discuss a slightly different but related subject of randomly branching filaments using vector models, based mainly on the works[2),3)] done in collaboration with Nishigaki.*) Let me first explain motivations for studying this problem and possible relation of this system to random surfaces, described by the matrix models.

The main motivations from particle-physicists' point of view in attacking the matrix models are twofold: First, we hope to learn possible clues to non-perturbative understanding of string theory as a unification theory of all fundamental interactions including gravity. Secondly, we also want to learn how to describe the quantum physics in the regime where spacetime geometry itself and even its topology are wildly fluctuating. The matrix models can be regarded as a toy model to both of these questions. To the former, it is a toy model in the sense that the dimension of the target spacetime is assumed to be small in order to be soluble. To the latter, it is in the sense that the base spacetime is 2 dimensional. From both points of view, the structure of the models has now been fairly well understood in the case where the dimensions of the target space are less than (or equal to) 1. The technical difficulty in going beyond 1 dimension is that one cannot then reduce the models to free fermion systems. Furthermore, there are indications from the study of the continuum and perturbative Polyakov strings that the dimension $=1$ of the target space is not just a technical barrier, but that is a real physical boundary beyond which the non-pertur-

*) Section 5 includes some new results which have not been mentioned in Refs. 2) and 3). For other related works, see Refs. 4), 5), 6), 10), 14) and 15).

bative phase of the system is drastically different from the cases where the target space dimensions are less than or equal to one.

For example, the partition function of random surfaces for fixed area A with sphere topology in D dimensional flat target space is expected to behave for large A as,

$$Z(A)\sim e^{-xA}A^{-3+\gamma_0}\,, \tag{1·1}$$

according to the famous formula of KPZ-DDK,[7)~9)] where the so-called string susceptibility exponent is

$$\gamma_0=\frac{1}{12}(D-1-\sqrt{(D-1)(D-25)})\,. \tag{1·2}$$

For $D<1$, this shows that γ_0 is negative, while for $D>1$ γ_0 becomes complex with *positive* real part. Although this certainly signals that the KPZ-DDK approach is invalid for $D>1$, it may be suggestive to study the meaning of the positive real part of γ_0. In fact, it might be interpreted as a signal of the tendency of the formation of filamentary bridges between surfaces. Consider the partition function $\tilde{Z}(A)$ with sphere topology of total area A under the condition that it must have at least one filamentary bridge. In terms of the original unconditional partition function Z, $\tilde{Z}$ is given as

$$\begin{aligned}\tilde{Z}(A)&\sim\int dA_1dA_2\delta(A_1+A_2-A)A_1A_2Z(A_1)Z(A_2)\\&\sim A^{2\gamma_0-3}e^{-xA}\,.\end{aligned} \tag{1·3}$$

Thus, as soon as $\mathrm{Re}\gamma_0>0$, the contributions of filamentary surfaces to statistical sum are expected to dominate.

Secondly, if we treat the 2-dimensional cosmological constant to be small, the D-dimensional Polyakov string theory suggests that the lightest mass of the system in the D-dimensional noncritical string theory is $m^2=-(D-1)/12$, which is for $D>1$ a tachyon. In other words, the usual perturbative vacuum and the general world sheets are unstable against the formation of the tachyonic string bridges and loops. It is not unreasonable to regard this phenomenon as related to the instability against the formation of filaments.

The existence of a physical phase boundary at $D=1$ might also be related to the fact that the cosmological term defined perturbatively in this approach becomes complex when $D>1$, suggesting again inappropriateness of the naive continuum treatment.

For these reasons, it seems worth while to study models in which one only takes into account the filamentary surfaces. We will in fact see several interesting features which can be compared with known $D=c<1$ random surface models. It would also be interesting to see a possible connection of the scaled theory of random filaments to the ordinary local field theory, which might be useful in order to investigate an analogous connection of the matrix model to the string field theory.

Schematically, such a simplest model can be described as

$$Z\sim\sum_{\text{filaments}}e^{\alpha\chi(f)-tA(f)}\prod_{\text{matter configurations}}e^{H(f,\text{matter})}, \qquad (1\cdot4)$$

where $\chi(f)$=Euler number, $A(f)$=area and $H(f$, matter) is the appropriate action for the matter fields x^μ living on the filaments. The most natural form of the matter action will be $\int d\tau(dx/d\tau)^2$, which is obtained from the Polyakov action by dimensional reduction. If we only consider the filaments with constant diameter, for simplicity, $A(f)$ is replaced by the length of the filaments, $A(f)=L(f)$=length. Thus, t can be regarded as the one-dimensional cosmological constant. The Euler number is equal to $2(1-h(f))$ with $h(f)$ being the number of the independent loops in the random filaments. Note that we allow arbitrary branching of the filaments, while the fundamental statistical weight is their total length once the genus is fixed, apart from, of course, the difference of the matter configurations.

§ 2. Vector model as a system of discretized random filaments

It is easy to realize the partition function of the type (1·4) by a class of vector models as given by

$$Z=\int\prod_{A=1}^{n}d^N\Phi_A\exp\left[-\beta\left\{\sum_{A=1}^{n}V(\Phi_A{}^2)-\frac{1}{2}\sum_{A\neq B}g_{AB}\Phi_A{}^2\Phi_B{}^2\right\}\right], \qquad (2\cdot1)$$

where the field Φ_A is an N-component real vector with A being the label of sites in a discretized target space, and $\Phi_A{}^2$ is the corresponding $0(N)$ invariant length. Of course, when the target space is a continuum space as we will discuss in the final section, the summation in (2·1) is replaced by integral. We note that there is no usual kinetic term of the form $\Phi_A\cdot\Phi_B$. Suppose that the potential V takes the form

$$V(\Phi^2)=\frac{1}{2}\Phi^2-\frac{1}{4}\lambda(\Phi^2)^2 \qquad (2\cdot2)$$

and expand (2·1) with respect to the coupling constants λ and g_{AB}. In this Feynman graph expansion, the Feynman rule is the following (after rescaling $\Phi\to(1/\sqrt{\beta})\Phi$):

propagator	1
$(\Phi_A)^2$ vertex	$\frac{\lambda}{\beta}$
$(\Phi_A)^2(\Phi_B)^2$ vertex	$\frac{g_{AB}}{\beta}$

A typical diagram f looks like as depicted in Fig. 1(a). Figure 1(b) is the dual of Fig. 1(a) and is interpreted as a configuration of the random filament. This is analogous to the situation of matrix models where the random surfaces are dual to the Feynman diagrams. The contribution of a diagram f to the connected part $F=\log Z$ is

$$F(f)=N^{v(f)}\prod_{(AB)\in f}\frac{g_{AB}}{\beta}, \quad (g_{AA}\equiv\lambda) \qquad (2\cdot3)$$

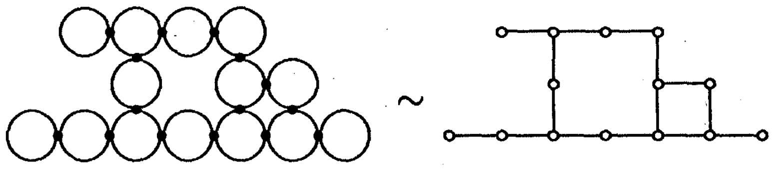

Fig. 1. (a) An example of the Feynman diagram for the potential (2·2).
(b) The dual diagram corresponding to (a).

where the product is over all bonds and $v(f)$ is the number of vertices, in the *dual* diagram, which is equal to the number of *index* loops in the original Feynman diagram picture. Note that the number of index loops is not in general equal to the number of loops in the sense of the usual Feynman diagrams. The genus of the filament f is given by the topological relation $h(f)=1-v(f)+b(f)$, with $b(f)$ being the number of bonds in the dual diagram. In the Feynman diagram, $b(f)$ is nothing but the number of 4-point vertices. For the examples of Fig. 1, it is a 16-loop diagram in terms of the original Feynman diagram, while, in the dual picture, $b(f)=15$, $v(f)=14$, $h(f)=1-14+15=2$. Thus, (2·3) is rewritten as

$$F(f)=N^{1-h(f)}\left(\frac{N}{\beta}\right)^{b(f)}\prod_{(AB)\in f}g_{AB}\,. \tag{2·4}$$

The model (1·4) is realized by identifying as $a=(1/2)\log N$, $t=-(1/l)\log(N/\beta)$, with l being the one-dimensional lattice constant, and $H=-\sum_{(AB)\in f}\log g_{AB}$.

§ 3. Double scaling limit of vector models

Now let us consider the possibility of the scaling limit. For simplicity, we first consider the case $n=1$, namely, "pure 0+1 dimensional gravity".

$$Z=\int d^N\Phi\exp\left\{-\beta\left(\frac{1}{2}\Phi^2+\frac{\lambda}{4}(\Phi^2)^2\right)\right\}. \tag{3·1}$$

3.1. *Perturbative analysis*

The sum of connected diagrams in the perturbative expansion takes the form,

$$F=\sum_{h=0}F^{(h)}=\sum_{h}\sum_{b}N^{1-h}\left(\frac{N}{\beta}\right)^{b}(-\lambda)^b f_{b,h}\,, \tag{3·2}$$

where $f(b,h)$ is the number of inequivalent dual diagrams with b bonds and h loops. For $h=0$, the saddle point approximation gives

$$F^{(0)}=\frac{1}{2}\log\frac{-1+\sqrt{1+8\lambda N/\beta}}{4\lambda}+\frac{1-\sqrt{1+8\lambda N/\beta}}{16\lambda N/\beta}-\frac{1}{4}\,. \tag{3·3}$$

From this expression, the asymptotic behavior of $f_{b,0}$ is determined to be

$$f_{b,0}\sim b^{-5/2}4^b\,, \tag{3·4}$$

which amounts to $\gamma_0=1/2$. Equivalently, the singularity of the $h=0$ free energy is given as,

$$F^{(0)} \sim N\left(1-\frac{N}{\beta}\right)^{3/2}, \tag{3·5}$$

where we have chosen $\lambda=-1/4$ such that the singularity occurs at $N/\beta=1$. This result is obtained with the Φ^4 potential.

By extending the above analysis to general polynomial potentials, we find singularities of the type,

$$F^{(0)} \sim N\left(1-\frac{N}{\beta}\right)^{1+1/k}, \tag{3·6}$$

$$\gamma_0=1-\frac{1}{k}, \quad (k=(1), 2, 3, \cdots) \tag{3·7}$$

corresponding to the critical potentials determined by $1-2x^2V'(x^2)=(k-x^2)^k/k^k$. Note that the susceptibility exponent γ_0 is always positive (or zero when $k=1$, gaussian model). It is not difficult to go to higher orders with respect to $1/N$. We find $F^{(1-h)} \sim N^{1-h}(1-N/\beta)^{(1+1/k)(1-h)}$. Thus, as in the case of the matrix model, nontrivial double scaling limit $N\to\infty$, $N/\beta\to 1-$, keeping $N(1-N/\beta)^{1+1/k}$ fixed, exists. The "renormalized" cosmological constant t can be defined by $1-N/\beta=lt$ with $N \sim l^{-1-1/k}$ in the limit $l\to 0$. The genus$=0$ free energy therefore behaves as

$$F^{(0)} \propto t^{1+1/k}, \tag{3·8}$$

in contrast with the behavior $F^{(0)} \propto t^{2+1/k}$ in the one-matrix model.

The saddle point analysis can be easily extended to multi-vector models. In particular, at least for translationally invariant systems, the saddle point equations always take the same form as that of the one-vector model:

$$\frac{N}{\beta}=2wV'(w)+gw^2 \tag{3·9}$$

with $g=\sum_B g_{AB}$. This indicates a universal nature of the critical behaviors of the one-vector model.

If we generalize the model such that the number of the components of different vectors are independently varied, then the target space cannot be translation invariant and the model can describe spins coupled with external fields, as mentioned in Ref. 2) and studied recently in Ref. 10).

3.2. *Non-perturbative analysis*

The above structure can be demonstrated much more elegantly by developing an exact recursion-equation approach to the vector models. It is an analog of the orthogonal polynomial method in the matrix models. The partition function (3·1) can be rewritten as a single integral,

$$Z_N=\int_0^\infty dx x^{N-1}\exp\left\{-\beta\left(\frac{1}{2}x^2+\frac{1}{4}\lambda x^4\right)\right\}. \tag{3·10}$$

From the identity (equation of "motion"),

$$0=\int_0^\infty \frac{d}{dx}\left[x^N \exp\left\{-\beta\left(\frac{1}{2}x^2+\frac{1}{4}\lambda x^4\right)\right\}\right], \tag{3·11}$$

we obtain a recursion equation,

$$\frac{N}{\beta}Z_N - Z_{N+2} - \lambda Z_{N+4}=0\,. \tag{3·12}$$

In this equation, we analytically continue $\lambda>0$ to $\lambda=-1/4$.*) Then, by redefining $\tilde{Z}_N =2^{-N/2}Z_N$, (3·12) is reduced to

$$\left[1-\frac{N}{\beta}-(1-\Delta)^2\right]\tilde{Z}_N=0\,, \tag{3·13}$$

where Δ is a step translation operator $\Delta\tilde{Z}_N=\tilde{Z}_{N+2}$. In this form, we can take the continuum limit $\beta\to 0$ directly by using the scaling variables $1-N/\beta=2\beta^{-p}t$, provided $0<p<1$. Then (3·13) reduces to the form

$$\left(2\beta^{-p}t-\beta^{2(p-1)}\frac{d^2}{dt^2}\right)Z(t)=0\,, \quad (\tilde{Z}_N=Z(t)) \tag{3·14}$$

which leads to $p=2/3$ and to the differential equation

$$\left(2t-\frac{d^2}{dt^2}\right)Z(t)=0\,. \tag{3·15}$$

The perturbative behavior with respect to $1/N$ of the free energy can be easily rederived from this equation. Define $v(t)=(d/dt)\log Z(t)$. Equation (3·15) then is equivalent to a Riccati-type equation,

$$2t-(v'+v^2)=0\,, \tag{3·16}$$

from which the following asymptotic expansion with respect to $t^{-3/2}$ can be derived.**)

$$\log Z(t)=\frac{2\sqrt{2}}{3}t^{3/2}-\frac{1}{4}\log t+\frac{1}{2\sqrt{2}}t^{-3/2}+\frac{7}{96}t^{-3}+\cdots\,. \tag{3·17}$$

The derivation of the differential equations at the higher critical points is completely in parallel with the above. The result is $p=k/(k+1)$, and

$$\left(2t-\frac{d^k}{dt^k}\right)Z(t)=0\,, \tag{3·18}$$

$$\left(\frac{d}{dt}+v\right)^k\cdot 1=2t\,. \tag{3·19}$$

Extension of the above analyses to multi-vector models of our type is straightforward. Instead of a single partition function of the one-vector case, we have to consider a set of correlation functions. Let us regard the set of the correlation

*) This in turn requires to deform the integration contour in the integral (3·10).

**) After the conference, I was informed that a similar Riccati-type equation has been used in order to investigate the large-order behavior of the $1/N$-expansion of vector models by Hikami and Brézin.[16] I would like to thank S. Hikami for bringing this work to my attention.

functions as an element of a vector space, and express it by Ψ_N. Then the exact recursion equation can always be expressed as a linear condition of the following generic form,

$$\mathcal{A}(\Delta)\Psi_N=\frac{N}{\beta}\Psi_N\,, \tag{3·20}$$

where $\mathcal{A}(\Delta)$ is a matrix operator and depends on Δ polynomially. The critical points arise when one (or more) eigenvalue of the matrix $\mathcal{A}(\Delta)$ behaves as $(1-\Delta)^k$ as $\Delta\to 1$. This shows that at least when the dimension of the vector space of Ψ is finite, the critical behaviors of multi-vector models are classified by the same universality classes of the one-vector model, independently of the structure of the target spaces.

Here, as such a simplest example, we demonstrate the case of two-vector model (namely, an Ising model on the random filaments) with a quartic potential of coupling constant λ and the hopping coupling constant $g_{AB}=-\mu$. The vector Ψ can be chosen to be

$$\Psi_N=2^{-N}\begin{pmatrix} Z_{N+2,N} \\ Z_{N,N} \\ Z_{N+4,N}\end{pmatrix}, \tag{3·21}$$

and the matrix operator $\mathcal{A}$ is given by

$$\mathcal{A}=\begin{pmatrix} 4(\lambda+\mu)\Delta & 4\Delta & 0 \\ 1 & 4\mu\Delta & \lambda \\ 4\Delta & 16\lambda\Delta^2 & 4\mu\Delta \end{pmatrix}. \tag{3·22}$$

Three eigenvalues are $2\sqrt{\Delta}(2(\lambda+\mu)\sqrt{\Delta}+1)$, $2\sqrt{\Delta}(2(\lambda+\mu)\sqrt{\Delta}-1)$ and $-4\Delta(\lambda-\mu)$, the first of which leads to the $k=2$ critical behavior when $\lambda+\mu=-1/4$, and the remaining two cases to the trivial critical behavior $k=1$ when $\lambda+\mu=3/4$ and $\lambda-\mu=-1/4$, respectively.

§ 4. Non-perturbative properties of the solutions

Now, let us discuss several non-perturbative properties of the solutions described by the differential equations (3·18) (or (3·19)). The infinite number of the critical points can be treated at once by introducing a "universal filament equation" which interpolates all the different critical points,

$$\left(t-\sum_{l=1}^{\infty}x_l\frac{d^l}{dt^l}\right)Z(t)=0\,. \tag{4·1}$$

The new infinite number of the parameters x_l can be regarded as the coupling constants corresponding to the scaling operators,

$$\mathcal{O}_l=\beta^{(l-k)/(k+1)}\int_0^{2\Phi^2}\frac{du}{u}\left[(1-u/2)^l-1\right]. \tag{4·2}$$

The k-th critical point is described by setting $x_k=1$, $x_l=0$ $(l\neq k)$.

4.1. *General solution*

Since (4·1) is linear, it is easy to write down the general solution in integral representation.

$$Z(t)=\sum_i w_i \int_{C_i} ds \exp[ts-\sum_{l=1}^{\infty} x_l s^{l+1}], \tag{4·3}$$

where C_i's are integration contours in the complex s-plane such that the integral is convergent, and w_i's are arbitrary constants. When $x_l=0$ for $l>k$ $(x_k>0)$, there are k independent countours. The "effective action" on the exponential in this expression coincides, in the scaling limits, with the effective action for the $1/N$-expansion around the saddle point in the original vector models.

The first general property of the solution is that $Z(t)$ is always an entire function in the complex t-plane, and hence only possible singularities with respect to t of the specific heat $v'(t)=(d^2/dt^2)\log Z(t)$ are double poles at the zeroes of $Z(t)$. Secondly, for k=even, the contour which coincides with the real axis is not allowed. Thus, every independent solution in (4·3) with even k is complex and has infinite oscillation for large negative t. For example, for $k=2$, the asymptotic behavior is $|t|^{-1/4}\exp\{-i((2/3)|t|^{3/2}+\pi/4)\}$. As a consequence, in particular, every real solution has an infinite number of zeroes on the real axis. All solutions without such singularities are complex. These behaviors are caused by the instabilities of the critical potential for even k. On the other hand, when k=odd, the real axis is always an allowed contour. The corresponding solution does not have zero for real t corresponding to the stable critical potential.

In terms of the perturbative expansion, the difference of even and odd k can be rephrased as the Borel non-summability (even k) and summability (odd k), respectively. The Borel-transformed amplitudes in the former cases contain branch-point singularities on the finite positive real axis. Most of these behaviors are strikingly in parallel to those in the $c<1$ matrix models.

4.2. *Flow and Virasoro structure*

From the point of view of finding possible new directions towards the formulation of the non-perturbative string theory, it is of some interest to seek for general mathematical framework in which the universal behaviors of various models can be unifiedly described. I would like to make two comments related, possibly, to this question. The first is that a Virasoro-algebra structure is naturally contained in Eq. (4·1).

The change of the partition function (4·1) under the flow of the coupling constants x_l is described by

$$\frac{\partial Z}{\partial x_l}=-\frac{\partial^{l+1}}{\partial t^{l+1}}Z\,. \tag{4·4}$$

This allows us to rewrite a single ordinary differential equation (4·1) into an infinite number of partial differential equations $L_n Z(t;\{x_l\})=0$ $(n\geq -1)$ with

$$L_n=\delta_{n,0}-x_0\delta_{n,-1}-(n+1)\frac{\partial}{\partial x_{n-1}}+\sum_{l=0}^{\infty}(l+1)x_l\frac{\partial}{\partial x_{l+n}}, \quad (x_0=-t) \tag{4·5}$$

satisfying the Virasoro algebra $[L_n, L_m]=(n-m)L_{n+m}$.

The origin of the Virasoro condition is the analyticity at $s=0$ of the effective action in the integral representation (4·1) with respect to the complex variable s, whose form is invariant under the conformal transformation $s\to s+\epsilon_n s^{n+1}$. A similar structure has been pointed out in matrix models.[11] It should also be interesting to remark that the partition function is a coherent state of the Virasoro algebra, expressed as

$$Z(t)=\sum w_i\int_{C_i}ds\,\exp(sL_{-1})\cdot 1\,. \tag{4·6}$$

4.3. *Connection with the KP hierarchy*

The next point is that the system of the filament equations can be neatly embedded in the well-known KP hierarchy of integrable non-linear differential equations. In essence, the KP hierarchy is summarized by the following infinite number of differential equations, often called the Sato equation,

$$\frac{\partial K}{\partial x_n}=-(KD^nK^{-1})_-K\,,\qquad (n\geq 1,\, D=d/dt) \tag{4·7}$$

where K is a pseudo-differential operator of the form $K=1+k_{-1}(t,x)D^{-1}+k_{-2}(t,x)D^{-2}+\cdots$, and the notation $(\cdot)_-$ indicates the negative power part of a general pseudo-differential operator. For example, the famous KdV equation is obtained from the Sato equation by imposing a condition $(KD^2K^{-1})_-=0$.

The universal filament equation (4·1) can be expressed in the form,

$$1+\sum_l^{\infty} lx_l(KD^{l-1}K^{-1})_{-1}=0\,, \tag{4·8}$$

under the condition,

$$(KD)_-=0\,. \tag{4·9}$$

The τ function $\tau(t,x)$ of the KP hierarchy which can be, in the present situation, defined by $k_{-1}=-D\log\tau$, is identified with the partition function $Z(t)$. The flow equation (4·4) is then reduced to the Sato equation. (For more details, refer to Ref. 3).)

It is now well known that the mathematical structure of the matrix models corresponding to (p,q)-minimal conformal models can be most efficiently described by the KP hierarchy with a reduction condition $(KD^qK^{-1})_-=0$. See, for example, Refs. 12), 13) and 11). The above observation implies that the matrix models and our vector models can be interpreted as different elements of one and the same Universal Grassmann Manifold, which is the solution space of the general KP hierarchy.

§ 5. Continuum target space

In the case of continuum target space, the eigenvalue spectrum of the matrix operator $\mathcal{A}$ can accumulate at 1 in the limit $\Delta\to 1$. Therefore, the analysis becomes more complicated. In this case, it is more convenient to directly derive the integral representation from the original path integral for the partition function, although this method is less rigorous than the recursion equation approach.

Let us consider the model of the following type,

$$Z_N=\int\prod_t[d^N\Phi(t)]\exp\Big[-\beta\Big\{\int\frac{d^Dt}{(2\pi)^{D/2}}V(\Phi^2(t))$$
$$+\frac{\lambda}{4}\int\frac{d^Dt_1}{(2\pi)^{D/2}}\int\frac{d^Dt_2}{(2\pi)^{D/2}}\Phi^2(t_1)\Phi^2(t_2)e^{-|t_1-t_2|^2/2}\Big\}\Big] \tag{5·1}$$

with D dimensional flat target space whose coordinates are t^μ ($\mu=1,2,\cdots,D$). This model corresponds to assuming that the action of the embedded filaments is proportional to $\int d\tau(\partial t^\mu/\partial\tau)^2+$cosmological term, which is the natural 1 dimensional approximation of the Polyakov action. For example, let us first consider the simplest nontrivial case where the potential is $V(\Phi^2)=(1/2)\Phi^2$. Then, performing the expansion with respect to λ, we find the system of random filaments as defined in § 3. The expansion takes the form,

$$(N\delta^D(0))^{1-h(f)}\Big(\frac{\lambda N\delta^D(0)}{\beta}\Big)^{b(f)}\int\prod\frac{d^Dt}{(2\pi)^{D/2}}\prod e^{-|t_i-t_j|^2/2}\,. \tag{5·2}$$

The singular factor $\delta^D(0)$ is originated from the δ-function factor $\delta^D(t)$ in the propagator, since the free term in the present Feynman graph expansion is *ultra* local, $\Phi^2(t)/2$. However, this is harmless because it appears only in the combination $N\delta^D(0)$. The double scaling limit can be defined as in the discrete case, provided that N is replaced by $N\delta^D(0)$.

The critical behavior can be studied first in the sphere approximation, which corresponds to the saddle point approximation with respect to the integral over the invariant length $x(t)=\sqrt{\Phi^2(t)}$. The saddle point equation is

$$\frac{N\delta^D(0)}{\beta}-2V'(x(t))x^2(t)-\lambda x^2(t)\int\frac{d^Dt_1}{(2\pi)^{D/2}}x^2(t_1)e^{(1/2)|t_1-t|^2}=0\,. \tag{5·3}$$

The first singular term is the contribution of the integration measure. The translation invariant solution $x(t)=x$ becomes critical such that

$$1-\frac{N\delta^D(0)}{\beta}=\frac{(k-x^2)^k}{k^k}\,, \tag{5·4}$$

when the potential $V(x^2(t))$ is adjusted so that the right-hand side of the (5·4) is equal to $1-2V'(x^2)x^2-\lambda x^4$. Then, the critical points are classified in the same universality classes as those for the discretized target space. In particular, for the present gaussian potential, this requires $\lambda=-1/4$, leading to the $k=2$ critical point.

The higher order terms are generated by making the expansion around the saddle point solution by setting $x(t)=x+\tilde{x}(t)$. We then arrive at the following integral for the critical region $1-(N\delta^D(0))/\beta\sim 0$,

$$Z_N\sim\int\prod[d\tilde{x}(t)]\exp\Big[-\int\frac{d^Dt}{(2\pi)^{D/2}}\frac{\beta}{3}\Big(1-\frac{N\delta^D(0)}{\beta}\Big)^{3/2}$$

$$-\int\frac{d^Dt}{(2\pi)^{D/2}}\left\{\frac{\beta}{2}\left(\frac{\partial\tilde{x}}{\partial t}\right)^2+\beta\left(1-\frac{N\delta^D(0)}{\beta}\right)^{1/2}\tilde{x}^2\right\}$$

$$-\int\frac{d^Dt}{(2\pi)^{D/2}}\frac{2}{3\sqrt{2}}\beta(\tilde{x}^3+0(\tilde{x}^4))$$

$$\left.+\text{higher order terms in } 1-\frac{N\delta^D(0)}{\beta}\right]. \qquad (5\cdot5)$$

Here, the kinetic term $(\partial\tilde{x}/\partial t)^2$ in the second term in the effective action is obtained from the nonlocal interaction term of the original action by making a Taylor expansion:

$$4\lambda\beta x^2\times\int\frac{d^Dt_1}{(2\pi)^{D/2}}\int\frac{d^Dt_2}{(2\pi)^{D/2}}e^{-(1/2)|t_1-t_2|^2}\tilde{x}(t_1)\tilde{x}(t_2)$$

$$\sim-\beta\int\frac{d^Dt}{(2\pi)^{D/2}}\left(\tilde{x}^2(t)-\frac{1}{2}\left(\frac{\partial\tilde{x}}{\partial t}\right)^2+\text{higher derivative terms}\right). \qquad (5\cdot6)$$

The higher derivative terms can be neglected by the scaling argument, which we will explain soon.

Now to obtain a nontrivial finite result in the critical limit $1-(N\delta^D(0))/\beta\to0$, $\beta\to0$, we scale the variables as

$$1-\frac{N\delta^D(0)}{\beta}=(ma)^4\,, \qquad (5\cdot7)$$

$$t\to a^{-1}t\,, \qquad (5\cdot8)$$

$$\tilde{x}(t)\to a^2\tilde{x}(t)\,, \qquad (5\cdot9)$$

$$\beta=(a\mu)^{D-6}\,, \qquad (5\cdot10)$$

where m and μ are free mass parameters which have the dimension of mass in the target space. The one-dimensional lattice and cosmological constants are a^4 and m^4, respectively. The effective action then reduces to a finite form in the continuum limit $a\to0$.*)

$$S_{\text{eff}}=\mu^{D-6}\int\frac{d^Dt}{(2\pi)^{D/2}}\left[\frac{1}{2}\left(\frac{\partial\tilde{x}}{\partial t}\right)^2-\frac{2}{3\sqrt{2}}\tilde{x}^3+\frac{1}{\sqrt{2}}m^4\tilde{x}\right], \qquad (5\cdot11)$$

where we have made a field translation $\tilde{x}\to\tilde{x}+$constant after the scaling, such that the constant term, namely, the first term on the exponential in (5·5) of the action disappears from the action. The effective action reduces to the one for the discretized target space if the field is assumed to be constant.

Neglecting the higher derivative terms in the kinetic term and the higher power terms than 3 (and the terms containing derivatives) in the potential are justified by the appearance of the positive powers with respect to a in the scaling limit. This explicitly shows that the exact form of the nonlocal interaction term in (5·1) is not important in the scaling limit, as long as the kernel of the potential decreases fast

*) Here the continuum limit is in the sense of the discretized filament. Note, however, that we have introduced the cutoff parameter a such that it has the dimension of length in the external target space.

enough such that the integrals of the higher Taylor-expansion terms are finite. The form of the effective action (5·11) is therefore universal. Similar results have been obtained[14),15)] for different vector models, i.e., linear σ-models with the kinetic term of the form $\Phi_A \cdot \Phi_B$.

Viewed as a local field theory, the action (5·11) is superrenormalizable for $D<6$. This condition coincides with that for the possibility of the double scaling limit $\beta \to \infty$, $a \to 0$, as follows from (5·10). The superrenormalizability assures that the system can be rendered to be completely finite by a redefinition of a few coupling constants in the potential.

On the other hand, the relation (5·7) suggests that the fractal dimension of the random filaments at this critical point is 4. The argument goes as follows: If we rescale the mass parameters (m, μ) by factor two, $m \to 2m$, $\mu \to 2\mu$, the relation (5·7) indicates that the average length of the random filaments decreases by factor 2^{-4}, while in the target space this rescaling means that the correlation length will decrease by factor 2^{-1}. Then, by definition, Hausdorff dim.$=\log 2^4/\log 2=4$.

The higher critical points can also be reached similarly by introducing the higher order terms in the potential. In general, we find the effective action of the form in the scaling limit,

$$S_{\text{eff}} \sim \mu^{D-(2(k+1))/(k-1)} \int \frac{d^D t}{(2\pi)^{D/2}} \left[\frac{1}{2} \left(\frac{\partial \tilde{x}}{\partial t} \right)^2 + \tilde{x}^{k+1} + m^{2k/(k-1)} \tilde{x} \right]. \quad (k=2, 3, 4, \cdots) \tag{5·12}$$

Again, this action is superrenormalizable when $D<2(k+1)/(k-1)$ which coincides with the condition for the possibility of the double scaling limit, since $\beta=(a\mu)^{D-2(k+1)/(k-1)}$. The corresponding fractal dimesion is $2k/(k-1)$, since $1-(N\delta^D(0)/\beta)=(ma)^{2k/(k-1)}$. Thus the fractal dimension decreases as we go to higher critical points. In particular, it reduces to the value, 2, of the ordinary random walk in the limit $k \to \infty$.*)

The same remarks, as in the case of discrete target space, apply for the differences of non-perturbative behaviors between the odd and even criticalities. We note that the effective actions for general k in the continuum target space reduce to those in the discrete target space when only the constant mode of the field $\tilde{x}$ is retained.

§ 6. Conclusion

There are of course many new questions arising from our study. In the Introduction, we have explained a motivation for studying the vector models that it might provide some suggestions about the nature of the random surfaces with higher target space dimensions. However, we do not yet have any concrete connections of the present models to random surface models, although we have found many interesting parallelisms in the properties of the solutions in the scaling limit. In the following, we mention a few speculative remarks which seem to be worth for future study.

*) The same observation has been made in Ref. 4) by a different method.

1. At the trivial critical point $k=1$, the susceptibility exponent γ_0 equals zero, which coincides with the value of the $c=1$ matrix model apart from the logarithmic correction. As is well known, the universal critical properties of the $c=1$ model is derived from studying the upside-down harmonic potential. It is tempting to speculate a possibility, due to this property, that the $c=1$ matrix model can somehow be regarded as a special vector model with N^2 components.

2. As was shown in § 5, the vector models in the scaling limit are equivalent with superrenormalizable local field theories, and hence the dimensions of the target space are restricted, depending upon the criticalities, such that the highest possible dimension is less than 6 for nontrivial criticalities. Is there anything corresponding to this property in the case of matrix models?

3. Different vector models such as studied in Refs. 5) and 6) give identical results as those of the present models, which are very natural as a model for random filaments but in fact are rather unconventional as vector models, in the double scaling limit. It is desirable to understand this universality in a more rigorous way. For that purpose, it may be useful to further study the properties of our matrix recursion equations (3·20) for the case of infinite dimensional vector space of Ψ.

4. In Refs. 12) and 13), we have formulated a very simple action principle from which the whole structure of the $c<1$ matrix models can be derived, in terms of the framework of the KP hierarchy. It will be helpful for studying the general framework for the theory of random surfaces with higher dimensional target space, if one can extend the action principle such that it encompasses the vector model.

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