Where Has the Conformal Symmetry of Anyons Gone in a Magnetic Field?

Tadashi AWAJI and Masahiro HOTTA

Department of Physics, Tohoku University, Sendai 980

It is known that a many-body system of free anyons has the conformal symmetry and that the symmetry is broken explicitly when an external magnetic field is applied to the system. However, we can show that a modified conformal symmetry still exists in the presence of the magnetic field and plays a crucial role in anyon energy spectrum.

§ 1. Introduction

Recently, a $2+1$-dimensional Chern-Simons (C-S) gauged Schrödinger field theory in the presence of a uniform external magnetic field has attracted much attention. This is because it describes the fractional quantum Hall effect (FQHE) in condensed matter physics.[2] A Hall current with a fractional-valued Hall coefficient can take place in this model when an electric field is applied into the system. On the other hand, another interesting aspect of this model has been pointed out. Anyons,[1] which are particles with an intermediate statistics between bosonic and fermionic one, appear when the C-S gauged Schrödinger field is second-quantized. A many-body system of anyons is expected to have peculiar properties clearly different from those of ordinary particle systems.

Though the model with such striking aspects has been investigated vigorously so far, we still believe that more detailed analysis of the anyon and the C-S gauged Schrödinger field will reveal newer aspects of the theories and give us deeper understanding of the anyon physics and the FQHE.

It is known that the conformal symmetry exists in the C-S gauged Schrödinger field theory without the external magnetic field.[11),12] The conformal symmetry gets broken when the magnetic field appears in the system, because the magnetic field possesses mass dimension 2. However, this does not mean necessarily that the number of symmetries of the system decreases. Surprisingly, it has been shown that the conformal symmetry modifies its form when the magnetic field is applied and survives as a certain exact space-time symmetry.[15]

On the other hand, free anyons also have the conformal symmetry, and the magnetic field breaks the symmetry. In this paper, we shall discuss a modified conformal symmetry of anyons in the presence of the magnetic field and show a notable role of the modified conformal symmetry in determining the energy spectrum. Moreover, we shall discuss the possibility of dynamical realization of the 2-dimensional conformal symmetry of anyons in the magnetic field.

This paper is organized as follows. In § 2, we shall review the modified conformal symmetry of the C-S gauged Schrödinger field theory in the presence of the uniform external magnetic field. It is also shown that the 2-dimensional conformal symmetry is realized dynamically in a subset of solutions of equations of motion. In § 3, it is presented that the modified conformal symmetry also exists in a many-body system of anyons in the magnetic field. Then, we reveal a role of the modified conformal symmetry in the energy spectrum of anyons. In § 4, we discuss the energy spectrum of anyons quantized semi-classically and, using the results, investigate the possibility of dynamical realization of the 2-dimensional conformal symmetry in a subspace of the Hilbert space of anyons.

§ 2. Modified conformal symmetry in a C-S gauged Schrödinger field theory

In this section, a C-S gauged Schrödinger field theory in the presence of a uniform external magnetic field B is discussed. The action reads

$$S=\int d^3x\left[\Psi^\dagger iD_0\Psi-\frac{1}{2m}|D_k\Psi|^2+\frac{g}{2}|\Psi|^4+\frac{1}{4\pi\nu}\epsilon^{\alpha\beta\gamma}a_\alpha\partial_\beta a_\gamma\right], \tag{1}$$

where $D_\mu=\partial_\mu-ia_\mu-ieA_\mu$, $\partial_\mu=\partial/\partial x^\mu$, and

$$eA_0=0\,,\quad eA_k=-\frac{eB}{2}\epsilon_{kl}x^l\,, \tag{2}$$

and the Latin indices run over 1, 2 and the Greek indices over 0, 1, 2. Ψ is a complex bosonic field, and a_μ is C-S abelian gauge field. We include in the action a two-body delta-functional potential of Ψ with a coupling constant g. The action with $\nu=3, 5, \cdots$ describes FQHE with filling $1/3, 1/5, \cdots$.[2]

When $B=0$, the dilation and special conformal symmetries exist. The dilation is

$$\begin{aligned}
&t'=\Omega^2 t\,,\\
&x'^k=\Omega x^k\,,\\
&\Psi'=\Omega^{-1}\Psi\,,\\
&a'_\nu=\frac{\partial x^\mu}{\partial x'^\nu}a_\mu\,.
\end{aligned} \tag{3}$$

The special conformal transformation is

$$\begin{aligned}
&t'=\frac{t}{1-ct}\,,\\
&x'^k=\frac{x^k}{1-ct}\,,\\
&\Psi'=(1-ct)e^{i(1/2)mc(r^2/1-ct)}\Psi\,,\\
&a'_\nu=\frac{\partial x^\mu}{\partial x'^\nu}a_\mu\,.
\end{aligned} \tag{4}$$

The action (1) with $B=0$ does not change its form under the transformations (3) and (4).[11),12)] However, when $B\neq 0$, these symmetries are explicitly broken because B has mass dimension 2. This does not mean straightforwardly that the number of space-time symmetries of the system decreases. Amazingly, it has been shown that these symmetries modify their forms when the magnetic field is applied, and survive as exact space-time symmetries.[15)] The modified dilation is given by

$$t'=\frac{2}{\omega}\tan^{-1}\left(\Omega^2\tan\left(\frac{\omega}{2}t\right)\right),$$

$$\boldsymbol{x}'=(\Omega^{-1}+\Omega)\eta(t)\boldsymbol{x}+(\Omega^{-1}-\Omega)\eta(t)\begin{bmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{bmatrix}\boldsymbol{x},$$

$$\Psi'=\frac{1}{\sqrt{2\eta(t)}}\exp\left[i\frac{m}{4}\omega r^2(\Omega^{-2}-\Omega^2)\eta(t)\sin(\omega t)\right]\Psi,$$

$$a'_\nu=\frac{\partial x^\mu}{\partial x'^\nu}a_\mu, \tag{5}$$

where $\eta(t)=(\Omega^{-2}+\Omega^2+(\Omega^{-2}-\Omega^2)\cos(\omega t))^{-1}$ and Ω is a parameter independent of time. The modified special conformal transformation is given by

$$t'=\frac{2}{\omega}\tan^{-1}\left(\frac{\tan\left(\frac{\omega}{2}t\right)}{1-\frac{2c}{\omega}\tan\left(\frac{\omega}{2}t\right)}\right),$$

$$x'^k=\frac{1}{\left(1-\frac{2c}{\omega}\tan\left(\frac{\omega}{2}t\right)\right)^2+\tan^2\left(\frac{\omega}{2}t\right)}$$

$$\times\left[\delta^{kl}\left(1-\frac{2c}{\omega}\tan\left(\frac{\omega}{2}t\right)+\tan^2\left(\frac{\omega}{2}t\right)\right)+\epsilon^{kl}\frac{2c}{\omega}\tan^2\left(\frac{\omega}{2}t\right)\right]x^l,$$

$$\Psi'=\left[\left(\cos\left(\frac{\omega}{2}t\right)-\frac{2c}{\omega}\sin\left(\frac{\omega}{2}t\right)\right)^2+\sin^2\left(\frac{\omega}{2}t\right)\right]^{1/2}$$

$$\times\exp\left[-i\frac{mc}{2}\xi(t)\lambda(t)r^2\tan^2\left(\frac{\omega}{2}t\right)\right]\exp\left[i\frac{mcr^2}{2}\frac{1+\tan^2\left(\frac{\omega}{2}t\right)}{1-\frac{2c}{\omega}\tan\left(\frac{\omega}{2}t\right)}\right]\Psi,$$

$$a'_\nu=\frac{\partial x^\mu}{\partial x'^\nu}a_\mu, \tag{6}$$

where

$$\lambda(t)=3-\frac{6c}{\omega}\tan\left(\frac{\omega}{2}t\right)+\left(1+\frac{4c^2}{\omega^2}\right)\tan^2\left(\frac{\omega}{2}t\right) \tag{7}$$

and

$$\xi(t)=\left[\left(1-\frac{2c}{\omega}\tan\left(\frac{\omega}{2}t\right)\right)\left[\left(1-\frac{2c}{\omega}\tan\left(\frac{\omega}{2}t\right)\right)^2+\tan^2\left(\frac{\omega}{2}t\right)\right]\right]^{-1} \tag{8}$$

and c is a free parameter. The transformations (5) and (6) really make the action (1) unchanged.[15] Therefore, they are exact space-time symmetries.*) When $\omega\to 0$, Eqs. (5) and (6) reduce smoothly to Eqs. (3) and (4).

Note that there is another kind of space-time symmetry in the action (1).[15] The transformation is defined as

$$t'=\frac{2}{\omega}\tan^{-1}\left(\frac{\omega}{2}\tau+\tan\left(\frac{\omega}{2}t\right)\right),$$

$$\boldsymbol{x}'=\frac{1}{1+\left(\frac{\omega}{2}\tau+\tan\left(\frac{\omega}{2}t\right)\right)^2}$$

$$\times\begin{bmatrix} 1+\frac{\omega}{2}\tau\tan\left(\frac{\omega}{2}t\right)+\tan^2\left(\frac{\omega}{2}t\right) & \frac{\omega}{2}\tau \\ -\frac{\omega}{2}\tau & 1+\frac{\omega}{2}\tau\tan\left(\frac{\omega}{2}t\right)+\tan^2\left(\frac{\omega}{2}t\right)\end{bmatrix}\boldsymbol{x},$$

$$\Psi'=\frac{1}{\sqrt{2\xi(t)}}\exp\left[i\frac{m\omega}{4}r^2\xi(t)\left(\frac{\omega^2\tau^2}{4}\sin(\omega t)-\omega\tau\cos(\omega t)\right)\right]\Psi,$$

$$a'_\nu=\frac{\partial x^\mu}{\partial x'^\nu}a_\mu, \tag{9}$$

where

$$\xi(t)=\frac{1}{2}\left[\cos^2\left(\frac{\omega}{2}t\right)+\left(\sin\left(\frac{\omega}{2}t\right)+\frac{\omega\tau}{2}\cos\left(\frac{\omega}{2}t\right)\right)^2\right]^{-1} \tag{10}$$

and τ is a free parameter. Transformation (9) also does not change the action (1). When ω vanishes, transformation (9) reduces to the ordinary time translation.

It has also been pointed out that transformations (5), (6) and (9) form an $SO(2,1)$ group.[15] We call the $SO(2,1)$ group *the modified conformal group* and the symmetry corresponding to the group *the modified conformal symmetry* in this paper. The modified conformal symmetry shows us many dynamical aspects of the system,[15] and exists also in quantum problem of anyons, as discussed later.

Equations of motion derived from the action (1) are

$$iD_0\Psi=-\frac{1}{2m}D_k{}^2\Psi-g(\Psi^\dagger\Psi)\Psi, \tag{11}$$

$$-\frac{1}{2\pi\nu}(\partial_\alpha a_\beta-\partial_\beta a_\alpha)=\epsilon_{\alpha\beta\gamma}J^\gamma, \tag{12}$$

where $J^0=\Psi^\dagger\Psi$ and $J^k=-(i/2m)(\Psi^\dagger D_k\Psi-D_k\Psi^\dagger\Psi)$. Notably, these equations possess some analytic solutions when $g=2\pi|\nu|/m$.[13] The forms of the solutions are

$$\Psi=\left[\frac{2}{\pi|\nu|}\right]^{1/2}\cos^{-1}\left(\frac{\omega}{2}t\right)\frac{\left|\frac{dF}{d\eta}(\eta)\right|}{1+|F(\eta)|^2}e^{i\Lambda_F},$$

*) These symmetries are *imported* from the system without B through a general coordinate transformation, which has appeared in Ref. 14). See Ref. 15) for details.

$$a_k=-\nu\int\partial_k\theta(\boldsymbol{x}-\boldsymbol{y})J^0(t,\boldsymbol{y})d^2y+\partial_k\Lambda_F\,,$$
$$a_0=\nu\int\partial_k\theta(\boldsymbol{x}-\boldsymbol{y})J^k(t,\boldsymbol{y})d^2y+\partial_0\Lambda_F\,,\tag{13}$$

where

$$\eta=\frac{e^{i(\omega/2)t}z}{\cos\left(\frac{\omega}{2}t\right)}\,,\tag{14}$$

$z=x+iy$, $F(\eta)$ is an arbitrary analytic function of η, and $\theta(\boldsymbol{x})=\tan^{-1}(y/x)$. The phase Λ_F in Eq. (13) is almost a gauge degree of freedom except its singular contribution, which cannot be gauged away. For example, when $F(\eta)=\eta^N$, the corresponding Λ_F is given by

$$\Lambda_F=-\frac{\nu}{|\nu|}(N-1)\theta(\boldsymbol{x})+\text{regular term}\,,\tag{15}$$

where only the regular term can be gauged away.

It is a remarkable aspect of the C-S gauged Schrödinger field theory that the 2-dimensional conformal symmetry is also realized dynamically in a set of the analytic solutions (13), though the theory is defined in the 3-dimensional space-time. Consider a solution Ψ in the form of Eq. (13), and transform it under the dynamical 2-dimensional conformal transformation:

$$t'=t\,,$$
$$z'=\cos\left(\frac{\omega}{2}t\right)e^{-i(\omega/2)t}f\left(\frac{e^{i(\omega/2)t}z}{\cos\left(\frac{\omega}{2}t\right)}\right),$$
$$\Psi'=\left|\frac{\partial z}{\partial z'}\right|e^{i(\Lambda_{F'}(t',z')-\Lambda_F(t,z))}\Psi\,,\tag{16}$$

where $f(\eta)$ is an arbitrary analytic function and $F'(f(\eta))=F(\eta)$. Then it can be shown that Ψ' is also a solution in the form of Eq. (13).

In this way, the C-S gauged Schrödinger field theory in the magnetic field has the modified conformal symmetry and, the 2-dimensional dynamical conformal symmetry exists in a class of the solution of the equations of motion.

§ 3. Modified conformal symmetry of anyons in a magnetic field

In § 2, we discussed the C-S Schrödinger field theory. If the field is second-quantized, we get a quantum field theory equivalent with a many-body quantum mechanics of anyons. In this section, we discuss the N-body system of anyons in the uniform external magnetic field B. The Schrödinger equation with B reads

$$i\frac{\partial}{\partial t}\Psi=H_B\Psi=-\frac{1}{2m}\sum_{\alpha=1}^{N}\left[\partial_{\alpha k}+i\frac{eB}{2}\epsilon_{kl}x_\alpha{}^l-i\nu\sum_{\beta\neq\alpha}\epsilon_{kl}\frac{(x_\alpha-x_\beta)^l}{|x_\alpha-x_\beta|^2}\right]^2\Psi\,.\tag{17}$$

Because one anyon cannot be put upon another anyon in order not to disturb the notion of anyon-path homotopy, the wavefunction of anyons must vanish when two coordinates of anyons coincide with each other. This is called the hardcore condition of anyons.

In the system of anyons in B, there also exist the modified conformal symmetries, corresponding to the symmetries (5), (6) and (9) of the C-S gauged Schrödinger field theory in B.[16] The quantum generators of the modified conformal group can be written down explicitly. For the symmetry corresponding to transformation (9), the generator is written as

$$\begin{aligned}\tilde{H}=&\cos^2\left(\frac{\omega}{2}t\right)\left[H_B+\frac{\omega}{2}\left(J+\frac{\nu}{2}N(N-1)\right)\right]\\&+\frac{\omega}{4}\cos\left(\frac{\omega}{2}t\right)\sin\left(\frac{\omega}{2}t\right)\sum_{\alpha=1}^{N}(x_\alpha{}^k(t)p_\alpha{}^k(t)+p_\alpha{}^k(t)x_\alpha{}^k(t))\\&-\frac{m\omega^2}{8}\left(2\cos^2\left(\frac{\omega}{2}t\right)-1\right)\sum_{\alpha=1}^{N}(x_\alpha{}^k(t))^2\,,\end{aligned}\tag{18}$$

where $J=\sum_{\alpha=1}^{N}\epsilon^{kl}x_\alpha{}^k(t)p_\alpha{}^l(t)$ and

$$x_\alpha{}^k(t)=e^{itH_B}x_\alpha{}^k e^{-itH_B}\,,\quad p_\alpha{}^k(t)=e^{itH_B}(-i\partial/\partial x_\alpha{}^k)e^{-itH_B}\,.\tag{19}$$

For the modified dilation symmetry, the generator is

$$\tilde{D}=\frac{2}{\omega}\tan\left(\frac{\omega}{2}t\right)\tilde{H}-\frac{1}{4}\sum_{\alpha=1}^{N}\left[x_\alpha{}^k(t)p_\alpha{}^k(t)+p_\alpha{}^k(t)x_\alpha{}^k(t)+m\omega(x_\alpha{}^k(t))^2\tan\left(\frac{\omega}{2}t\right)\right].\tag{20}$$

For the modified special conformal symmetry, it is

$$\tilde{K}=-\frac{4}{\omega^2}\tan^2\left(\frac{\omega}{2}t\right)\tilde{H}+\frac{4}{\omega}\tan\left(\frac{\omega}{2}t\right)\tilde{D}+\frac{m}{2\cos^2\left(\frac{\omega}{2}t\right)}\sum_{\alpha=1}^{n}(x_\alpha{}^k(t))^2\,.\tag{21}$$

The transformations generated by $\tilde{H}$, $\tilde{D}$ and $\tilde{K}$ really make the form of the Schrödinger equation (17) unchanged. By virtue of these symmetries, the charges are conserved,

$$\frac{d}{dt}O=i[H_B,0]+\frac{\partial O}{\partial t}=0\,,\tag{22}$$

where $O=\tilde{D}$, $\tilde{K}$ or $\tilde{H}$.

It is remarkable that they form an $SO(2,1)$ algebra,

$$\begin{aligned}&[\tilde{H},\tilde{D}]=i\tilde{H}\,,\\&[\tilde{D},\tilde{K}]=i\tilde{K}\,,\\&[\tilde{H},\tilde{K}]=2i\tilde{D}\,.\end{aligned}\tag{23}$$

Note that all of them are commutative with the conserved angular momentum J. When ω vanishes, $\tilde{H}$, $\tilde{D}$ and $\tilde{K}$ reduce to the ordinary Hamiltonian, dilational charge and special conformal charge of free anyons, forming the $SO(2,1)$ algebra.[11),12)] Jackiw has shown that the quantum mechanics of two-body free anyons with fixed angular momentum is completely described by a single, irreducible, unitary and infinite-dimensional representation of the $SO(2,1)$ algebra with $\omega=0$.[11)] We can extend his argument for anyons in the magnetic field, as follows.

It is a very important property of the modified conformal symmetry that $\tilde{H}$ and $\tilde{K}$ satisfy

$$R\equiv\tilde{H}+\frac{\omega^2}{4}\tilde{K}=H_B+\frac{\omega}{2}\left(J+\frac{\nu}{2}N(N-1)\right). \tag{24}$$

The eigenstates of R are the energy eigenstates with a fixed angular momentum J. As seen in Eq. (24), the modified conformal algebra contains the Hamiltonian in the presence of B. Therefore, the symmetry can play a crucial role in determining the energy spectrum of anyon systems. We also recombine the generators of the algebra such that

$$L_\pm=\left(\frac{\omega}{4}\tilde{K}-\frac{1}{\omega}\tilde{H}\right)\pm i\tilde{D}\,. \tag{25}$$

Their commutation relations are calculated as

$$[R, L_\pm]=\pm\omega L_\pm\,, \tag{26}$$

$$[L_+, L_-]=-\frac{2}{\omega}R\,. \tag{27}$$

Equation (26) means that $L_+(L_-)$ is a raising (lowering) operator of R by ω. On the other hand, it is shown that R is bounded below, since R is explicitly written as

$$R=\sum_{\alpha=1}^{N}\left[\frac{1}{2m}\left(p_\alpha^k(t)-\nu\sum_{\beta\neq\alpha}\epsilon^{kl}\frac{(x_\alpha(t)-x_\beta(t))^l}{|x_\alpha(t)-x_\beta(t)|^2}\right)^2+\frac{m\omega^2}{8}x_\alpha^k(t)^2\right]\geq 0\,. \tag{28}$$

Therefore, there is a state $|d\rangle$ which satisfies

$$L_-|d\rangle=0\,, \tag{29}$$

$$R|d\rangle=d\omega|d\rangle\,. \tag{30}$$

$|d\rangle$ is the highest weight state of the modified conformal group. From Eq. (24), the highest weight state $|d\rangle$ is an energy eigenstate with its eigenvalue $E=\omega[d-(1/2)J-(\nu/4)N(N-1)]$. Every energy eigenstate can be reproduced by operating L_+ repeatedly to the highest weight states with arbitrary angular momentum. Now, what we must do is only to solve Eqs. (29) and (30) for $|d\rangle$. For this purpose, we write down two equations from Eqs. (29) and (30),

$$\left(L_-+\frac{1}{\omega}R\right)|d\rangle=\left(\frac{\omega}{2}\tilde{K}-i\tilde{D}\right)|d\rangle=d|d\rangle\,, \tag{31}$$

$$2m\omega L_-|d\rangle=-2m\left(\tilde{H}-\frac{\omega^2}{4}\tilde{K}+i\omega\tilde{D}\right)|d\rangle=0\,. \tag{32}$$

Here, we introduce a state $|\tilde{d}\rangle$ by

$$|d\rangle=e^{-(\omega/2)\tilde{K}}|\tilde{d}\rangle\,. \tag{33}$$

Using identities of the $SO(2,1)$ algebra such that

$$e^{(\omega/2)\tilde{K}}\tilde{D}e^{-(\omega/2)\tilde{K}}=\tilde{D}-i\frac{\omega}{2}\tilde{K}\,, \tag{34}$$

$$e^{(\omega/2)\tilde{K}}\tilde{H}e^{-(\omega/2)\tilde{K}}=\tilde{H}-i\omega\tilde{D}-\frac{\omega^2}{4}\tilde{K}\,, \tag{35}$$

we get two constraint equations for $|\tilde{d}\rangle$,

$$-i\tilde{D}|\tilde{d}\rangle=d|\tilde{d}\rangle\,, \tag{36}$$

$$-2m\tilde{H}|\tilde{d}\rangle=0\,. \tag{37}$$

Writing down Eqs. (33), (36) and (37) by the coordinate representation in which $|d\rangle$ and $|\tilde{d}\rangle$ correspond to Ψ_d and $\tilde{\Psi}_d$, we get

$$\Psi_d=\exp\left[-\frac{m\omega}{4}\sum_a x_a{}^{k2}\right]\tilde{\Psi}_d\,, \tag{38}$$

$$\frac{1}{2}\sum_{a=1}^{n}(x_a{}^k\partial_{ak}+1)\tilde{\Psi}_d=d\tilde{\Psi}_d\,, \tag{39}$$

$$\sum_{\alpha=1}^{N}\left(\partial_{\alpha k}-i\nu\sum_{\beta\neq\alpha}\epsilon^{kl}\frac{(x_\alpha-x_\beta)^l}{|x_\alpha-x_\beta|^2}\right)^2\tilde{\Psi}_d=0\,. \tag{40}$$

Equation (39) means that

$$\tilde{\Psi}_d(t,\Omega x_a{}^k)=\Omega^{2d-N}\tilde{\Psi}_d(t,x_a{}^k)\,. \tag{41}$$

Therefore, after removing the factor $e^{-(\omega/2)\tilde{K}}$, all of the highest weight states have a definite scaling property under the spatial dilation. When one introduces a multi-valued wavefunction,

$$\Psi_{\rm multi}=\exp[i\nu\sum_{\alpha>\beta}\theta(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)]\Psi \tag{42}$$

with $\theta(\boldsymbol{x})=\tan^{-1}(y/x)$, Eq. (40) is reduced to the Laplace equation,

$$\sum_{\alpha=1}^{N}\partial^2_{\alpha k}\tilde{\Psi}_{\rm multi}=0\,. \tag{43}$$

Therefore, the energy spectrum problem of anyons in the magnetic field can be entirely described by the Laplace equation with multi-valuedness defined by Eq. (42).

Next, we analyze explicitly two-body energy eigenfunctions of anyons corresponding to the highest weight states of the modified conformal group. First, fixing the angular momentum, we write down the highest weight conditions for two anyons. Removing the contribution of the center-of-mass degrees of freedom and using the relative coordinates $\boldsymbol{r}=\boldsymbol{x}_1-\boldsymbol{x}_2=(r\cos\theta,\,r\sin\theta)$, we get from the angular momentum conservation and Eqs. (39) and (40),

$$-i\frac{\partial}{\partial\theta}\Phi_{dJ}=J\Phi_{dJ}\,, \tag{44}$$

$$r\frac{\partial}{\partial r}\Phi_{dJ}=(2d-1)\Phi_{dJ}\,, \tag{45}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial}{\partial\theta}+i\nu\right)^2\right]\Phi_{dJ}=0\,, \tag{46}$$

where J is the relative angular momentum and takes an integer value. From Eqs. (44) and (45), the eigenfunction is expressed in the form as

$$\Phi_{dJ}=r^{2d-1}e^{iJ\theta}\,. \tag{47}$$

Substituting Eq. (47) into Eq. (46), we obtain

$$2d-1=\pm|J+\nu|\,. \tag{48}$$

From the hardcore condition of anyons, the eigenfunction must vanish at $\boldsymbol{r}=\boldsymbol{0}$. Therefore, the positive sign in Eq. (48) should be selected. Finally, we get an energy eigenfunction,

$$\Phi_J=r^{|J+\nu|}e^{iJ\theta} \tag{49}$$

with

$$E=\omega\left(\frac{1}{2}+\frac{1}{2}(|J+\nu|-J-\nu)\right). \tag{50}$$

If one wants the ordinary expression of the eigenfunction, it needs a normalization constant and the magnetic exponential prefactor $e^{-(\omega/2)\tilde{K}}$. Using a step operator L_+, we also obtain all of the energy eigenfunctions,

$$\Phi_J{}^{(n)}=r^{|J+\nu|}L_n{}^{|J+\nu|}\left(\frac{m_r\omega}{2}r^2\right)e^{iJ\theta} \tag{51}$$

with

$$E=\omega\left(n+\frac{1}{2}+\frac{1}{2}(|J+\nu|-J-\nu)\right), \tag{52}$$

where $L_n{}^{\alpha}(x)$ is the Laguerre polynomials, and $n=0,1,2,\cdots$, and $m_r=m/2$ is the reduced mass. This coincides with the well-known result.[3),8)] Consequently, it can be said that the energy spectrum of two anyons in the uniform external magnetic field is completely determined only from kinematics of the rotational symmetry and the modified conformal symmetry.

§ 4. Semi-classical energy spectrum of anyons in a magnetic field

In § 3, we discussed the modified conformal symmetry of anyons in the magnetic field. On the other hand, in the C-S system in § 2, there is another type of symmetry; the 2-dimensional dynamical conformal symmetry in a subset of solutions of the equations of motion. Here, one question arises. Does the 2-dimensional conformal symmetry also exist in the system of anyons in the magnetic field? We can find a clue about this problem by investigating the energy spectrum quantized semi-classically in the two- and three-body problems.

First, we start with a classical action of two anyons in the magnetic field B. Removing the center-of-mass degrees of freedom, and adopting a relative coordinate of two anyons $\boldsymbol{r}=\boldsymbol{x}_2-\boldsymbol{x}_1$, the action reads

$$S=\int dt\left[\frac{m_r}{2}\dot{r}^{k2}-\frac{e_rB}{2}\epsilon_{kl}\dot{r}^k r^l-\nu\dot{r}^k\partial_k\theta(\boldsymbol{r})\right], \tag{53}$$

where $m_r=m/2$ and $e_r=e/2$. The equation of motion derived from the action (53) has general solutions,

$$\boldsymbol{r}=r_0\begin{bmatrix}\cos(\omega t)\\ -\sin(\omega t)\end{bmatrix}+\boldsymbol{\epsilon}, \tag{54}$$

where r_0 is a radius of a circle of a cyclotron motion, and $\boldsymbol{\epsilon}$ is a coordinate vector of the center of the circle. Energy E and angular momentum J of the solution are such that

$$E=\frac{m_r}{2}(\omega r_0)^2, \tag{55}$$

$$J=\frac{m_r\omega}{2}(\boldsymbol{\epsilon}^2-r_0^2)-\nu. \tag{56}$$

In order to apply the Bohr-Sommerfeld semi-classical quantization to the system, we calculate the action of classical orbits in Eq. (54). Because the flux stays at the origin $\boldsymbol{r}=\boldsymbol{0}$, the classical orbits separate into two classes, whether the origin is in the circle of the cyclotron motion (Case A) or not (Case B). In Case A, $J+\nu<0$ and in Case B, $J+\nu>0$ from Eq. (56). The action of the orbits in one period $T=2\pi/\omega$ is

$$\begin{aligned} S|_0^T&=[-\nu\theta(\boldsymbol{r}(t))]_0^T\\ &=\left[\begin{array}{ll} 2\pi\nu & \text{(Case A)}\\ 0 & \text{(Case B)}. \end{array}\right. \end{aligned} \tag{57}$$

Therefore, the energy spectrum derived from the Bohr-Sommerfeld quantization is such that

$$\begin{aligned} E&=\frac{2\pi\times\text{integer}-S}{T}\\ &=\omega\left[\text{integer}+\frac{1}{2}(|J+\nu|-J-\nu)\right]. \end{aligned} \tag{58}$$

This energy spectrum is only an approximation. So, in general, the exact spectrum needs higher quantum corrections. However, it must be emphasized that ν-dependence of the energy spectrum in Eq. (58) is the same as that of the exact result in Eq. (52). Very notably, higher quantum corrections than the semi-classical one are *not* needed in the ν-dependence. Remembering that the exact two-body energy spectrum is derived from the modified conformal symmetry as shown in § 3, it can be said that *the modified conformal symmetry protects the two-body spectrum from higher* *quantum corrections than the semi-classical one.* This interesting aspect tempts us to quantize three anyons semi-classically.

The three-body quantization can be performed in the same way. The action reads

$$S=\int dt\left[\frac{m}{2}\sum_{\alpha=1}^{3}[\dot{x}_\alpha{}^k]^2-\frac{eB}{2}\epsilon^{kl}\sum_{\alpha=1}^{3}\dot{x}_\alpha{}^k x_\alpha{}^l-\nu\sum_{\alpha=1}^{3}\dot{x}_\alpha{}^k\sum_{\beta\neq\alpha}\partial_k\theta(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)\right], \tag{59}$$

where $\dot{x}=dx/dt$. Classical solutions with three anyons in cyclotron motion are classified into 4 distinct classes (Case 1~Case 4) according to their action values in one period. Case 1 is with $S=0$, Case 2 with $S=2\pi\nu$, Case 3 with $S=4\pi\nu$ and Case 4 with $S=6\pi\nu$. The typical arrangement of three anyon orbits in each case is displayed in Fig. 1. For Case 1, the typical situation is that any orbit does not enclose one another. For Case 2, there is one orbit including another one and the third orbit encloses nothing and is enclosed by no other orbits. For Case 3, there is an orbit including two other orbits and the two in it do not enclose each other. For Case 4, the first circle encloses the second one, and the second one encloses the third one. We notice only the ν-dependence of the semi-classical energy spectrum $\Delta E_{\rm sc}(\nu)$. The results are such that

$$\Delta E^{\rm sc}_{\rm Case\ 1}(\nu)=0\,, \tag{60}$$

$$\Delta E^{\rm sc}_{\rm Case\ 2}(\nu)=-\nu\omega\,, \tag{61}$$

$$\Delta E^{\rm sc}_{\rm Case\ 3}(\nu)=-2\nu\omega\,, \tag{62}$$

$$\Delta E^{\rm sc}_{\rm Case\ 4}(\nu)=-3\nu\omega\,. \tag{63}$$

It is rather mysterious that the energy eigenfunctions have been analytically obtained only in Case 1 and Case 4, and that the ν-dependence of the energy spectrum for the analytic results coincides exactly with that in Eqs. (60) and (63). For example,

$$\Psi_{\rm Case\ 1}=\prod_{\alpha>\beta}^{3}|z_\alpha-z_\beta|^\nu \times\exp\left(-\frac{m\omega}{4}\sum_{\alpha=1}^{3}|z_\alpha|^2\right) \tag{64}$$

is an energy eigenfunction of Case 1 with $E=3\omega/2$ ($\Delta E=0$), and

$$\Psi_{\rm Case\ 4}=\prod_{\alpha>\beta}^{3}|z_\alpha-z_\beta|^{-\nu}(z_\alpha-z_\beta)^2 \times\exp\left(-\frac{m\omega}{4}\sum_{\alpha=1}^{3}|z_\alpha|^2\right) \tag{65}$$

is an energy eigenfunction of Case 4 with

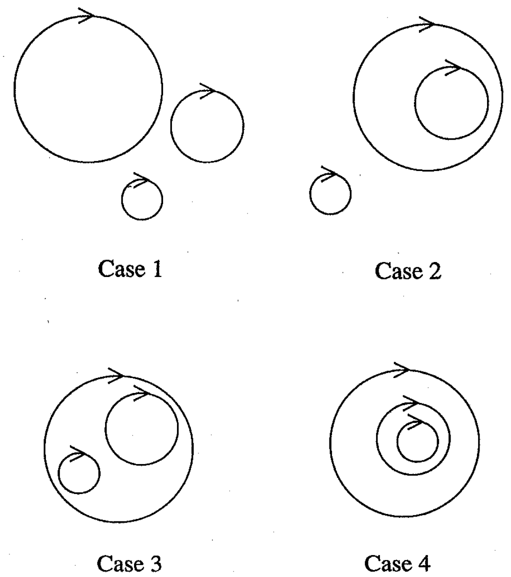

Fig. 1. Typical arrangement of classical orbits of three anyons in the uniform external magnetic field, corresponding to Cases 1~4 in § 4. The circles express anyon orbits of the cyclotron motion in the x-y plane.

$E=\omega((15/2)-3\nu)(\Delta E=-3\nu\omega)$.[4),8)] Other exact wavefunctions in Case 1 and Case 4 have been created with step operators acting on the states in Eqs. (64) and (65).[9)] In this way, the ν-dependence in Case 1 and Case 4 is really protected from higher quantum corrections than the semi-classical one. Then, what protects it?

We have not found its complete answer yet. However, there are some clues to this question. Even if the number of anyons increases to more than three, extension of Eqs. (64) and (65) to the many-body problem is always possible, and the exact wavefunctions can be written down.[7),8)] The ν-dependence of the energy spectrum remains protected from higher quantum corrections even though the system has *many* dynamical variables. It may be another clue that the energy eigenfunctions analytically obtained can be expressed by the vertex operator formalism,[10),8)] which often appears in the *conformal field theory*. From these, we conjecture that a certain larger symmetry than the modified conformal symmetry protects the semi-classical ν-dependence from higher quantum corrections in Case 1 and Case 4, and that the large symmetry, perhaps with infinite number of generators, may be the 2-dimensional dynamical conformal symmetry like that of the C-S system in Eq. (16). Thus far, this is only a conjecture and needs more effort for its establishment.

We want to comment also on Case 2 and Case 3. In these cases the exact energy eigenfunctions are missing so far. It seems that those energy eigenfunctions and their energy spectrum have more complicated forms than those in Case 1 and Case 4. We have calculated only their ν-dependence of the semi-classical energy spectrum in Eqs. (61) and (62). On the other hand, there are recent perturbative and numerical works calculating a part of the energy spectrum in Case 2 and Case 3. The semi-classical picture is available to grasp the behavior of those results, as shown below.

Recently, three-body energy spectrum of anyons in a harmonic potential has been calculated perturbatively[5)] and numerically.[6)] It is known that energy eigenfunctions of three anyons in the magnetic field B have the same forms of those in a harmonic potential with a frequency $\omega/2$,[14)] and that the energy E_B of anyons in the magnetic field B is related to the energy E_H of the harmonic potential problem as

$$E_B=E_H-\frac{\omega}{2}(J+3\nu)\,. \tag{66}$$

Therefore, it is possible to translate those results from the harmonic potential case to the magnetic field case. From the numerical results,[6)] we can extract the ν-dependence of the energy spectrum in the presence of the magnetic field, $\Delta E_{\mathrm{num}}(\nu)$. The results are such that

$$\Delta E^{\mathrm{num}}_{\mathrm{Case}\ 1}(\nu)=0\,, \tag{67}$$

$$\Delta E^{\mathrm{num}}_{\mathrm{Case}\ 2}(\nu)=\omega(-\nu+\delta_2(\nu))\,, \tag{68}$$

$$\Delta E^{\mathrm{num}}_{\mathrm{Case}\ 3}(\nu)=\omega(-2\nu+\delta_3(\nu))\,, \tag{69}$$

$$\Delta E^{\mathrm{num}}_{\mathrm{Case}\ 4}(\nu)=\omega(-3\nu)\,, \tag{70}$$

where δ_2 and δ_3 are nontrivial corrections and $\delta_2(0)=\delta_2(1)=\delta_3(0)=\delta_3(1)=0$. The numerical energy spectrum in Case 2 (Case 3) decreases by $\omega(2\omega)$ when ν changes from 0 to 1. This behavior has been precisely predicted in our semi-classical results of Eqs. (61) and (62). We think δ_2 and δ_3 as higher quantum corrections than the semi-classical one.

For example, in the harmonic potential problem there is an anyon state which is led to the fermion grand state when $\nu\to 1$. The anyon state is translated into a state Ψ_{FG} of the magnetic field problem, which energy decreases by 2ω when ν changes from 0 to 1. Hence, the state Ψ_{FG} belongs to Case 3 of our semi-classical picture. The higher quantum correction δ_3 of Ψ_{FG} behaves near $\nu=1$ as

$$\delta_3(\nu)=\frac{1}{2}(\nu-1)+c(\nu-1)^2+o((\nu-1)^3)\,, \tag{71}$$

where c is a positive constant and $c\sim 0.65$.[5)]

Because the first term in Eq. (71) is the same order of the semi-classical contribution in Eq. (62), the semi-classical picture is not so good near $\nu=1$ in Case 3. This fact makes the above mysterious feature in Case 1 and Case 4 more striking.

§ 5. Summary and discussion

In this paper, we discussed the modified conformal symmetry in the presence of the uniform external magnetic field for the C-S gauged Schrödinger field and the anyons. We revealed the role of the modified conformal symmetry in the anyon energy spectrum. Especially, we pointed out that the two-body energy spectrum of anyons in the magnetic field can be completely determined by kinematics of the modified conformal symmetry and the rotation symmetry.

It was also shown that, in the C-S gauged Schrödinger field theory, there is the 2-dimensional dynamical conformal symmetry in a subset of the solutions of equations of motion. We conjectured that the 2-dimensional conformal symmetry appears dynamically also in the quantum mechanics of anyons in the magnetic field, and keeps higher quantum corrections than the semi-classical one from contributing to the energy spectrum in some cases.

Here, we want to comment on the modified conformal symmetry of anyons in an external harmonic potential. As mentioned in § 4, the quantum mechanics of anyons in the harmonic potential is equivalent with that in the external magnetic field. This fact tells us the modified conformal symmetry also exists in the system of anyons in the external harmonic potential. Especially, it is also true that the two-anyon energy spectrum in the external harmonic potential is entirely determined by kinematics of the modified conformal symmetry and the rotation symmetry.[17)]

Acknowledgements

We would like to thank R. Jackiw for giving us useful comments. We are also grateful to A. Iwazaki for discussing related subjects. We thank H. Murayama for reading through the manuscript and giving us helpful comments.

References

1) J. M. Leinaas and J. Myrheim, Nuovo Cim. **37B** (1977), 1.
F. Wilczek, Phys. Rev. Lett. **49** (1982), 957.
2) R. B. Laughlin, Phys. Rev. Lett. **50** (1983), 1395.
F. D. M. Haldane, Phys. Rev. Lett. **51** (1983), 605.
B. Halperin, Phys. Rev. Lett. **52** (1984), 1583.
Z. F. Ezawa and A. Iwazaki, Phys. Rev. **B43** (1991), 2637.
S. C. Zhang, T. H. Hansen and S. Kivelson, Phys. Rev. Lett. **62** (1989), 82.
N. Read, Phys. Rev. Lett. **62** (1989), 86.
S. M. Girvin and R. Prange, *The Quantum Hall Effect* (Springer-Verlag, NY, 1990).
3) F. Wilczek, Phys. Rev. Lett. **48** (1982), 1144.
Y. S. Wu, Phys. Rev. Lett. **52** (1984), 2103.
M. D. Johnson and G. S. Canright, Phys. Rev. **B41** (1990), 6870.
4) Y. S. Wu, Phys. Rev. Lett. **53** (1984), 111.
5) A. Khare and J. McCabe, Phys. Lett. **B269** (1991), 330.
J. McCabe and S. Ouvry, Phys. Lett. **260B** (1991), 113.
A. Comrd, J. McCabe and S. Ouvry, Phys. Lett. **260B** (1991), 372.
C. Chou, Phys. Rev. **D44** (1991), 2533, Erratum: *Multianyon Spectra and Wavefunctions.*
6) M. Sporre, J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **67** (1991), 1813.
M. V. N. Murthy, J. Law, M. Brack and R. K. Bhaduri, Phys. Rev. Lett. **67** (1991), 1817.
7) C. Chou, Phys. Lett. **A155** (1991), 245.
8) G. V. Dunne, A. Lerda and C. A. Trugenberger, Mod. Phys. Lett. **A6** (1991), 2819.
9) G. V. Dunne, A. Lerda, S. Sciuto and C. A. Trugenberger, MIT Preprint CTP#1978 (June 1991).
K. H. Cho and C. Rim, Chonbug Univ. Preprint SNUTP-91-21 (1991).
10) S. Fubini, Mod. Phys. Lett. **A6** (1991), 347, 487.
M. Stone, Int. J. Mod. Phys. **B5** (1991), 509.
11) R. Jackiw, Ann. of Phys. **201** (1990), 83.
12) R. Jackiw and S. -Y. Pi, Phys. Rev. **D42** (1990), 3500.
13) Z. F. Ezawa, M. Hotta and A. Iwazaki, Phys. Rev. **D44** (1991), 452.
R. Jackiw and S. -Y. Pi, Phys. Rev. Lett. **67** (1991), 41; Phys. Rev. **D44** (1991), 2524.
14) S. Takagi, Prog. Theor. Phys. **85** (1991), 463, 723; **86** (1991), 783.
15) Z. F. Ezawa, M. Hotta and A. Iwazaki, Phys. Rev. **D44** (1991), 3906.
M. Hotta, Prog. Theor. Phys. **86** (1991), 1289.
16) More detailed analysis can be seen in T. Awaji and M. Hotta, Tohoku Univ. Preprint TU-386 (October, 1991).
17) This was emphasized to us by R. Jackiw.
18) M. Sporre, J. J. M. Verbaarschot and I. Zahed, SUNY-NTG-91/40 (October, 1991).

Note added: After the first draft of this paper had been completed, we received a preprint of Sporre, et al.[18] A semi-classical argument like that in § 4 appears in it. (They also cite a recent preprint in which, they say, the same semi-classical discussion as theirs can be seen. However, we have not obtained the cited paper yet.) In § 4, the semi-classical quantization of anyons is discussed for the main purpose of investigating some symmetry. On the other hand, they argue it only for intuitive understanding of the anyon energy spectrum numerically obtained. Therefore, we think that our point of view and theirs are different. We strongly believe that our semi-classical argument reveals a new aspect of anyon physics which they did not notice.