Quantum Conserved Charges and S-Matrices in $N=2$ Supersymmetric Sine-Gordon Theory

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We study the quantum conserved charges and S-matrices of $N=2$ supersymmetric sine-Gordon theory in the framework of perturbation theory formulated in $N=2$ superspace. The quantum affine algebras $\widehat{sl_q(2)}$ and super topological charges play important roles in determining the $N=2$ soliton structure and S-matrices of soliton-(anti) soliton as well as soliton-breather scattering.

§1. Introduction

Recently there has been increasing interest in massive integrable field theories in two dimensions which can be regarded as perturbed systems of conformal field theories. Following the idea orginally due to Zamolodchikov,[1] we apply a certain perturbation term $\lambda\int\Phi d^2w$ where λ is the coupling constant, upon the action of the conformal field theory as follows:

$$S=S_{\rm CFT}-\lambda\int d^2w\Phi(w,\bar{w})\,. \tag{1·1}$$

In this perturbed system the conformal invariance no longer holds, but the theory is still solvable as a massive theory, because of the existence of infinitely many conservation laws. The soliton theories like KdV, modified KdV and sine-Gordon theories belong to such a class of solvable models. They possess factorizable S-matrices due to an infinite number of conservation laws. Although these soliton theories are related with each other, here we shall focus our attention on the sine-Gordon theory.

Let us now recapitulate what have been known so far about sine-Gordon theory as a perturbed conformal field theory. First of all, we consider the bosonic ($N=0$) sine-Gordon theory. It has been realized that the conformal minimal model perturbed by $\Phi_{(1,3)}$ operator leads to the integrable restriction of sine-Gordon theory.[2)~6)] In the Feigin-Fuchs representation, the $\Phi_{(1,3)}$ operator is given by $e^{-i\beta\phi}$ with $(1/2)\beta^2=m/(m+1)$ for the central charge of the minimal unitary series: $c=1-(6/m(m+1))$, where $m=3, 4, \cdots$. Together with the screening operator $e^{i\beta\phi}$ the $\Phi_{(1,3)}$ forms the cos $\beta\phi$ interaction term of sine-Gordon theory.

Some time ago Sasaki and Yamanaka[7] studied the higher-spin conserved charges which are polynomials in Virasoro operators at the classical levels as well as at quantum levels. Eguchi and Yang derived the conserved charges based on the perturbation theory à la Zamolodchikov. It was also realized by LeClair, Smirnov and Eguchi-Yang that the Hilbert space is truncated for the rational values of the coupling constant.[2)~6)] It has turned out that quantum group symmetry[8),9)] plays an important role for the truncation.

Now it should be interesting to see whether incorporation of supersymmetry will affect the solvability of sine-Gordon theory. The $N=1$ supersymmetric version of sine-Gordon theory can also be regarded as the $N=1$ superconformal minimal model perturbed by $\Phi_{(1,3)}$ operator. The conservation laws and S-matrices are discussed by Sasaki-Yamanaka,[7)] Mathieu,[11)] Ahn-Bernard-LeClair[12)] and Schoutens.[13)]

Now the question is the following: What about the $N=2$ supersymmetric extension of sine-Gordon theory? As we will see, there exist somewhat different features for $N=2$ case in costrast to $N=0$ and 1 cases.

The classical equation of motion for the $N=2$ sine-Gordon theory is given by the coupled equations:[14)]

$$\bar{D}_+ D_+ \phi^+ = g\sin\beta\phi^- ,$$
$$\bar{D}_- D_- \phi^- = g\sin\beta\phi^+ , \qquad (1\cdot 2)$$

where ϕ^+ and ϕ^- are the chiral and anti-chiral superfields of $N=2$ supersymmetric theory. $Z=(z, \theta^+, \theta^-)$ and $\bar{Z}=(\bar{z}, \bar{\theta}^+, \bar{\theta}^-)$ are holomorphic and anti-holomorphic parts of the $N=2$ supercoordinates, respectively. The chiral and anti-chiral superfields satisfy the constraints $D_-\phi^+=\bar{D}_-\phi^+=0$ and $D_+\phi^-=\bar{D}_+\phi^-=0$, where the $N=2$ supercovariant derivatives are defined as $D_\pm=\partial/\partial\theta^\pm+(1/2)\theta^\mp\partial_z$ and similar expressions for $\bar{D}_+$ and $\bar{D}_-$. The chiral and anti-chiral superfields are composed of a complex boson $\varphi^\pm$ and a complex fermion $\psi^\pm$, $\bar{\psi}^\pm$ and the auxiliary field $F^\pm$ as

$$\phi^\pm = \varphi^\pm + \theta^\pm\psi^\mp + \bar{\theta}^\pm\bar{\psi}^\mp + \theta^\pm\bar{\theta}^\pm F^\pm . \qquad (1\cdot 3)$$

In terms of the component fields the equations of motion read

$$\partial_z\partial_{\bar{z}}\varphi^+ = -g^2\sin\varphi^+\cos\varphi^- - g^2\psi^+\bar{\psi}^+\sin\varphi^- ,$$
$$\partial_{\bar{z}}\psi^- = g\bar{\psi}^+\cos\varphi^- ,$$
$$\partial_z\bar{\psi}^- = -g\psi^+\cos\varphi^- , \qquad (1\cdot 4)$$

where we set $\beta=1$ for simplicity. On the basis of real components $\varphi^\pm=(1/\sqrt{2})\times(\varphi_1\pm i\varphi_2)$ one of the bosonic parts of the equations of motion, for φ_1, in the limit of vanishing fermion fields becomes a sine-Gordon equation and the other one, for φ_2, becomes a hyperbolic sine-Gordon (sinh-Gordon) equation.

The conservation laws were studied at the classical level,[14),15)] and some lower-spin conserved charges were explicitly constructed as polynomials in super-Virasoro generator (super energy-momentum tensor) in Feigin-Fuchs-Miura form.

More systematic method to investigate classical conservation laws is provided by Lie superalgebraic approach. Recently Inami and Kanno studied the $N=2$ super KdV and sine-Gordon theories in this framework.[16)] They have shown that the $N=2$ supersymmetric sine-Gordon corresponds to the $A(1,1)^{(1)}$ Lie superalgebra.

Now what about the conservation laws at the quantum level? Does the factorization of S-matrix into bosonic and supersymmetric parts[17),12),13)] also hold in $N=2$ case? For this purpose we shall base our argument on the perturbation theory. In this article, we shall also present the argument on exactness of first order perturbation.

In the next section, we discuss conservation laws in the framework of $N=2$ perturbation theory based on the superspace formalism. In § 3, we present the quantum conserved charges which generate the quantum group symmetry denoted as $\widehat{sl_q(2)}$. In § 4, we study the super topological charges of $N=2$ supersymmetry and their relation with the topological charge which belongs to the quantum group algebras. This analysis is important for determining the $N=2$ soliton structure. We present the S-matrices for soliton-(anti) soliton as well as soliton-breather scattering in $N=2$ sine-Gordon theory in § 5. The final section is devoted to conclusion and future problems.

§ 2. $N=2$ perturbation theory and conservation laws

The action which leads to the classical equations of motion for the $N=2$ sine-Gordon theory is constructed by adding a chiral perturbation to the $N=2$ free action S_{free} as

$$S=S_{\text{free}}-\lambda\int d^2z\Phi(z,\bar{z}),$$
$$\Phi(z,\bar{z})=\int d^2\theta^+(e^{i\beta\phi^+}+e^{-i\beta\phi^+})+\int d^2\theta^-(e^{i\beta\phi^-}+e^{-i\beta\phi^-})$$
$$=2\left(\int d^2\theta^+\cos\beta\phi^+ +\int d^2\theta^-\cos\beta\phi^-\right), \tag{2·1}$$

where ϕ^+ and ϕ^- are free chiral and anti-chiral superfields satisfying the conditions: $\bar{D}_+D_+\phi^+=0$ and $\bar{D}_-D_-\phi^-=0$. Hence they are decomposed into holomorphic and anti-holomorphic parts as $\phi^\pm=S^\pm+\bar{S}^\pm$.

To the lowest order in perturbation theory, we get

$$\partial_{\bar{z}_1}A(Z_1,\bar{Z}_1)=\lambda\partial_{\bar{z}_1}\left\{\int d^2z_2d^2\theta_2^-A(Z_1)(e^{i\beta\phi^-(Z_2)}+e^{-i\beta\phi^-(Z_2)})+(-\to+)\right\}. \tag{2·2}$$

Let us suppose that the operator product expansion of an operator A and the chiral perturbation term $\exp[i\beta S^-(Z_2)]$ is given by

$$A(Z_1)e^{i\beta S^-(Z_2)}\sim\frac{\theta_{12}^-}{Z_{12}}\times(\text{residue}), \tag{2·3}$$

where Z_{12} and $\theta_{12}^\pm$ stand for the invariant distances in $N=2$ superspace

$$Z_{12}=z_1-z_2-\frac{1}{2}(\theta_1^+\theta_2^-+\theta_1^-\theta_2^+),\quad \theta_{12}^\pm=\theta_1^\pm-\theta_2^\pm. \tag{2·4}$$

Now if the residue behaves as

$$\text{residue}\sim D_+X+D_-X' \tag{2·5}$$

for some operators X and X', then the following charge

$$Q=\oint dzd\theta^{+}d\theta^{-}A(Z) \qquad (2\cdot 6)$$

turns out to be a conserved charge, provided that (2·6) is invariant under the interchanges: (i) $+\leftrightarrow-$ and (ii) $\beta\leftrightarrow-\beta$.

Now we can classify conserved charges at the quantum level into two categories. The first category includes regularized Virasoro polynomials which form an infinite set of higher-spin conserved charges that assure the integrability of the theory. The second one contains extra non-local conserved charges which do not have classical analogues in contrast to the first category and lead to the quantum group symmetry as will be discussed in the later section.

In Refs. 14) and 15), we have presented some lower-spin conserved polynomial charges in terms of the super stress tensor in Feigin-Fuchs-Miura form:

$$T=:D_{+}S^{+}D_{-}S^{-}:-i\alpha(\partial S^{+}-\partial S^{-}), \qquad (2\cdot 7)$$

which consists of the $N=2$ superconformal generators: the $U(1)$ charge, the supercurrents and the energy-momentum tensor (Virasoro generator). In the above equation we choose $\alpha=1/\beta$ so that the vertex operator $e^{i\beta S^{\pm}}$ should have the conformal weight 1/2 as the chiral screening operators. Then it turns out that the coefficients of the Virasoro polynomials depend upon "central charge" given by $c=3(1-2/\beta^{2})$.

Here one important observation is in order. Although we applied the $N=2$ superconformal field theory (SCFT) technique, our $N=2$ sine-Gordon theory should not be regarded as a perturbed $N=2$ SCFT minimal models, in contrast to the $N=0$ and $N=1$ cases. This is because, if so, a part of the chiral perturbation term $e^{-i\beta S^{\pm}}$ would possess a negative conformal weight $-1/2$ which, of course, cannot be accepted. We should rather interpret the present $N=2$ sine-Gordon theory as a super-renormalizable theory with zero background charge, for which the coupling constant λ has a mass dimension.

§ 3. Quantum conserved charges

As we have mentioned, the quantum theory has extra non-local charges of the vertex operator type in addition to the polynomial charges. The vertex type charges have no classical analogues. In our $N=2$ sine-Gordon we found the following extra quantum charges:

$$Q^{\pm}(\beta)=\oint dzd\theta^{+}d\theta^{-}:e^{\pm i(1/\beta)(S^{+}+S^{-})}:=\oint dzJ^{\pm}(z), \qquad (3\cdot 1)$$

which can be seen to satisfy the condition (2·5). As for Q^{-}, for example, we find

$$:\exp\left[-\frac{i}{\beta}(S^{+}+S^{-})\right](Z_{1})::\exp(i\beta S^{-})(Z_{2}):$$

$$\sim\frac{\theta_{1\bar{2}}}{Z_{12}}\times D_{-}\left\{\frac{1}{1-\beta^{2}}\exp\left[-\frac{i}{\beta}S^{+}+i\left(-\frac{1}{\beta}+\beta\right)S^{-}\right](Z_{2})\right\}. \qquad (3\cdot 2)$$

In Eq. (3·1) the non-local conserved currents $J^{\pm}(z)$ are found to be

$$J^{\pm}(z)=\left(\pm\frac{1}{\sqrt{2}\beta}\partial_z\varphi_2+\frac{1}{\beta^2}\psi^+\psi^-\right)e^{\pm(i\sqrt{2}/\beta)\varphi_1} \tag{3·3}$$

and similarly we have the anti-holomorphic non-local currents

$$\bar{J}^{\pm}(\bar{z})=\left(\mp\frac{1}{\sqrt{2}\beta}\partial_{\bar{z}}\bar{\varphi}_2+\frac{1}{\beta^2}\bar{\psi}^+\bar{\psi}^-\right)e^{\mp(i\sqrt{2}/\beta)\bar{\varphi}_1}. \tag{3·4}$$

Under the perturbation term $\lambda\int d^2w\Phi(w,\bar{w})$ where

$$\Phi(w,\bar{w})=\sum_{a,b=+,-}\Phi^{(ab)}(w)\bar{\Phi}^{(ab)}(\bar{w}) \tag{3·5}$$

with

$$\Phi^{(+\pm)}(w)=\pm i\beta\psi^+(w)e^{\pm i\beta\varphi^-}(w)\,,\quad \Phi^{(-\pm)}(w)=\pm i\beta\psi^-(w)e^{\pm i\beta\varphi^+}(w)\,,$$
$$\bar{\Phi}^{(+\pm)}(\bar{w})=\pm i\beta\bar{\psi}^+(\bar{w})e^{\pm i\beta\bar{\varphi}^-}(\bar{w})\,,\quad \bar{\Phi}^{(-\pm)}(\bar{w})=\pm i\beta\bar{\psi}^-(\bar{w})e^{\pm i\beta\bar{\varphi}^+}(\bar{w})\,, \tag{3·6}$$

they satisfy the following conservation laws

$$\partial_{\bar{z}}J^{\pm}=\partial_z H^{\pm}\,,\quad \partial_z\bar{J}^{\pm}=\partial_{\bar{z}}\bar{H}^{\pm}\,, \tag{3·7}$$

where

$$H^{\pm}=\lambda\sum_{ab}h^{\pm(ab)}(z)\bar{\Phi}^{(ab)}(\bar{z})\,,$$
$$\bar{H}^{\pm}=\lambda\sum_{ab}\Phi^{(ab)}(z)\bar{h}^{\pm(ab)}(\bar{z})\,, \tag{3·8}$$

together with the operator product expansion

$$J^{\pm}(z)\Phi^{(ab)}(w)\sim\frac{1}{z-w}\partial_w h^{\pm(ab)}(w)\,,$$
$$\bar{J}^{\pm}(\bar{z})\bar{\Phi}^{(ab)}(\bar{w})\sim\frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\bar{h}^{\pm(ab)}(\bar{w})\,, \tag{3·9}$$

and the non-vanishing $h^{\pm(ab)}(w)$'s are found to be

$$h^{+(\pm-)}(w)=\frac{\pm i}{\beta}\psi^{\pm}(w)e^{i(\sqrt{2}/\beta-\beta/\sqrt{2})\varphi_1\mp(\beta/\sqrt{2})\varphi_2}(w)\,,$$
$$h^{-(\pm+)}(w)=\frac{\pm i}{\beta}\psi^{\pm}(w)e^{-i(\sqrt{2}/\beta-\beta/\sqrt{2})\varphi_1\pm(\beta/\sqrt{2})\varphi_2}(w)\,, \tag{3·10}$$

and we also have similar expressions for $\bar{h}^{\pm(ab)}(\bar{w})$'s. From the conservation laws (3·7) we redefine conserved charges as

$$Q_{\pm}=\int dzJ^{\pm}+\int d\bar{z}H^{\pm}\,,$$
$$\bar{Q}_{\pm}=\int d\bar{z}\bar{J}^{\pm}+\int dz\bar{H}^{\pm}\,. \tag{3·11}$$

It turns out that these quantum conserved charges generate the q-deformation of $sl(2)$

affine Kac-Moody algebra denoted as $\widehat{sl_q(2)}$ which has been discussed by Bernard and LeClair[10] for the $N=0$ sine-Gordon case.

For the commutation relation between Q_+ and Q_-, for example, we find by using the perturbation theory

$$Q_+\bar{Q}_- - q^{-2}\bar{Q}_- Q_+ = \lambda\int_t dx\partial_x K\,, \tag{3·12}$$

where the quantum group deformation parameter q is given by

$$q=-e^{-i\pi/\gamma}\,, \quad \gamma=\beta^2 \tag{3·13}$$

and the $1/\gamma$ is equal to the spin of the conserved charges $Q_\pm$. Let us note that the γ and β are related with each other for $N=0$ and $N=1$ cases as

$$\gamma=\frac{\beta^2}{2-\beta^2}\,, \quad (N=0) \qquad \gamma=\frac{2\beta^2}{1-\beta^2}\,. \quad (N=1) \tag{3·14}$$

The quantity K on the right-hand side of (3·12) is found to be

$$K=\left(-\frac{1}{\beta^2}\right)\Big\{\exp\Big[i\Big(\frac{\sqrt{2}}{\beta}-\frac{\beta}{\sqrt{2}}\Big)\phi_1(x,t)-\frac{\beta}{\sqrt{2}}\phi_2(x,t)+i\phi_3(x,t)\Big]$$
$$+\exp\Big[i\Big(\frac{\sqrt{2}}{\beta}-\frac{\beta}{\sqrt{2}}\Big)\phi_1(x,t)+\frac{\beta}{\sqrt{2}}\phi_2(x,t)-i\phi_3(x,t)\Big]\Big\}\,, \tag{3·15}$$

where we have bosonized the fermions as $\psi^\pm=e^{\pm i\varphi_3}$ and $\bar{\psi}^\pm=e^{\pm i\bar{\varphi}_3}$ in order to avoid the subtlety of the Grassmann variables. In Eq. (3·15) we denote $\phi_i(x,t)=\varphi_i(z)+\bar{\varphi}_i(\bar{z})$ for $i=1, 2$ and 3.

§ 4. Topological charges and quantum group symmetry

In this section, we shall show that the quantum conserved charges we found in the previous section and the topological charge we now discuss generate a quantum group symmetry.

First we note that the action (2·1) is invariant under the following shifts of the fields (m being an integer):

1) $\phi_1\to\phi_1+\dfrac{2m\pi\sqrt{2}}{\beta}\,,\quad \phi_2,\ \phi_3$: fixed,

2) $\phi_1\to\phi_1+\dfrac{(2m+1)\pi\sqrt{2}}{\beta}\,,\quad \phi_2$: fixed, $\quad \phi_3\to\phi_3+\pi \pmod{2\pi}$.

Then we find that there exists a topological current for our $N=2$ case:

$$\mathcal{T}^\mu(x,t)=\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\{\sqrt{2}\phi_1(x,t)\}\,, \tag{4·1}$$

where $\epsilon^{\mu\nu}$ is the Levi-Civita symbol in two dimensions.

Therefore we can define the topological charge as

$$\mathcal{T}=\int_{-\infty}^{+\infty}\mathcal{T}^0 dx=\frac{\beta}{2\pi}\sqrt{2}\{\phi_1(x=+\infty)-\phi_1(x=-\infty)\}\,. \tag{4·2}$$

The right-hand side of (3·12) turns out to be

$$\lambda\int_t dx\partial_x K=2\lambda/(-\beta^2)\left[1-\exp\left\{i\left(\frac{\sqrt{2}}{\beta}-\frac{\beta}{\sqrt{2}}\right)\phi_1(x=-\infty)\right\}\cos\phi_3(x=-\infty)\right], \tag{4·3}$$

where we have taken the soliton configurations $\phi_1(x=+\infty)=\phi_3(x=+\infty)=0$ which is derived from the translational invariance, and $\phi_2(x=+\infty)=\phi_2(x=-\infty)=0$ because the hyperbolic sine-Gordon component damps at $\pm\infty$. Therefore, the topological charge is determined by $\phi_1(x=-\infty)$ as

$$\mathcal{T}=-\frac{\beta}{2\pi}\sqrt{2}\phi_1(x=-\infty)\,. \tag{4·4}$$

Hence we have the following boundary conditions in accordance with the invariance of the action:

i) $\mathcal{T}$ is an odd integer, $\phi_3(x=-\infty)=\pi \pmod{2\pi}$ and

ii) $\mathcal{T}$ is an even integer, $\phi_3(x=-\infty)=0 \pmod{2\pi}$.

Therefore we find the quantum commutation relation as follows

$$Q_\pm\bar{Q}_\pm-q^2\bar{Q}_\pm Q_\pm=0\,,$$
$$Q_\pm\bar{Q}_\mp-q^{-2}\bar{Q}_\mp Q_\pm=a(1-q^{\pm 2\mathcal{T}})\,,$$
$$[\mathcal{T},Q_\pm]=\pm 2Q_\pm\,,\quad [\mathcal{T},\bar{Q}_\pm]=\pm 2\bar{Q}_\pm\,, \tag{4·5}$$

where we have defined a constant $a=2\lambda/(-\beta^2)$. From this commutation relation we can easily see that the extra charges generate the quantum affine algebra $\widehat{sl_q(2)}$ with vanishing center, which has been discussed by Bernard and LeClair.[10)]

In the above derivation of (4·5), we have computed the algebras of the non-local quantum conserved charges to the first order in perturbation theory. Here one should note that the first order correction is the only possible correction for general values of β and hence the above result is actually exact. This statement was not explicitly shown in the previous articles.[14),15),22)] This can be shown by writing down all the possible perturbation terms and considering their scale dimensions as well as their symmetry. The proof goes as follows. Let us take the case of $\partial_{\bar{z}}J^\pm$, for illustration. We find that the left (holomorphic) and right (anti-holomorphic) conformal dimensions of $\partial_{\bar{z}}J^\pm$ is given by $(1+1/\beta^2, 1)$. Then the possible λ^n (n-th order) term should have the form

$$\psi\bar{\psi}\exp\left(\frac{i}{\beta}\varphi^+ +\frac{i}{\beta}\varphi^- +il\beta\varphi^+ +im\beta\varphi^-\right)\exp(il\beta\bar{\varphi}^+ +im\beta\bar{\varphi}^-) \tag{4·6}$$

with $k\,\partial_z$'s and $k'\partial_{\bar{z}}$'s being multiplied in front. In the above equation $\psi(\bar{\psi})$ denotes either $\psi^+(\bar{\psi}^+)$ or $\psi^-(\bar{\psi}^-)$. Terms without fermion fields would not lead to conserved charges that are invariant under supersymmetry transformations. By noting that the scale dimension of λ is $(1/2, 1/2)$, we have the following equations

$$\frac{1}{2}+lm\beta^2+\frac{1}{\beta^2}+l+m+\frac{n}{2}+k=1+\frac{1}{\beta^2},$$

$$\frac{1}{2}+lm\beta^2+\frac{n}{2}+k'=1. \tag{4·7}$$

From the above equations we get $lm=k'=0$ and $n=1$. Therefore only the first order term is allowed from the dimensional counting. This means that the lowest order term gives in fact the exact result. This kind of argument also applies to more complicated cases like (4·5).

We now consider the topological charges of $N=2$ supersymmetry which are different from the ordinary sine-Gordon topological charge given in (4·2). The supercharges are holomorphic without the perturbation. But now we should consider the effect of the perturbation terms. In the same methods as the bosonic topological charge, it is easily checked that the superalgebra has the topological modification.[18)]

The holomorphic and anti-holomorphic supercurrents are written in terms of components,

$$G^{\pm}(z)=\psi^{\pm}(z)\partial_z\varphi^{\pm}(z),$$

$$\bar{G}^{\pm}(\bar{z})=\bar{\psi}^{\mp}(\bar{z})\partial_{\bar{z}}\bar{\varphi}^{\mp}(\bar{z}), \tag{4·8}$$

which obey the conservation laws in perturbation theory as in the extra non-local currents:

$$\partial_{\bar{z}}G^{\pm}=\partial_z F^{\pm}, \quad \partial_z\bar{G}^{\pm}=\partial_{\bar{z}}\bar{F}^{\pm}, \tag{4·9}$$

where

$$F^{\pm}=\lambda\sum_{ab}f^{\pm(ab)}(z)\bar{\Phi}^{(ab)}(\bar{z}),$$

$$\bar{F}^{\pm}=\lambda\sum_{ab}\Phi^{(ab)}(z)\bar{f}^{\pm(ab)}(\bar{z}) \tag{4·10}$$

with $f^{\pm(ab)}(z)$'s and $\bar{f}^{\pm(ab)}(\bar{z})$'s being defined through the operator product expansion:

$$G^{\pm}(z)\Phi^{(ab)}(w)\sim\frac{1}{z-w}\partial_w f^{\pm(ab)}(w),$$

$$\bar{G}^{\pm}(\bar{z})\bar{\Phi}^{(ab)}(\bar{w})\sim\frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\bar{f}^{\pm(ab)}(\bar{w}). \tag{4·11}$$

Now we define the supercharges as follows

$$Q^{\pm}=\int dzG^{\pm}+\int d\bar{z}F^{\pm}, \quad \bar{Q}^{\pm}=\int d\bar{z}\bar{G}^{\pm}+\int dz\bar{F}^{\pm} \tag{4·12}$$

and non-vanishing $f^{\pm(ab)}$'s and $\bar{f}^{\pm(ab)}$'s are given by

$$f^{+(-\pm)}=-e^{\pm i\beta\varphi^{+}}, \quad f^{-(+\pm)}=-e^{\pm i\beta\varphi^{-}},$$

$$\bar{f}^{+(+\pm)}=-e^{\pm i\beta\bar{\varphi}^{-}}, \quad \bar{f}^{-(-\pm)}=-e^{\pm i\beta\bar{\varphi}^{+}}. \tag{4·13}$$

Perturbation theory leads to the following anti-commutation relation

$$\{Q^{\pm}, \bar{Q}^{\mp}\}=\lambda\int dx\partial_x K^{\pm\mp}, \tag{4·14}$$

where

$$K^{\pm\mp}=\sum_{ab}f^{\pm(ab)}(z)\bar{f}^{\mp(ab)}(\bar{z})=e^{i\beta(\varphi^{\pm}+\bar{\varphi}^{\pm})}+e^{-i\beta(\varphi^{\pm}+\bar{\varphi}^{\pm})}$$

$$=\exp\{i(\beta/\sqrt{2})(\phi_1(x,t)\pm i\phi_2(x,t))\}+\exp\{-i(\beta/\sqrt{2})(\phi_1(x,t)\pm i\phi_2(x,t))\}, \tag{4·15}$$

while we get

$$\{Q^{\pm}, \bar{Q}^{\pm}\}=0. \tag{4·16}$$

With the same soliton configuration as discussed before we obtain

$$\{Q^{\pm}, \bar{Q}^{\mp}\}=\lambda[2-(e^{i(\beta/\sqrt{2})\phi_1(-\infty)}+e^{-i(\beta/\sqrt{2})\phi_1(-\infty)})]=\mathcal{T}'. \tag{4·17}$$

Hence we note that the super topological charge $\mathcal{T}'$ is determined by the topological charge $\mathcal{T}$ as

$$\mathcal{T}'=2\lambda(1-(-1)^{\mathcal{T}}). \tag{4·18}$$

Therefore, when $\mathcal{T}$ is an odd integer we get $\mathcal{T}'=4\lambda$, on the other hand, if $\mathcal{T}$ is an even integer we have $\mathcal{T}'=0$. Since the soliton and the anti-soliton possess the topological charge $\mathcal{T}=\pm1$, they must have non-zero super topological charge $\mathcal{T}'=4\lambda$. We will see the physical consequence of this fact in the next section.

In terms of the real basis Q_1 and Q_2 satisfying

$$\{Q_{1,2}, \bar{Q}_{1,2}\}=2T_{1,2}, \tag{4·19}$$

$Q^{\pm}$ and $\bar{Q}^{\pm}$ are given by

$$Q^{\pm}=\frac{1}{\sqrt{2}}(Q_1\pm iQ_2), \quad \bar{Q}^{\pm}=\frac{1}{\sqrt{2}}(\bar{Q}_1\mp i\bar{Q}_2) \tag{4·20}$$

and therefore we have $\mathcal{T}'=T_1-T_2=2T_1=-2T_2$.

§ 5. Soliton structure and S-matrix of $N=2$ sine-Gordon theory

In the present section, we examine the $N=2$ soliton structure and construct the S-matrices of the $N=2$ sine-Gordon theory.

We first study the realization of $N=2$ supersymmetry. Let us first consider the realization of $N=1$ supersymmetry as discussed by Zamolodchikov.[19] The tricritical Ising model possesses $N=1$ supersymmetry and is described by ϕ^6 potential in the Landau-Ginzburg picture. The suitably perturbed model which is still solvable and has $N=1$ supersymmetry is represented as the deformed ϕ^6 potential which has three minima. We denote these minima by numbers, -1, 0, 1. The fundamental particles correspond to the soliton or kink states which are classical configurations going from one minimum to another. We denote a kink state connecting the degenerate vacua a and b ($a, b=0, \pm1, |a-b|=1$) by $|K_{ab}(\theta)\rangle$ where θ represents the rapidity of the state. Therefore we have four solitons which we denote by $|K_{10}\rangle$, $|K_{-10}\rangle$, $|K_{01}\rangle$ and $|K_{0-1}\rangle$. We now introduce supercharges denoted as Q and $\bar{Q}$. The supercharge Q acts on this kink state as

$$Q|K_{ab}(\theta)\rangle=i(a+ib)e^{\theta/2}K_{-ab}(\theta)\,, \tag{5·1}$$

where $a^2+b^2=1$, and we have similar expression for $\bar{Q}$ by replacing $i(a+ib)$ with $-i(a-ib)$. We easily see that the superalgebra is realized as

$$Q^2=e^{\theta}=P\,,\quad \bar{Q}^2=e^{-\theta}=\bar{P}\,,\quad \{Q,\bar{Q}\}=2T\,, \tag{5·2}$$

where we have taken the mass to unity and T is the super-topological charge. Any multi-particle state can be given by

$$|K_{ab}(\theta_1)K_{bc}(\theta_2)K_{cd}(\theta_3)\cdots\rangle\,. \tag{5·3}$$

The supersymmetry is realized on the multi-particle state as

$$Q|K_{a_1a_2}(\theta_1)K_{a_2a_3}(\theta_2)\cdots K_{a_{N-1}a_N}(\theta_{N-1})\rangle$$
$$=\sum_{i=1}^{N} i(a_i+ia_{i+1})e^{\theta_i/2}|K_{-a_1-a_2}\cdots K_{-a_{i-1}-a_i}K_{-a_ia_{i+1}}K_{a_{i+1}a_{i+2}}\cdots K_{a_{N-1}a_N}\rangle\,. \tag{5·4}$$

Now we proceed to the $N=2$ supersymmetry which can be realized as a tensor product of two tri-critical Ising soliton states. The fundamental particles are written in the form:

$$|K_{ab,cd}\rangle\equiv|K_{ab}K_{cd}\rangle\,. \tag{5·5}$$

In $N=2$ case, we have supercharges Q_1 and Q_2 and conjugated ones. We assume that the first supercharge operates on the first factor of the right-hand side of the equation just as tri-critical Ising model. But when the second supercharge operates on the second factor, we require that the sign of the components of the first factor should always change. For example, the supercharge acts on the one-particle states:

$$Q_1|K_{ab,cd}\rangle=i(a+ib)e^{\theta/2}|K_{-ab,cd}\rangle\,,\qquad \bar{Q}_1|K_{ab,cd}\rangle=-i(a-ib)e^{-\theta/2}|K_{-ab,cd}\rangle\,,$$
$$Q_2|K_{ab,cd}\rangle=i(c+id)e^{\theta/2}|K_{-a-b,-cd}\rangle\,,\quad \bar{Q}_2|K_{ab,cd}\rangle=-i(c-id)e^{-\theta/2}|K_{-a-b,-cd}\rangle\,. \tag{5·6}$$

Therefore the supercharges satisfy

$$Q_1{}^2=Q_2{}^2=e^{\theta}=P\,,\quad \bar{Q}_1{}^2=\bar{Q}_2{}^2=e^{-\theta}=\bar{P}\,,$$
$$\{Q_{1,2},\bar{Q}_{1,2}\}=2T_{1,2}\,,\quad T_1=-(a^2-b^2)\,,\quad T_2=-(c^2-d^2)\,, \tag{5·7}$$

and they, otherwise, anticommute with each other. The reason why we demand that the sign in the first factor change when we operate the second charge is to assure the anticommutativity of Q_1 and Q_2.

Now the bosonic and fermionic $N=1$ tri-critical Ising soliton states and their antiparticle states are constructed as:[12),13),20)]

$$|B\rangle=\frac{1}{\sqrt{2}}(|K_{-10}\rangle+|K_{10}\rangle)\,,\quad |F\rangle=\frac{1}{\sqrt{2}}(|K_{-10}\rangle-|K_{10}\rangle)\,.$$

$$|\bar{B}\rangle=\frac{1}{\sqrt{2}}(|K_{0-1}\rangle+|K_{01}\rangle)\,,\quad |\bar{F}\rangle=\frac{1}{\sqrt{2}}(|K_{0-1}\rangle-|K_{01}\rangle)\,. \tag{5·8}$$

As we mentioned, our quantum conserved charges are integrals of the highest component of their supermultiplet by construction. Therefore they commute with the supercharges. This means that we have two commuting symmetries, i.e., $N=2$ supersymmetry and quantum affine algebra $\widehat{sl_q(2)}$. Hence the S-matrix can be written as a product of "minimal supersymmetric S-matrix" and the ordinary sine-Gordon S-matrix as in the $N=1$ case discussed by Ahn et al.[12),20)] and by Schoutens.[13)]

$$S_{N2\mathrm{SG}}\sim S_{MN2}\times S_{\mathrm{SG}}\,, \tag{5·9}$$

where $S_{N2\mathrm{SG}}$, S_{MN2} and S_{SG} represent the S-matrices of $N=2$ supersymmetric sine-Gorbon, of the minimal $N=2$ supersymmetric model, and of ordinary bosonic sine-Gordon,[21)] respectively. We expect the "minimal supersymmetric S-matrix" is that of the coset model $SU(2)_2\times SU(2)_2/SU(2)_4$ which is the perturbed $c=1$ model discussed in Refs. 12) and 20). Alternatively, it may be realized at a special value of the coupling constant, $\beta=2/\sqrt{3}$, of the bosonic sine-Gordon theory which corresponds to $c=1$ $N=2$ super CFT.[15)]

The fundamental particles of our model can be represented as the product of RSOS solitons of the coset model and kinks of the sine-Gordon part. We can show that they actually form the super multiplets, though we need suitable truncation to obtain the minimal number of particles.

Fundamental particles that form supermultiplets are

$$|BB\rangle, |BF\rangle, |FB\rangle, |FF\rangle, |B\bar{B}\rangle, |B\bar{F}\rangle, |F\bar{B}\rangle, |F\bar{F}\rangle \tag{5·10}$$

and their conjugates. So we have sixteen fundamental particles. But some consideration about topological charges leads to reduction of the number of fundamental particles. Because of the form of the potential without fermions, we expect kink and anti-kink solutions as in the case of bosonic and $N=1$ supersymmetric cases. And the topological charges are discussed in Refs. 12) and 13) and in the beginning of this paper. Since kink and anti-kink solutions have odd topological charge, their super-topological charges should be non-zero. This restricts $|XY\rangle$ in Eq. (5·10) to one of

$$|B\bar{B}\rangle, |B\bar{F}\rangle, |F\bar{B}\rangle, |F\bar{F}\rangle, \tag{5·11}$$

which possess $T_1=-T_2=-1$ and hence $\mathcal{T}'=-2$ and their conjugates with $T_1=-T_2=1$ and $\mathcal{T}'=2$. For remaining states the super-topological charges vanish. In fact, particles of (5·11) and their conjugates form a closed set, i.e., we can restrict on-shell states to the set and other states cannot appear in the final state. Of course, this is another expression of the conservation laws of super topological charges. Therefore we have four supermultiplets; those of

$$|A^{\pm}B\bar{B}\rangle, |A^{\pm}\bar{B}B\rangle\,, \tag{5·12}$$

where we have denoted kink and anti-kink states of ordinary sine-Gordon theory by A^+ and A^-.[21)]

Now we parametrize the S-matrices for tri-critical Ising solitons scattering

$K_{ab}(\theta_1)+K_{bc}(\theta_2)\to K_{ad}(\theta_2)+K_{dc}(\theta_1)$ as

$$|K_{ab}(\theta_1)K_{bc}(\theta_2)\rangle_{\text{in}}=\sum_d S_{bc}^{da}(\theta_{12})|K_{ad}(\theta_2)K_{dc}(\theta_1)\rangle_{\text{out}}, \tag{5·13}$$

where $a, b, c, d=0, \pm 1$, and $\theta_{12}=\theta_1-\theta_2$ is the rapidity difference. Assuming the commutativity of the supercharges and S-matrix Zamolodchikov derived the S-matrix.[19]

The S-matrix can be parametrized as

$$|K_{0a}(\theta_1)K_{a0}(\theta_2)\rangle_{\text{in}}=A_0(\theta_{12})|K_{0a}(\theta_2)K_{a0}(\theta_1)\rangle_{\text{out}}+A_1(\theta_{12})|K_{0-a}(\theta_2)K_{-a0}(\theta_1)\rangle_{\text{out}},$$
$$|K_{a0}(\theta_1)K_{0a}(\theta_2)\rangle_{\text{in}}=B_0(\theta_{12})|K_{a0}(\theta_2)K_{0a}(\theta_1)\rangle_{\text{out}},$$
$$|K_{a0}(\theta_1)K_{0-a}(\theta_2)\rangle_{\text{in}}=B_1(\theta_{12})|K_{a0}(\theta_2)K_{0-a}(\theta_1)\rangle_{\text{out}}, \tag{5·14}$$

where

$$A_0(\theta)=e^{C\theta}\cosh\left(\frac{\theta}{4}\right)S(\theta), \qquad A_1(\theta)=-ie^{C\theta}\sinh\left(\frac{\theta}{4}\right)S(\theta),$$
$$B_0(\theta)=\sqrt{2}e^{-C\theta}\cosh\left(\frac{\theta-i\pi}{4}\right)S(\theta), \quad B_1(\theta)=\sqrt{2}e^{-C\theta}\cosh\left(\frac{\theta+i\pi}{4}\right)S(\theta) \tag{5·15}$$

with

$$C=\frac{1}{2\pi i}\log 2,$$
$$S(\theta)=\frac{1}{\sqrt{\pi}}\prod_{k=1}^{\infty}\frac{\Gamma(k-\theta/2\pi i)\Gamma(-1/2+k+\theta/2\pi i)}{\Gamma(1/2+k-\theta/2\pi i)\Gamma(k+\theta/2\pi i)}. \tag{5·16}$$

As for the bosonic sine-Gordon theory, the S-matrix is well known.[21] For kink A^+ and anti-kink A^- we have

$$S_{\text{SG}ab;cd}=\frac{U(\theta)}{i\pi}S_{ab;cd}, \tag{5·17}$$

where $a, b, c, d=\pm$ and $S_{\text{SG}ab;cd}$ is the scattering amplitude for the process: $A^a+A^b\to A^c+A^d$ and $U(\theta)$ is given as

$$U(\theta)=\Gamma\left(\frac{1}{\gamma}\right)\Gamma\left(1+i\frac{\theta}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}-i\frac{\theta}{\gamma}\right)\prod_{n=1}^{\infty}\frac{R_n(\theta)R_n(i\pi-\theta)}{R_n(0)R_n(i\pi)},$$
$$R_n(\theta)=\frac{\Gamma\left(\frac{2n}{\gamma}+\frac{i\theta}{\gamma}\right)\Gamma\left(1+\frac{2n}{\gamma}+\frac{i\theta}{\gamma}\right)}{\Gamma\left(\frac{2n+1}{\gamma}+\frac{i\theta}{\gamma}\right)\Gamma\left(1+\frac{2n-1}{\gamma}+\frac{i\theta}{\gamma}\right)} \tag{5·18}$$

with γ being given by Eq. (3·13). Non-vanishing $S_{ab;cd}$'s are

$$S_{++;++}=S_{--;--}=\sinh\left[\frac{1}{\gamma}(i\pi-\theta)\right],$$
$$S_{+-;+-}=S_{-+;-+}=\sinh\frac{\theta}{\gamma}, \quad S_{+-;-+}=S_{-+;+-}=i\sin\frac{\pi}{\gamma}. \tag{5·19}$$

Here we note that the $S_{ab;cd}$ is the well-known solution of Yang-Baxter equation for the six-vertex model.

Now we present explicit results for minimal $N=2$ supersymmetric parts of S-matrices. First we write down the in-states of tri-critical Ising soliton and anti-soliton scattering in terms of the out-states. This can be provided by the matrices $\mathcal{A}$ and $\mathcal{B}$ given below. And then, by taking the tensor product of these in-states we can construct the S-matrices for soliton-antisoliton scattering for $N=2$ supersymmetry as follows:[22)]

$$|X\bar{Y}(\theta_1)\bar{V}W(\theta_2)\rangle_{\rm in}=|X(\theta_1)\bar{V}(\theta_2)\rangle_{\rm in}|\bar{Y}(\theta_1)W(\theta_2)\rangle_{\rm in}$$
$$=\sum_{X',\bar{Y}',\bar{V}',W'}\mathcal{B}_{X\bar{V},X'\bar{V}'}(\theta_{12})\mathcal{A}_{\bar{Y}W,\bar{Y}'W'}(\theta_{12})|X'\bar{Y}'(\theta_2)\bar{V}'W'(\theta_1)\rangle_{\rm out}\,, \tag{5·20}$$

where the indices $X'(\bar{Y}')$ and $W'(\bar{V}')$ run over $B(\bar{B})$ and $F(\bar{F})$. In the above equation the tri-critical Ising soliton amplitudes $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A}=\begin{pmatrix} A_+ & A_+ & 0 & 0 \\ A_+ & A_+ & 0 & 0 \\ 0 & 0 & A_- & A_- \\ 0 & 0 & A_- & A_- \end{pmatrix},\quad \mathcal{B}=\begin{pmatrix} B_+ & B_- & 0 & 0 \\ B_- & B_+ & 0 & 0 \\ 0 & 0 & B_+ & B_- \\ 0 & 0 & B_- & B_+ \end{pmatrix}, \tag{5·21}$$

where the rows and columns of $\mathcal{A}$ are arranged in the order: $\bar{B}B$, $\bar{F}F$, $\bar{B}F$ and $\bar{F}B$, while those of $\mathcal{B}$ are in the order: $B\bar{B}$, $F\bar{F}$, $B\bar{F}$ and $F\bar{B}$. In (5·21) $A_\pm(\theta)$ and $B_\pm(\theta)$ are obtained from (5·15) as

$$A_+(\theta)=\frac{1}{2}(A_0(\theta)+A_1(\theta))=\frac{1}{2}e^{C\theta}\left(\cosh\frac{\theta}{4}-i\sinh\frac{\theta}{4}\right)S(\theta)\,,$$
$$A_-(\theta)=\frac{1}{2}(A_0(\theta)-A_1(\theta))=\frac{1}{2}e^{C\theta}\left(\cosh\frac{\theta}{4}+i\sinh\frac{\theta}{4}\right)S(\theta)\,,$$
$$B_+(\theta)=\frac{1}{2}(B_0(\theta)+B_1(\theta))=e^{-C\theta}\cosh\frac{\theta}{4}S(\theta)\,,$$
$$B_-(\theta)=\frac{1}{2}(B_0(\theta)-B_1(\theta))=-ie^{-C\theta}\sinh\frac{\theta}{4}S(\theta)\,. \tag{5·22}$$

For illustration, we consider the scattering of $B\bar{B}$ and $\bar{B}B$ as follows:

$$|B\bar{B}(\theta_1)\bar{B}B(\theta_2)\rangle_{\rm in}=|B(\theta_1)\bar{B}(\theta_2)\rangle_{\rm in}|\bar{B}(\theta_1)B(\theta_2)\rangle_{\rm in}$$
$$=B_+A_+|(B\bar{B})(\bar{B}B)\rangle_{\rm out}+B_-A_+|(F\bar{B})(\bar{F}B)\rangle_{\rm out}$$
$$+B_+A_+|(B\bar{F})(\bar{B}F)\rangle_{\rm out}+B_-A_+|(F\bar{F})(\bar{F}F)\rangle_{\rm out}\,. \tag{5·23}$$

Hence, for example, we obtain the following result:

$$S(B\bar{B}+\bar{B}B\rightarrow B\bar{B}+\bar{B}B)=S(B\bar{B}+\bar{B}B\rightarrow B\bar{F}+\bar{B}F)$$
$$=B_+A_+=\frac{1}{4}\left(1+\cosh\frac{\theta}{2}-i\sinh\frac{\theta}{2}\right)S^2(\theta)\,. \tag{5·24}$$

The S-matrices for the processes: $\bar{X}Y+V\bar{W}\to\bar{X}'Y'+V'\bar{W}'$ are obtained by crossing symmetry. The total S-matrices are given as a product of above $N=2$ supersymmetry parts with the ordinary sine-Gordon S-matrix factors as follows:

$$S_{N2SG}(\theta)=S_{MN2}(\theta)\times S_{SG}(x=e^{\theta/\gamma},\ q=-e^{-i\pi/\gamma})\,. \tag{5·25}$$

Here we have a comment on the correspondence between the coset and the bosonic sine-Gordon theories. As we mentioned before, the $N=2$ supersymmetry is realized for the special value of the coupling constant, $\beta=2/\sqrt{3}$. By setting $\gamma=2$ corresponding to the above value of β in Eqs. (5·17)~(5·19) (or setting $\gamma=16\pi$ in the S-matrix formula of Ref. 21) where the normalization of γ differs from ours by a factor 8π) for ordinary bosonic sine-Gordon theory, we obtain the scattering matrices for soliton A and anti-soliton $\bar{A}$ (here we use this notation instead of $A^{\pm}$, following Zamolodchikov's notation):[21]

$$|A(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}=S_T(\theta_{12})|\bar{A}(\theta_2)A(\theta_1)\rangle_{\rm out}+S_R(\theta_{12})|A(\theta_2)\bar{A}(\theta_1)\rangle_{\rm out}\,,$$
$$|A(\theta_1)A(\theta_2)\rangle_{\rm in}=S_0(\theta_{12})|A(\theta_2)A(\theta_1)\rangle_{\rm out},\ |\bar{A}(\theta_1)\bar{A}(\theta_2)\rangle_{\rm in}=S_0(\theta_{12})|\bar{A}(\theta_2)\bar{A}(\theta_1)\rangle_{\rm out}\,, \tag{5·26}$$

where S_T and S_R are transmission and reflection amplitudes for soliton-antisoliton scattering and S_0 denotes the amplitude for identical soliton scattering

$$S_T(\theta)=-i\sinh\frac{\theta}{2}S^2(\theta)\,,\quad S_0(\theta)=\cosh\frac{\theta}{2}S^2(\theta)\,,\quad S_R(\theta)=S^2(\theta)\,. \tag{5·27}$$

What is remarkable here is that we can recover these results by expressing the soliton states in terms of the coset soliton states as

$$\begin{aligned}|A\rangle&=|\bar{B}B\rangle-|\bar{F}B\rangle+|\bar{B}F\rangle+|\bar{F}F\rangle+|B\bar{B}\rangle+|F\bar{B}\rangle-|B\bar{F}\rangle+|F\bar{F}\rangle\\&=|K_{01}K_{-10}\rangle+|K_{01}K_{10}\rangle+|K_{-10}K_{0-1}\rangle+|K_{-10}K_{01}\rangle\\&\quad+|K_{0-1}K_{-10}\rangle-|K_{0-1}K_{10}\rangle-|K_{10}K_{0-1}\rangle+|K_{10}K_{01}\rangle\,.\end{aligned} \tag{5·28}$$

This strongly supports the validity of our calculation for S-matrices of $N=2$ SG soliton scattering.

Now let us briefly consider the $N=2$ sine-Gordon breathers which are bound states of the $N=2$ kinks. As in the case of $N=2$ kinks, the $N=2$ breathers can be constructed from two tri-critical Ising models and bosonic sine-Gordon theory. In the case of $N=1$ sine-Gordon theory we have $N=1$ bosonic or fermionic breathers with no topological charge denoted by ϕ_n and ψ_n which form a supermultiplet of tri-critical Ising model as discussed by Ahn.[20] The n-th breathers ϕ_n and ψ_n possess a mass: $m_n=2m\sin(n\pi\gamma/2)$ $(\gamma<1)$ where m is the soliton mass and set to unity. In our $N=2$ case, the breather can be represented as

$$|B_n x_n y_n\rangle\,, \tag{5·29}$$

where B_n in the n-th breather of ordinary sine-Gordon and x_n and y_n stand for either ϕ_n or ψ_n. Therefore we get the following $N=2$ breather multiplet:

$$|\Phi_n{}^{(00)}\rangle\equiv|B_n\phi_n\phi_n\rangle\,,\quad |\Phi_n{}^{(01)}\rangle\equiv|B_n\phi_n\psi_n\rangle\,,$$

$$|\Phi_n^{(10)}\rangle \equiv |B_n \psi_n \phi_n\rangle\,, \quad |\Phi_n^{(11)}\rangle \equiv |B_n \psi_n \psi_n\rangle\,. \tag{5·30}$$

We now examine how to construct S-matrices for breather-soliton scattering. Let us take $|\Phi_n^{(00)}\rangle$ and $|B\bar{B}\rangle$ as an initial state. For minimal $N=2$ supersymmetric part, we calculate the product of in-states of $|\phi_n B\rangle$ and $|\phi_n \bar{B}\rangle$, which are given as

$$\begin{aligned} |\phi_n B\rangle_{\rm in} &= X_n(\theta) 2^{-\Delta\theta_n/2\pi i}(\alpha_n |\phi_n B\rangle_{\rm out} + \beta_n |\psi_n F\rangle_{\rm out})\,, \\ |\phi_n \bar{B}\rangle_{\rm in} &= X_n(\theta) 2^{\Delta\theta_n/2\pi i}(\gamma_n |\phi_n \bar{B}\rangle_{\rm out} + \delta_n |\psi_n \bar{F}\rangle_{\rm out})\,, \end{aligned} \tag{5·31}$$

where $\Delta\theta_n = i\pi - in\pi\gamma$ and $X_n(\theta)$ is given by

$$X_n(\theta) = -2S\left(\theta + \frac{1}{2}\Delta\theta_n\right) S\left(\theta - \frac{1}{2}\Delta\theta_n\right) \tag{5·32}$$

with $S(\theta)$ being given in (5·16) and $\alpha_n = \gamma_n = \cosh[(2\theta - i\pi)/4]$, $\beta_n = \sqrt{m_n}$ and $\delta_n = -i\sqrt{m_n}$ as obtained by Ahn.[20] Hence we get

$$\begin{aligned} |\phi_n(\theta_1) B(\theta_2)\rangle_{\rm in} |\phi_n(\theta_1) \bar{B}(\theta_2)\rangle_{\rm in} = (X_n(\theta))^2 \{ &\alpha_n \gamma_n |\phi_n \phi_n B\bar{B}\rangle_{\rm out} + \beta_n \gamma_n |\psi_n \phi_n F\bar{B}\rangle_{\rm out} \\ &+ \alpha_n \delta_n |\phi_n \psi_n B\bar{F}\rangle_{\rm out} + \beta_n \delta_n |\psi_n \psi_n F\bar{F}\rangle_{\rm out} \}\,. \end{aligned} \tag{5·33}$$

Here, note that the factor $2^{-\Delta\theta_n/2\pi i}$ in $|\phi_n B\rangle_{\rm in}$ is canceled by the factor $2^{\Delta\theta_n/2\pi i}$ in $|\phi_n \bar{B}\rangle_{\rm in}$. Thus we obtain the following S-matrices.

$$\begin{aligned} &S(\phi_n \phi_n + B\bar{B} \to \phi_n \phi_n + B\bar{B}) = (X_n(\theta))^2 \cosh^2[(2\theta - i\pi)/4]\,, \\ &S(\phi_n \phi_n + B\bar{B} \to \phi_n \psi_n + B\bar{F}) = (X_n(\theta))^2 (-i\sqrt{m_n}) \cosh[(2\theta - i\pi)/4]\,, \\ &S(\phi_n \phi_n + B\bar{B} \to \psi_n \phi_n + F\bar{B}) = (X_n(\theta))^2 \sqrt{m_n} \cosh[(2\theta - i\pi)/4]\,, \\ &S(\phi_n \phi_n + B\bar{B} \to \psi_n \psi_n + F\bar{F}) = (X_n(\theta))^2 (-i m_n)\,. \end{aligned} \tag{5·34}$$

Other S-matrices for breather-solution as well as breather-breather scattering should also be calculated through a similar procedure.

§ 6. Conclusion

In this article we have investigated the $N=2$ sine-Gordon theory in the framework of perturbation theory. We obtained the quantum conserved charges which generate $sl(2)$ quantum affine Kac-Moody algebra $\widehat{sl_q(2)}$. The $N=2$ supersymmetry commutes with this quantum group symmetry.

On the basis of this two commuting quantum symmetries, we have constructed S-matrix of $N=2$ sine-Gordon theory as a product of S-matrix for the $N=2$ minimal supersymmetric part and that for the ordinary sine-Gordon part. One interesting point is the topological charges of the two symmetries. We derived the relation of the ordinary sine-Gordon topological charge and the topological charges of $N=2$ supersymmetry, which was originally discussed by Witten and Olive.[18] It turns out that the relation between the topological and super-topological charges plays an essential role when we restrict the fundamental particles from the abstract realization theory. One remarkable observation is that we can reproduce the soliton-antisoliton

S-matrices of ordinary bosonic sine-Gordon theory at a special value of the coupling constant, $\beta=2/\sqrt{3}$, which corresponds to the $N=2$ SUSY point.

Now some remarks on further problems are in order. Our model is the simplest model and we believe it can be extended to any $N=2$ supersymmetric models.[23)~25)] In this regards, it would be intriguing to extend the present analysis to the $N=2$ supersymmetric version of Toda field theories,[26)~31)] to see if the quantum group structure is described by commuting $N=2$ supersymmetry and quantum group symmetry $\widehat{sl_q(n)}$ for $n=3, 4, \cdots$. Another interesting subjects is to investigate the correspondence of $N=2$ sine-Gordon theory with some Thirring-type fermionic theories. This is motivated by the well-known correspondence between the $N=0$ sine-Gordon theory and the massive Thirring model.[32),33)] We have one more comment on supersymmetry (see Refs. 13), 17), 23) and 24)). We think it becomes clear that S-matrix of N-extended supersymmetric models can be constructed simply by making the N product of that of the $N=1$ supersymmetric model (tri-critical Ising model) and some bosonic part, up to CDD ambiguity. We are interested in whether there is a model for arbitrary N.

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