Quantum Hilbert Space of G_C Chern-Simons-Witten Theory and Gravity

Nobuharu HAYASHI

Institute of Physics, University of Tokyo, Komaba, Tokyo 153

Euclidean Chern-Simons-Witten theory with a complex Lie group G_C is discussed. When G_C is a complexification of $SU(2)$, this theory gives an important physical application. It describes Euclidean gravity with negative cosmological constant. In the present paper, we consider a quantum Hilbert space of the Chern-Simons-Witten theory with G_C on a torus, and show that it is finite-dimensional at special values of coupling constants. In such cases, we find that any physical state is given by a product of the holomorphic Weyl-Kac character of the Wess-Zumino-Novikov-Witten model and the anti-holomorphic one. Moreover, it is also shown that each of topological invariants of 3-dimensional manifolds which admit the genus one Heegaard splitting is factorized into a product of two parts. One of them coincides with a topological invariant of the Chern-Simons-Witten theory with the maximal compact subgroup of G_C and the other is closely related to its complex conjugate.

§ 1. Introduction

Great efforts have been devoted to a study of the Chern-Simons-Witten (CSW) theory. It is motivated by mathematical and physical observations. The mathematical observation is that the CSW theory is considered to provide topological invariants of a huge class of 3-manifolds.[1] One of the most important examples is the Jones polynomial.[2] The CSW theory is described by an action and a network of Wilson lines which have no Riemann metric dependence, so it is manifestly a topological theory. The topological invariants in the CSW theory mean knot and link invariants which correspond to vacuum expectation values of the Wilson loops.

The physical observations are concerned with the anyon physics and the quantum gravity. The anyon is a particle with a fractional spin and propagates in a 3-manifold. The trajectories are regarded as Wilson lines (or loops). Wave functions of anyons defined on a Riemann surface are given by solutions of the Knizhnik-Zamolodchikov equations.[3] The monodromies determine representations of the braid group.[4]

A relationship between the CSW theory and the quantum gravity is what we are most interested in here. In 3-dimensions, the CSW theory gives the first order formalism of gravity in which it is described by spin connections and dreibeins, and a gauge group of the CSW theory is chosen to provide a structure group of a local frame bundle over a 3-manifold.[5] As far as the Euclidean gravity is concerned, the CSW theories with the gauge groups $ISO(3)$, $SU(2)\times SU(2)$ and $SL(2, \boldsymbol{C})$ correspond to gravities with zero, positive and negative cosmological constant. We will call the CSW theory with any one of these gauge groups the (Euclidean) CSW gravity.

In the quantum field theory of the gravity, if it is considered in the ADM (Arnowitt, Deser and Misner) formalism, a state of the quantum Hilbert space is given by a solution to the Wheeler-DeWitt equation.[6] But it must be replaced by a parallel transport condition in the CSW gravity.[5] It guarantees independence of complex structures on a Riemann surface being an equal-time hyper-surface. The CSW gravity is said to be exactly solvable. This property owes to a fact that a phase space of the CSW theory is a moduli space of flat connections (a quotient space of gauge fields by the gauge group) in finite-dimensions and the parallel transport condition can be exactly solved at the quantum mechanics level. Thus the CSW gravity provides one of examples of non-perturbative quantum gravity.

A study of the quantum Hilbert space structure of the CSW gravity seems instructive in exploring non-perturbative aspects of the 3-dimensional quantum gravity. One can also include particles, i.e., the anyons. It seems natural to ask the following questions in our CSW approach. Is there an orthonormal basis of the quantum Hilbert space? Can we construct an analog of the Jones-Witten invariants[1] equal to vacuum expectation values of Wilson loops? As the first step to answer them, it is possible to investigate the quantum Hilbert space structure of the Euclidean CSW gravity with $SL(2, \boldsymbol{C})$ ($SL(2, \boldsymbol{C})$ CSW gravity). It describes the Euclidean gravity with negative cosmological constant whose square root is related to the inverse of a coupling constant of the $SL(2, \boldsymbol{C})$ CSW gravity. But as a matter of fact, there exists no problem in discussing a more general case in which the gauge group is given by the complexification G_C of an arbitrary compact Lie group G. Let us call the CSW theory with the complex Lie group G_C the G_C CSW theory. We will discuss the quantum Hilbert space structure of the G_C CSW theory in the present paper.

This paper is organized as follows. We review the G_C CSW theory and its geometric quantization procedure in § 2. We construct the physical quantum Hilbert space on a torus and discuss its unitary structure in § 3. Orthogonality of physical states belonging to a physical state basis on the torus is proved in § 4. We attempt to construct an analog of the Jones-Witten invariants of 3-manifolds in the G_C CSW theory in § 5. The last section is devoted to conclusions of this paper.

§ 2. Quantization of the G_C Chern-Simons-Witten theory

This section is devoted to a review on the classical G_C CSW theory and its geometric quantization.[7] Such a quantization procedure was employed by Axelrod, Della Pietra and Witten in quantizing the CSW theory with a simply connected and compact Lie group.[8] Then Witten immediately applied it to the CSW theory with a complex Lie group.[9] A 3-manifold $\Sigma\times\boldsymbol{R}$ is most convenient for quantizing such CSW theories. $\boldsymbol{R}$ plays a role of time parametrized by x^0, and Σ stands for a closed Riemann surface parametrized by real coordinates (x^1, x^2).

2.1. *The G_C Chern-Simons-Witten theory*

Let $\mathcal{G}_C$ be the Lie algebra of the complex group G_C. Let G and $\mathcal{G}$ be the maximal compact subgroup of G_C and its Lie algebra. We introduce a smooth $\mathcal{G}_C$-valued connection one form $\mathcal{A}$ and its complex conjugate $\bar{\mathcal{A}}$. They are expanded by a basis of the Lie algebra $\mathcal{G}$ as

$$\mathcal{A}=\sum_{a=1}^{\dim(G)}\mathcal{A}^a T_a\,,\quad \bar{\mathcal{A}}=\sum_{a=1}^{\dim(G)}\bar{\mathcal{A}}^a T_a\,. \qquad (2\cdot1)$$

The Lie algebra $\mathcal{G}$ admits an invariant positive definite Killing form. T_a are chosen to satisfy $\mathrm{Tr}(T_aT_b)=\delta_{ab}$.

We discuss an action of the G_C CSW theory defined by

$$I[\mathcal{A},\bar{\mathcal{A}}]=I[\mathcal{A}]+I[\bar{\mathcal{A}}]\,, \qquad (2\cdot2)$$

where

$$I[\mathcal{A}]=\frac{t}{8\pi}\int\mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right),$$

$$I[\bar{\mathcal{A}}]=\frac{\bar{t}}{8\pi}\int\mathrm{Tr}\left(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}}+\frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\right). \qquad (2\cdot3)$$

The coupling constants are $t=k+is$ and $\bar{t}=k-is$. The single valued property of $\exp(iI[\mathcal{A},\bar{\mathcal{A}}])$ in the path-integral quantization requires $k\in\boldsymbol{Z}$ (integers). On the other hand, the parameter s may be an arbitrary complex number.

From the physical point of view, it is important to consider a case $G_C=SL(2,\boldsymbol{C})$. This case is related to the 3-dimensional gravity. To see this, we substitute $\mathcal{A}=\omega+ie$ and $\bar{\mathcal{A}}=\omega-ie$ into the action (2·2) where ω and e are $\mathcal{G}$-valued connection one forms. It is decomposed into a sum of two parts:

$$I[\mathcal{A},\bar{\mathcal{A}}]=I_k[e,\omega]+I_s[e,\omega]\,, \qquad (2\cdot4)$$

where

$$I_k[e,\omega]=\frac{k}{4\pi}\int\mathrm{Tr}\left(\omega\wedge d\omega-e\wedge de+\frac{2}{3}\omega\wedge\omega\wedge\omega-2\omega\wedge e\wedge e\right),$$

$$I_s[e,\omega]=-\frac{s}{2\pi}\int\mathrm{Tr}\left(e\wedge(d\omega+\omega\wedge\omega)-\frac{1}{3}e\wedge e\wedge e\right). \qquad (2\cdot5)$$

If we identify $\omega_\mu{}^a$ and $e_\mu{}^a$ $(\mu=0,1,2)$ with the spin connections and the dreibeins respectively, we find that the action $I_k[e,\omega]$ stands for the exotic term and the action $I_s[e,\omega]$ the Einstein-Hilbert action. The exotic term gives rise to a framing problem of 3-manifolds.[9)]

First, as an exceptional case, we consider a case of the Lorentzian space-time. Let us suppose that G takes the non-compact real form $SL(2,\boldsymbol{R})$ of $SL(2,\boldsymbol{C})$ ($SL(2,\boldsymbol{R})$ is a double cover of $SO(2,1)$). The parameter s must be real for the action $I_s[e,\omega]$ to be the Einstein-Hilbert action. It can be identified with the inverse of the Newton constant. We realize that the cosmological constant term (the last term of $I_s[e,\omega]$) has positive signature provided that $\int\mathrm{Tr}(e\wedge e\wedge e)=\int d^3x\sqrt{-g}>0$. On the other hand, in the Euclidean space-time, $SL(2,\boldsymbol{C})$ is regarded as a complexification of $G=SU(2)$ (a double cover of $SO(3)$). The parameter s must be purely imaginary from the same reason as in the Lorentzian case. In this case, is can be identified with the inverse of the Newton constant. The cosmological constant term has negative signature provided that $\int \mathrm{Tr}(e\wedge e\wedge e)=\int d^3x\sqrt{g}>0$.

2.2. *Geometric quantization*

Let us quantize the G_C CSW theory described by the action (2·2) with s being purely imaginary in the Euclidean space-time. We consider the quantization in $\mathcal{A}_0=\bar{\mathcal{A}}_0=0$ gauge. Let $\mathcal{W}_C$ be a phase space of smooth complex connections on the G_C-bundle over Σ. $\mathcal{W}_C$ has a symplectic structure determined by the following symplectic form (a closed two form)

$$\omega=\frac{t}{8\pi}\int_\Sigma \mathrm{Tr}(\delta\mathcal{A}\wedge\delta\mathcal{A})+\frac{\bar{t}}{8\pi}\int_\Sigma \mathrm{Tr}(\delta\bar{\mathcal{A}}\wedge\delta\bar{\mathcal{A}})\,. \tag{2·6}$$

Here $(\delta\mathcal{A},\delta\bar{\mathcal{A}})$ is a variation of the complex connection one forms on Σ. According to the geometric quantization procedure, we start with a prequantum line bundle over $\mathcal{W}_C$. Let $(D/D\mathcal{A}_i{}^a)$ and $(D/D\bar{\mathcal{A}}_i{}^a)$ $(i=1,2)$ be connections on it. The symplectic form ω is a curvature two form and plays a role of the first Chern class of the prequantum line bundle. According to (2·6), the connections must satisfy the commutation relations on arbitrary sections of the prequantum line bundle

$$\left[\frac{D}{D\mathcal{A}_i{}^a(z)},\ \frac{D}{D\mathcal{A}_j{}^b(z')}\right]=-\frac{t}{8\pi}\epsilon_{ij}\delta^{ab}(z,z')\,,$$

$$\left[\frac{D}{D\bar{\mathcal{A}}_i{}^a(z)},\ \frac{D}{D\bar{\mathcal{A}}_j{}^b(z')}\right]=-\frac{\bar{t}}{8\pi}\epsilon_{ij}\delta^{ab}(z,z')\,. \tag{2·7}$$

To quantize the system, we need to choose a convenient polarization. Following Witten,[9] it is most convenient to choose the real polarization determined by

$$\frac{D}{D\mathcal{A}_z{}^a}\psi=\frac{D}{D\bar{\mathcal{A}}_{\bar{z}}^{\bar{a}}}\psi=0\,. \tag{2·8}$$

These conditions define a quantum line bundle as a subbundle of the prequantum line bundle. $\bar{\mathcal{A}}_{\bar{z}}^{\bar{a}}$ is a complex conjugate of $\mathcal{A}_z{}^a$. They are given by $\mathcal{A}_i{}^a dx^i=\mathcal{A}_z{}^a dz+\mathcal{A}_{\bar{z}}^{\bar{a}}d\bar{z}$ and $\bar{\mathcal{A}}_i{}^a dx^i=\bar{\mathcal{A}}_z{}^a dz+\bar{\mathcal{A}}_{\bar{z}}^{\bar{a}}d\bar{z}$. The holomorphic (anti-holomorphic) component dz $(d\bar{z})$ of dx^i is defined by a complex structure J on Σ. Solving the conditions (2·8), one can see that ψ is determined only by $(\omega_z{}^a,\omega_{\bar{z}}^{\bar{a}})$-dependence. Hence it is enough to obtain ψ at $e_z{}^a=e_{\bar{z}}^{\bar{a}}=0$. Thus $\mathcal{W}_C$ reduces to a phase space $\mathcal{W}$ of connections $\omega_z{}^a$ and $\omega_{\bar{z}}^{\bar{a}}$ on the G-bundle over Σ (G connections). Via such a reduction, the commutation relations (2·7) become

$$\left[\frac{D}{D\omega_z{}^a(z)},\frac{D}{D\omega_{\bar{z}}^{\bar{b}}(z')}\right]=-\frac{k}{4\pi}\delta^{ab}\delta(z,z')\,, \tag{2·9}$$

where $k=(t+\bar{t})/2$. These relations hold on arbitrary L_2-sections of a line bundle over $\mathcal{W}$.

As a common property of the CSW theories, $\mathcal{A}_0$ and $\bar{\mathcal{A}}_0$ play a role of the Lagrange multiplier fields which impose the Gauss law constraints $F_{z\bar{z}a}(\mathcal{A})=\bar{F}_{z\bar{z}a}(\bar{\mathcal{A}})=0$. Acting on a section of the line bundle over $\mathcal{W}_C$ and fixing $e_i{}^a$ at zero, the constraint operators are

$$\left(D_{\bar z}\frac{D}{d\omega_{\bar z}^{a}}-\frac{t}{8\pi}F_{z\bar z a}(\omega)\right)\psi=\left(D_z\frac{D}{D\omega_z{}^{a}}-\frac{\bar t}{8\pi}F_{z\bar z a}(\omega)\right)\psi=0\,, \tag{2·10}$$

where $D_{\bar z}=\partial_{\bar z}+\omega_{\bar z}$ $(D_z=\partial_z+\omega_z)$. The Gauss law constraints (2·10) impose gauge invariance on the section. The gauge invariance let us to identify connections related by gauge transformations: $\omega_{\bar z}\to\omega'_{\bar z}=g^{-1}\omega_{\bar z}g+ig^{-1}\partial_{\bar z}g$ $(\omega_z\to\omega'_z=\bar g^{-1}\omega_z\bar g+i\bar g^{-1}\partial_z\bar g)$. g and its complex conjugate $\bar g$ belong to a group $\hat G_C$ of smooth maps from Σ to G_C. A complex structure on $\mathcal{W}$ is induced by the complex structure on Σ. Once a complex structure J on Σ is picked, the above gauge transformation gives rise to a natural $\hat G_C$ action on $\mathcal{W}$. Thus we can introduce a quotient space $\mathcal{W}/\hat G_C$ $(=\mathcal{M}_J)$ by means of the complex structure on Σ.

The quantum Hilbert space consists of sections of a line bundle ove $\mathcal{M}_J$. To make the dependence on J manifest, let us denote it by $\mathcal{H}_J{}^Q$. We can make use of a theorem by Narasimhan and Seshadri,[8),10)] that the moduli space $\mathcal{M}$ of the flat G connections is isomorphic to the quotient space $\mathcal{M}_J$, i.e., $\mathcal{M}\cong\mathcal{M}_J$. According to this theorem, we are able to identify the quantum Hilbert space over $\mathcal{M}_J$ with that over $\mathcal{M}$ while $\mathcal{M}_J$ is dependent on the complex structures on Σ. Hence there must exist a way to identify sections belonging to the quantum Hilbert spaces defined on different complex structures on Σ. Let us introduce a projectively flat connection δ^Q on the quantum bundle $\mathcal{H}^Q\to\mathcal{T}$. Here $\mathcal{H}^Q$ represents a set of all the quantum Hilbert spaces $\mathcal{H}_J{}^Q$ on Σ. $\mathcal{T}$ is a space of all the complex structures on Σ, i.e., the Teichmüller space. $\mathcal{H}_J{}^Q$ consists of parallel sections ψ determined by $\delta^Q\psi=0$. This is considered to be an analog of the Wheeler-DeWitt equation. For a purely imaginary s, we need to use an exotic hermitian pairing $\mathcal{H}_{-\bar s}\otimes\mathcal{H}_s\cong\mathcal{H}_s\otimes\mathcal{H}_s\to \boldsymbol{C}$ to establish unitarity. $\mathcal{H}_s$ stands for the quantum Hilbert space for given s (at fixed k). We require that under an infinitesimal variation of the complex structure J on Σ, every inner product should transform as

$$\delta\langle\!\langle\chi|\psi\rangle\!\rangle=\langle\!\langle\delta^Q\chi|\psi\rangle\!\rangle+\langle\!\langle\chi|\delta^Q\psi\rangle\!\rangle\,. \tag{2·11}$$

We will explicitly see such a property in the case $\Sigma=T^2$ (torus) in the next section.

§ 3. Quantum Hilbert space on torus

The arguments given in the previous section are applicable to the quantum Hilbert space on any closed and orientable Riemann surface. But the inner product must be defined to a certain Riemann surface. This problem is complicated except in the genus one case. For simplicity, here, we restrict ourselves to the genus one case. Then we can explicitly find the parallel sections which satisfy the parallel transport condition $\delta^Q\psi=0$. It is also possible to provide the inner product of the quantum Hilbert space on a torus which is compatible with the parallel transport.

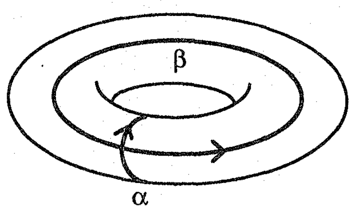

Fig. 1. α and β cycles of a torus.

3.1. *Moduli space of flat G connections*

The physical states are parallel sections of the line bundle over $\mathcal{M}_J$, i.e., the moduli space of the flat G connections. $\mathcal{M}_J$ is given by the conjugacy classes of the holonomy representations of $\pi_1(\Sigma)$ in G. The fundamental group of the torus is generated by two commutable elements, i.e., the α and β cycles (see Fig. 1), so $\pi_1(T^2)\cong \boldsymbol{Z}\oplus\boldsymbol{Z}$. Let us introduce the holonomy representations of the fundamental group $\pi_1(T^2)$. Let $U[\alpha]\in G$ and $U[\beta]\in G$ be elements of the holonomy associated with the α and β cycles. A relation of $\pi_1(T^2)$, $\alpha\beta=\beta\alpha$ puts a relation of the holonomy $U[\alpha]U[\beta]=U[\beta]U[\alpha]$. From this, the conjugacy classes of the holonomy determine the representations of $\pi_1(T^2)$ in G and may be used to parametrize the moduli space of the flat G connections. By conjugation, we can achieve that $U[\alpha]\in T$ and $U[\beta]\in T$ where T is the Cartan subgroup of G. Let us parametrize them by $U[\alpha]=\exp(2\pi i\theta_1)$ and $U[\beta]=\exp(2\pi i\theta_2)$. Here θ_1 and θ_2 belong to the Cartan subalgebra $\boldsymbol{t}$. $U[\alpha]$ and $U[\beta]$ must be still conjugated simultaneously by the Weyl group elements. Thus the space of the representations of $\pi_1(T^2)$ in G is given by $T\times T/W$ where W stands for the Weyl group. For θ_1 and θ_2, we must put an identification $(\theta_1, \theta_2)\simeq(w\theta_1+\lambda_1, w\theta_2+\lambda_2)$ where w is an element of the Weyl group and λ_1 and λ_2 are in the coroot lattice $\check{T}$. The moduli space of the flat G connections becomes $\mathcal{M}=T\times T/W\cong \boldsymbol{t}\times\boldsymbol{t}/(\check{T}\times\check{T})\ltimes W$. The notation $\ltimes$ represents the semi-direct product.

Let us parametrize the complex structure on the torus by a complex parameter $\tau=\tau_1+i\tau_2$ and its complex conjugate. τ belongs to the upper half plane specified by $\tau_2>0$. By means of τ and its complex conjugate $\bar{\tau}$, the holomorphic (anti-holomorphic) coordinate on the torus is given by $z=x_1+\tau x_2$ ($\bar{z}=x_1+\bar{\tau}x_2$). But periodicities along the α and β cycles require an identification $(x_1, x_2)\simeq(x_1+m_1, x_2+m_2)$ for $m_1, m_2\in\boldsymbol{Z}$. Using the complex structure on the torus, we are able to define a holomorphic (anti-holomorphic) component of the modular parameters of $\mathcal{M}_J$ by $u=\theta_2-\tau\theta_1$ ($\bar{u}=\theta_2-\bar{\tau}\theta_1$).

3.2. *Parallel transport conditions*

The quantum Hilbert space is spanned by the physical states, i.e., the gauge invariant sections satisfying the parallel transport condition $\delta^Q\psi=0$. As a matter of fact, δ^Q has a long and complicated expression.[9] It is convenient to rewrite this equation into a simpler form. It is useful to know a property of H (a determinant of the Laplacian on the torus). It satisfies the heat equations

$$\left(h\delta^{(1,0)}+\frac{1}{4}\nabla_{\underline{i}}\delta J^{\underline{ij}}\nabla_{\underline{j}}\right)H^{1/2}=\left(h\delta^{(0,1)}+\frac{1}{4}\nabla_{\bar{i}}\delta J^{\bar{i}\bar{j}}\nabla_{\bar{j}}\right)H^{1/2}=0\,. \tag{3·1}$$

$\nabla_{\underline{i}}$ and $\nabla_{\bar{i}}$ are components of connections on a vector bundle over $\mathcal{M}_J$ with respect to u^i and $\bar{u}_i$ directions. $\delta^{(1,0)}(\delta^{(0,1)})$ is a holomorphic (anti-holomorphic) component of a quantum trivial connection δ associated with an infinitesimal variation of the complex parameter $\tau(\bar{\tau})$. Infinitesimal variations of the complex structure $\delta J^{\underline{ij}}$ and $\delta J^{\bar{i}\bar{j}}$ are covariantly constant because they are independent of u and $\bar{u}$ as

$$\delta J^{\underline{ij}}=\delta J^{\underline{i}}{}_{\bar{m}}\omega^{\bar{m}\underline{j}}=i\frac{d\tau}{\pi}C^{ij}\,,\quad \delta J^{\bar{i}\bar{j}}=\delta J^{\bar{i}}_{\underline{m}}\omega^{\underline{m}\bar{j}}=-i\frac{d\bar{\tau}}{\pi}C^{ij}\,. \tag{3·2}$$

The matrix C^{ij} is the inverse of C_{ij} given by an inner product of a basis of the Cartan subalgebra. Multiplying $\delta^Q\psi=0$ by $H^{1/2}$, we find that they are rewritten into

$$\tilde{\delta}^{Q(1,0)}\tilde{\psi}=\left(\delta^{(1,0)}+\frac{1}{2t}\nabla_{\underline{i}}\delta J^{\underline{ij}}\nabla_{\underline{j}}-\frac{2}{t}H^{-1/2}\left(h\delta^{(1,0)}+\frac{1}{4}\nabla_{\underline{i}}\delta J^{\underline{ij}}\nabla_{\underline{j}}\right)H^{1/2}\right)\tilde{\psi}$$

$$=\left(\delta^{(1,0)}+\frac{1}{2t}\nabla_{\underline{i}}\delta J^{\underline{ij}}\nabla_{\underline{j}}\right)\tilde{\psi}=0\,,$$

$$\tilde{\delta}^{Q(0,1)}\tilde{\psi}=\left(\delta^{(0,1)}-\frac{1}{2\bar{t}}\nabla_{\bar{i}}\delta J^{\bar{i}\bar{j}}\nabla_{\bar{j}}+\frac{2}{\bar{t}}H^{-1/2}\left(h\delta^{(0,1)}+\frac{1}{4}\nabla_{\bar{i}}\delta J^{\bar{i}\bar{j}}\nabla_{\bar{j}}\right)H^{1/2}\right)\tilde{\psi}$$

$$=\left(\delta^{(0,1)}-\frac{1}{2\bar{t}}\nabla_{\bar{i}}\delta J^{\bar{i}\bar{j}}\nabla_{\bar{j}}\right)\tilde{\psi}=0\,. \tag{3·3}$$

Here $\tilde{\delta}^{Q(1,0)}(\tilde{\delta}^{Q(0,1)})$ is a holomorphic (anti-holomorphic) component of a quantum connection conjugated by $H^{1/2}$, i.e., $\tilde{\delta}^Q=H^{1/2}\delta^Q H^{-1/2}$. $\tilde{\psi}$ is a parallel section for $\tilde{\delta}^Q$ related to ψ by $\tilde{\psi}=H^{1/2}\psi$. The connections acting on $\tilde{\psi}$ are explicitly given by

$$\delta^{(1,0)}\tilde{\psi}=d\tau\left(\frac{d}{d\tau}+k\frac{dQ_0}{d\tau}\right)\tilde{\psi}\,,\quad \delta^{(0,1)}\tilde{\psi}=d\bar{\tau}\frac{d}{d\bar{\tau}}\tilde{\psi}\,,$$

$$\nabla_{\underline{i}}\tilde{\psi}=\left(\frac{\partial}{\partial u^i}+k\frac{\partial Q_0}{\partial u^i}\right)\tilde{\psi}\,,\qquad \nabla_{\bar{i}}\tilde{\psi}=\frac{\partial}{\partial\bar{u}^i}\tilde{\psi}\,. \tag{3·4}$$

For brevity, we used $Q_0=(\pi/2\tau_2)(u-\bar{u})^2$. The complex modular parameters $u^i(\bar{u}^i)$ are dependent on the parameter $\tau(\bar{\tau})$ by definition, so we have to use in (3·4)

$$\frac{d}{d\tau}=\frac{\partial}{\partial\tau}+\frac{u^i-\bar{u}^i}{2i\tau_2}\frac{\partial}{\partial u^i}\,,\quad \frac{d}{d\bar{\tau}}=\frac{\partial}{\partial\bar{\tau}}+\frac{u^i-\bar{u}^i}{2i\tau_2}\frac{\partial}{\partial\bar{u}^i}\,. \tag{3·5}$$

3.3. *Physical state basis*

We are ready to find solutions to (3·3). Multiplying them by $\tau_2^{-r/2}e^{\bar{t}Q_0/2}$ makes finding the solutions easy. We observe that

$$\tau_2^{-r/2}e^{\bar{t}Q_0/2}\tilde{\delta}^{Q(1,0)}\tilde{\psi}=\left(\delta^{(1,0)}_{t/2}+\frac{1}{2t}\nabla^{t/2}_{\underline{i}}\delta J^{\underline{ij}}\nabla^{t/2}_{\underline{j}}\right)\check{\psi}=0\,,$$

$$\tau_2^{-r/2}e^{\bar{t}Q_0/2}\tilde{\delta}^{Q(0,1)}\tilde{\psi}=\left(\delta^{(0,1)}_{-\bar{t}/2}-\frac{1}{2\bar{t}}\nabla^{-\bar{t}/2}_{\bar{i}}\delta J^{\bar{i}\bar{j}}\nabla^{-\bar{t}/2}_{\bar{j}}\right)\check{\psi}=0\,. \tag{3·6}$$

$\tilde{\psi}$ is related to $\check{\psi}$ by $\tilde{\psi}=\tau_2^{r/2}e^{-\bar{t}Q_0/2}\check{\psi}$. The connections appearing in (3·6) are computed to be

$$\delta^{(1,0)}_{t/2}\check{\psi}=d\tau\left(\frac{d}{d\tau}+\frac{t}{2}\frac{dQ_0}{d\tau}+\frac{r}{4i\tau_2}\right)\check{\psi}\,,$$

$$\nabla^{t/2}_{\underline{i}}\check{\psi}=\left(\frac{\partial}{\partial u^i}+\frac{t}{2}\frac{\partial Q_0}{\partial u^i}\right)\check{\psi}\,,$$

$$\delta^{(0,1)}_{-\bar{t}/2}\check{\psi}=d\bar{\tau}\left(\frac{d}{d\bar{\tau}}-\frac{\bar{t}}{2}\frac{dQ_0}{d\bar{\tau}}-\frac{r}{4i\tau_2}\right)\check{\psi}\,,$$

$$\nabla^{-\bar{t}/2}_{\bar{i}}\check{\psi}=\left(\frac{\partial}{\partial\bar{u}^i}-\frac{\bar{t}}{2}\frac{\partial Q_0}{\partial\bar{u}^i}\right)\check{\psi}\,. \tag{3·7}$$

The parameter r is the rank of G (the maximal compact subgroup of G_C). Substituting (3·7) into (3·6), we obtain a pair of heat equations

$$\left(\frac{\partial}{\partial\tau}-\frac{1}{4\pi i(t/2)}C^{ij}\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}\right)\check{\psi}=0\,,$$

$$\left(\frac{\partial}{\partial\bar{\tau}}+\frac{1}{4\pi i(-\bar{t}/2)}C^{ij}\frac{\partial}{\partial \bar{u}^i}\frac{\partial}{\partial \bar{u}^j}\right)\check{\psi}=0\,. \tag{3·8}$$

We notice that solutions to (3·8) take a form of a product of a holomorphic part and an antiholomorphic one. In the present paper, we consider only a case that a space spanned by the solutions to (3·8) is finite-dimensional. In such a case, it can be used the fact that solutions to the heat equations are given by theta functions. So we impose here on the coupling constants of the G_C CSW theory the following conditions:

$$\frac{t}{2}=l_L+h\,,\quad -\frac{\bar{t}}{2}=l_R+h\,, \tag{3·9}$$

where the parameters l_L and l_R take non-negative integers. (We cannot, of course, deny a situation in which l_L and l_R take fractional numbers, but we neglect such a possibility.) Let us remember that the moduli space of the flat G connections is isomorphic to $\boldsymbol{t}\times\boldsymbol{t}/(\check{T}\times\check{T})\ltimes W$. We can easily find the solutions invariant under the Weyl transformation. They are

$$\check{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}=\overline{\theta_{\bar{l}_R+h,\rho+\Lambda_R}(u,\tau)}\,\theta_{\bar{l}_L+h,\rho+\Lambda_L}(u,\tau)\,. \tag{3·10}$$

ρ is the Weyl vector (a half sum of positive roots). $\theta_{\bar{l}+h,\rho+\Lambda}$ represents a Weyl anti-symmetric invariant of the theta functions at level $l+h$. It is specified by a weight vector $\rho+\Lambda$ in $\hat{T}/(l+h)\check{T}\ltimes W$ where $\hat{T}$ stands for the weight lattice. It is convenient to introduce a fundamental chamber. We can use a fact that $\rho+\Lambda$ can be expanded by the fundamental weights as $\rho+\Lambda=\sum_{i=1}^r p_i\Lambda_i$ where p_i are positive integers. Then the fundamental chamber is given by $\mathcal{F}_l=\{\rho+\Lambda|\sum_{i=1}^r p_i<l+h\}$. The Weyl anti-symmmetric invariant is defined by

$$\theta_{\bar{l}+h,\rho+\Lambda}(u,\tau)=\sum_{m\in W}\epsilon(w)\theta_{l+h,w(\rho+\Lambda)}(u,\tau)\,. \tag{3·11}$$

$\epsilon(w)$ is a signature of an element w of the Weyl group. We use an expression of the theta function

$$\theta_{l,\gamma}(u,\tau)=\sum_{a\in\check{T}}\exp\left(i\pi l\tau\left(a+\frac{\gamma}{l}\right)^2+2\pi il\left\langle u,a+\frac{\gamma}{l}\right\rangle\right). \tag{3·12}$$

3.4. *Unitary structure*

Let us explicitly define the inner product of the physical states on the torus given by the exotic hermitian pairing. We discuss the inner product of the form

$$\langle\!\langle\psi'|\psi\rangle\!\rangle=\langle\psi'|H^{-1/2}e^{2\xi\Delta}H^{1/2}\psi\rangle=\int_{\mathcal{M}}d\mu e^{kQ_0}\bar{\psi}'\cdot H^{1/2}e^{2\xi\Delta}H^{1/2}\psi$$

$$=\int_{\mathcal{M}} d\mu e^{kQ_0}\bar{\tilde{\psi}}'\cdot e^{2\xi\Delta}\tilde{\psi}\,. \tag{3·13}$$

Here the integration measure is determined by the symplectic form $\omega_k=(ik\pi/\tau_2)C_{ij}\,du^i\wedge d\bar{u}^j$ and $d\mu=d^r u d^r\bar{u}/\tau_2{}^r=d^r\theta_1 d^r\theta_2(\sim\omega^r)$. The parameter ξ is determined so that $e^{-4k\xi}=-\bar{t}/t$.[9] In particular, in the case $k=0$ in which the exotic term $I_k[e,\omega]$ is absent, ξ must be replaced by $1/2is$ and the definition of the inner product (3·13) can be used even in such a case. Δ is the Laplacian on the moduli space of the flat G connections. It is given by

$$\Delta=kJ^i{}_m\omega^{mj}\nabla_i\nabla_j=-\frac{\tau_2}{\pi}C^{ij}(\nabla_{\underline{i}}\nabla_{\bar{j}}+\nabla_{\bar{i}}\nabla_{\underline{j}})\,. \tag{3·14}$$

The inner product (3·13) is invariant under the parallel transport on $\mathcal{T}$. To see this, we can use a fact that the quantum connection $\tilde{\delta}^Q$ is related to the trivial connection by

$$\tilde{\delta}^Q=e^{-\xi\Delta}\delta e^{\xi\Delta}\,. \tag{3·15}$$

Owing to this conjugation, the parallel transport conditions (3·3) become equivalent to $\delta^{(1,0)}\psi_0=\delta^{(0,1)}\psi_0=0$. Here ψ_0 is a parallel section for δ and is related to $\tilde{\psi}$ by $\psi_0=e^{\xi\Delta}\tilde{\psi}$. Then the inner product (3·13) can be rewritten by the parallel sections for δ. The result is

$$\langle\!\langle\psi'|\psi\rangle\!\rangle=\int_{\mathcal{M}} d\mu e^{kQ_0}\overline{\psi'}_0\cdot\psi_0\,. \tag{3·16}$$

This establishes the unitarity for purely imaginary s because $\delta^{(0,1)}$ is manifestly the adjoint of $\delta^{(1,0)}$.

It is easy to check (2·11). Under infinitesimal variations of the complex structure on the torus (i.e., τ and $\bar{\tau}$), (3·16) varies as

$$\begin{aligned}\delta\langle\!\langle\psi'|\psi\rangle\!\rangle&=\int_{\mathcal{M}} d\mu e^{kQ_0}(\overline{\delta\psi_0'}\cdot\psi_0+\overline{\psi_0'}\cdot\delta\psi_0)\\&=\int_{\mathcal{M}} d\mu e^{kQ_0}(\overline{\tilde{\delta}^Q\tilde{\psi}'}\cdot e^{2\xi\Delta}\tilde{\psi}+\bar{\tilde{\psi}}'\cdot e^{2\xi\Delta}\tilde{\delta}^Q\tilde{\psi})\\&=\int_{\mathcal{M}} d\mu e^{kQ_0}H(\overline{\delta^Q\psi'}\cdot H^{-1/2}e^{2\xi\Delta}H^{1/2}\psi+\overline{\psi'}\cdot H^{-1/2}e^{2\xi\Delta}H^{1/2}\delta^Q\psi)\\&=\langle\!\langle\delta^Q\psi'|\psi\rangle\!\rangle+\langle\!\langle\psi'|\delta^Q\psi\rangle\!\rangle\,.\end{aligned} \tag{3·17}$$

To obtain the second equality, we used partial integrals and the conjugation (3·15). If both of ψ' and ψ are parallel sections, the inner product becomes independent of the complex structures on the torus as it should be.

§ 4. Orthogonality of the physical states

This section is devoted to proving the orthogonality of the physical states given by (3·10). As can be seen from (3·10), each of them is given by a product of the holomorphic theta function and the anti-holomorphic one. It is specified by a pair of weight vectors (Λ_R, Λ_L) where $\rho+\Lambda_R$ and $\rho+\Lambda_L$ belong to the fundamental chambers $\mathcal{F}_{l_R}$ and $\mathcal{F}_{l_L}$ respectively. Let us pick two physical states (the parallel sections for the quantum connection $\tilde{\delta}^Q$) belonging to the physical state basis given by

$$\tilde{\psi}^{l_R l_L}_{\tilde{\Lambda}_R\tilde{\Lambda}_L}=\tau_2^{r/2}e^{-(\bar{\tau}/2)Q_0}\overline{\theta^{-}_{l_R+h,\rho+\tilde{\Lambda}_R}}\,\theta^{-}_{l_L+h,\rho+\tilde{\Lambda}_L}\,,$$

$$\tilde{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}=\tau_2^{r/2}e^{-(\bar{\tau}/2)Q_0}\overline{\theta^{-}_{l_R+h,\rho+\Lambda_R}}\,\theta^{-}_{l_L+h,\rho+\Lambda_L}\,. \tag{4·1}$$

The inner product of them is defined by (3·13). The orthogonality which we would like to prove here is expressed by

$$\langle\!\langle \psi^{l_R l_L}_{\tilde{\Lambda}_R\tilde{\Lambda}_L} | \psi^{l_R l_L}_{\Lambda_R\Lambda_L}\rangle\!\rangle=\mathrm{const}\,\delta_{\tilde{\Lambda}_R,\Lambda_R}\delta_{\tilde{\Lambda}_L,\Lambda_L}\,. \tag{4·2}$$

Remember that $\tilde{\psi}=H^{1/2}\psi$. We must also show that the right-hand side of (4·2) is independent of the complex structures on the torus as it should be. The orthogonality will arise from a twist-integral over the moduli space of the flat G connections.

We should pay attention to a region of the twist-integral.[11] In the previous section, it is shown that the moduli space of the flat G connections on the torus is isomorphic to $\boldsymbol{t}\times\boldsymbol{t}/(\check{T}\times\check{T})\ltimes W$. The moduli space is described by a fundamental region of the modular parameters. It is determined up to two translations in the coroot lattice and one Weyl transformation. Thus it can be chosen to satisfy $\theta_1\in\mathcal{N}_0$ and $\theta_2\in\cup_{w\in W}w\mathcal{N}_0$ because the Cartan subalgebra is regarded as $\boldsymbol{t}=\cup_{w\in W,\epsilon\in\check{T}}(\omega\mathcal{N}_0+\epsilon)$. $\mathcal{N}_0$ represents the open Weyl alcove.

4.1. *Proof of the orthogonality in $k\neq 0$ case*

First, we attempt to prove the orthogonality in the $k\neq0$ case in which the exotic term $I_k[e,\omega]$ is present. For brevity, let us denote the physical states $\tilde{\psi}^{l_R l_L}_{\tilde{\Lambda}_R\tilde{\Lambda}_L}$ and $\tilde{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}$ by $\tilde{\psi}'$ and $\tilde{\psi}$ respectively. Then by means of partial integrals and the commutation relations $[\nabla_{\underline{i}}, \nabla_{\bar{j}}]=-i\omega_{\underline{i}\bar{j}}=(k\pi/\tau_2)C_{ij}$, we easily find that

$$\begin{aligned}\langle\!\langle\psi'|\psi\rangle\!\rangle&=\int_{\mathcal{M}}d\mu e^{kQ_0}\overline{\tilde{\psi}'}\cdot e^{2\xi\Delta}\tilde{\psi}\\&=e^{2\xi kr}\int_{\mathcal{M}}d\mu e^{kQ_0}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{c(n,m)}{n\,!}(4k\xi)^n\left(\frac{\tau_2}{k\pi}\right)^m\\&\quad\times\left(\prod_{l=1}^{m}C^{i_l j_l}\right)\left(\prod_{l=1}^{m}\frac{\partial}{\partial u^{i_l}}\right)\overline{\tilde{\psi}'}\cdot\left(\prod_{l=1}^{m}\frac{\partial}{\partial\bar{u}^{j_l}}\right)\tilde{\psi}\,.\end{aligned} \tag{4·3}$$

Here $c(n,m)$ are determined by $\exp(x(\partial/\partial x))\equiv\sum_{n=0}^{\infty}\sum_{m=0}^{n}(c(n,m)/n!)x^m(\partial/\partial x)^m$. Introducing external sources, it is possible to rewrite (4·3) into a more compact form:

$$\begin{aligned}&\langle\!\langle\psi'|\psi\rangle\!\rangle\\&=e^{2\xi kr}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{c(n,m)}{n!}(4k\xi)^n\left(\frac{\tau_2}{k\pi}\right)^m\left(\frac{\bar{t}}{4\tau_2}\right)^{2m}\left(\frac{\partial}{\partial\eta}\cdot\frac{\partial}{\partial\zeta}\right)^m\boldsymbol{Z}_k(\eta,\zeta)|_{\eta=\zeta=0}\\&=\exp\left(2k\xi r+4k\xi\left(\zeta+\frac{\bar{t}^2}{16\pi k\tau_2}\frac{\partial}{\partial\eta}\right)\cdot\frac{\partial}{\partial\zeta}\right)\boldsymbol{Z}_k(\eta,\zeta)|_{\eta=\zeta=0}\,,\end{aligned} \tag{4·4}$$

where

$$\boldsymbol{Z}_k(\eta,\zeta)\equiv\int_{\mathcal{M}}d\mu e^{kQ_0}\overline{\exp\left(i\frac{4\tau_2}{\bar{t}}\eta\cdot\frac{\partial}{\partial\bar{u}}\right)\tilde{\psi}'}\cdot\exp\left(i\frac{4\tau_2}{\bar{t}}\zeta\cdot\frac{\partial}{\partial\bar{u}}\right)\tilde{\psi}$$

$$=\int_{\mathcal{M}}d\mu e^{kQ_0}\overline{\tilde{\psi}'\left(u,\ \bar{u}+i\frac{4\tau_2}{\bar{t}}\eta\right)}\cdot\tilde{\psi}\left(u,\ \bar{u}+i\frac{4\tau_2}{\bar{t}}\zeta\right). \qquad (4\cdot5)$$

The external sources η and ζ are assumed to be in the Cartan subalgebra, i.e., $(\eta,\zeta)\in\boldsymbol{t}\times\boldsymbol{t}$. We attempt to expand $\boldsymbol{Z}_k(\eta,\zeta)$ by eigenfunctions of the differential operator $4k\xi(\zeta+(\bar{t}^2/16\pi k\tau_2)(\partial/\partial\eta))\cdot(\partial/\partial\zeta)$. The Fourier expansion of $\boldsymbol{Z}(\eta,\zeta)$ enables us to carry out this strategy. Let us expand it as

$$\boldsymbol{Z}_k(\eta,\zeta)=\int_{t\times t}d^rpd^rq\exp(2\pi i\langle p,\eta\rangle+2\pi i\langle q,\zeta\rangle)\boldsymbol{Z}_k(p,q). \qquad (4\cdot6)$$

Owing to the assumption $(\eta,\zeta)\in\boldsymbol{t}\times\boldsymbol{t}$, the Fourier coefficient can be computed through well-defined Gaussian integrals of η and ζ. The result of this computation is

$$\boldsymbol{Z}_k(p,q)=\left(-\frac{\bar{t}}{4}\right)^r\sum_{\alpha,\beta,\gamma,\delta\in\check{T}}\ \sum_{\tilde{w}_R,\tilde{w}_L,w_R,w_L\in W}\epsilon(\tilde{w}_R)\epsilon(\tilde{w}_L)\epsilon(w_R)\epsilon(w_L)$$

$$\times\exp\left(\frac{\pi\bar{t}}{4\tau_2}p^2+\frac{n\bar{t}}{4\tau_2}q^2+2\pi i\left\langle p,\lambda_R'-\frac{\bar{t}}{2}\alpha+i\frac{\bar{t}}{4\tau_2}(u-\bar{u})\right\rangle\right.$$

$$\left.+2\pi i\left\langle q,\gamma_R'-\bar{t}/2\gamma+i\frac{\bar{t}}{4\tau_2}(u-\bar{u})\right\rangle+\cdots\right). \qquad (4\cdot7)$$

The dots represent a part independent of the momentums p and q. For brevity, we employed the notations

$$\lambda_R'=\tilde{w}_R\lambda_R=\tilde{w}_R(\rho+\tilde{\Lambda}_R),\quad \gamma_R'=w_R\gamma_R=w_R(\rho+\Lambda_R). \qquad (4\cdot8)$$

In addition to these, in the later argument, we will also use the following notations:

$$\lambda_L'=\tilde{w}_L\lambda_L=\tilde{w}_L(\rho+\tilde{\Lambda}_L),\quad \gamma_L'=w_L\gamma_L=w_L(\rho+\Lambda_L). \qquad (4\cdot9)$$

Substituting the Fourier expansion (4·6) into the expression (4·4), we immediately find that

$$\langle\!\langle\psi'|\psi\rangle\!\rangle=e^{2k\xi r}\int_{t\times t}d^rpd^rq\boldsymbol{Z}_k(p,q)\times\Omega, \qquad (4\cdot10)$$

where

$$\Omega=\exp\left(4k\xi\left(\zeta+\frac{\bar{t}^2}{16\pi k\tau_2}\frac{\partial}{\partial\eta}\right)\cdot\frac{\partial}{\partial\zeta}\right)\exp(2\pi i\langle p,\eta\rangle+2\pi i\langle q,\zeta\rangle)|_{\eta=\zeta=0}. \qquad (4\cdot11)$$

The next step is to compute the factor Ω. After moving the factor $e^{2\pi i\langle p,\eta\rangle}$ to the left side of all the remnant exponential functions and then setting $\eta=0$, we obtain

$$\Omega=\exp\left(4k\xi\left(\zeta+i\frac{\bar{t}^2p}{8k\tau_2}\right)\cdot\frac{\partial}{\partial\zeta}\right)e^{2\pi i\langle q,\zeta\rangle}|_{\zeta=0}$$

$$=\exp\left(\frac{\pi\bar{t}^2}{4k\tau_2}\langle p,\,q\rangle+4k\xi\zeta\cdot\frac{\partial}{\partial\zeta}\right)e^{2\pi i\langle q,\zeta\rangle}\,|_{\zeta=(i\bar{t}^2/8k\tau_2)p}\,. \tag{4·12}$$

The eigenfunction of $4k\xi\zeta\cdot(\partial/\partial\zeta)$ with an eigenvalue $4k\xi n$ is proportional to $\langle q,\,\zeta\rangle^n$. Without paying attention to normalization of the eigenfunction, we can arrive at

$$\Omega=\exp\left(\frac{\pi\bar{t}^2}{4k\tau_2}\langle p,\,q\rangle+4k\xi\zeta\cdot\frac{\partial}{\partial\zeta}\right)e^{2\pi i\langle q,\zeta\rangle}\,|_{\zeta=(i\bar{t}^2/8k\tau_2)p}$$

$$=\exp\left(\frac{\pi\bar{t}^2}{4k\tau_2}\langle p,\,q\rangle\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\pi\bar{t}^2}{4k\tau_2}\right)^n e^{4k\xi n}\langle p,\,q\rangle^n=\exp\left(\frac{\pi t}{2\tau_2}\langle p,\,q\rangle\right). \tag{4·13}$$

At the last equality, the definition $e^{4k\xi}=-t/\bar{t}$ is used.

Putting this result together with (4·10), we see that (4·10) becomes

$$\langle\!\langle\psi'|\psi\rangle\!\rangle=e^{2k\xi r}\left(-\frac{\bar{t}}{4}\right)^r\sum_{\alpha,\beta,\gamma,\delta\in\check{T}}\ \sum_{\tilde{w}_R,\tilde{w}_L,w_R,w_L\in W}\epsilon(\tilde{w}_R)\epsilon(\tilde{w}_L)\epsilon(w_R)\epsilon(w_L)$$

$$\times\int_{\mathcal{M}}d\mu\int_{t\times t}d^rpd^rq\exp\left(\frac{\pi t}{4\tau_2}(p+q)^2+2\pi i\,\left\langle p,\,\lambda_R'-\frac{\bar{t}}{2}\alpha+\frac{\bar{t}}{2}\theta_1\right\rangle\right.$$

$$\left.+2\pi i\,\left\langle q,\,\gamma_R'-\frac{\bar{t}}{2}\gamma+\frac{\bar{t}}{2}\theta_1\right\rangle\,+\cdots\right)$$

$$=Ce^{2k\xi r}\left(-\frac{\bar{t}}{4}\right)^r\left(-\frac{\tau_2}{\bar{t}}\right)^{(r/2)}\delta_{\lambda_R'-(\bar{t}/2)\alpha,\gamma_R'-(\bar{t}/2)\gamma}$$

$$\times\sum_{\alpha,\beta,\gamma,\delta\in\check{T}}\ \sum_{\tilde{w}_R,\tilde{w}_L,w_R,w_L\in W}\epsilon(\tilde{w}_R)\epsilon(\tilde{w}_L)\epsilon(w_R)\epsilon(w_L)$$

$$\times\int_{\mathcal{M}}d\mu\exp\left(\frac{\pi\tau_2}{\bar{t}}(\bar{t}\theta_1+\lambda_R'+\gamma_R'-\frac{\bar{t}}{2}(\alpha+\gamma))^2+\cdots\right). \tag{4·14}$$

To obtain the last equality, one needs a change of the variables given by $2p_1=p+q$ and $2p_2=p-q$. The integration of p_1 is a well-defined Gaussian integral because of $\bar{t}<0$, and the integration for p_2 gives a Kronecker delta function.[11] But in the p_2-integral, a periodicity $p_2\simeq p_2+\epsilon$ for any $\epsilon\in\check{T}$ gives rise to the infinite C which should be divided somewhere (for instance, see the last subsection of § 5). From now on, let C include trivial numerical constants independent of the coupling $(t,\,\bar{t})$ and the complex structures $(\tau,\,\bar{\tau})$ arising from the subsequent integrals. The Kronecker delta function in (4·14) imposes $\tilde{w}_R(\lambda_R-(\bar{t}/2)\alpha')=w_R(\gamma_R-(\bar{t}/2)\gamma')$. Here it is used that there always exist $\alpha'\in\check{T}$ and $\beta'\in\check{T}$ satisfying $\alpha=\tilde{w}_R\alpha'$ and $\gamma=w_R\gamma'$. Moreover, there is a fact that $w(\mathcal{N}_0+\epsilon)\cap\mathcal{N}_0\neq\emptyset$ $(\epsilon\in\check{T})$ if and only if $w=1$ and $\epsilon=0$. It means that $\tilde{w}_R=w_R$, $\lambda_R=\gamma_R$ and $\alpha=\gamma$. Thus the Kronecker delta function in (4·14) becomes equivalent to $\delta_{\tilde{w}_R,w_R}\delta_{\lambda_R,\gamma_R}\delta_{\alpha,\gamma}$. According to the conditions $\tilde{w}_R=w_R$ and $\alpha=\gamma$, (4·14) becomes

$$\langle\!\langle \psi'|\psi\rangle\!\rangle = Ce^{2k\xi r}(-\bar{t})^r\left(-\frac{\tau_2}{\bar{t}}\right)^{(r/2)}\delta_{\lambda_R,\gamma_R}\sum_{\alpha,\beta,\gamma,\delta\in\check{T}}\sum_{\tilde{w}_L,w_R,w_L\in W}\epsilon(\tilde{w}_L)\epsilon(w_L)$$
$$\times\int_{\mathcal{M}}d\mu\exp\Big(-2k\pi\tau_2\theta_1^2+\frac{\pi\tau_2}{\bar{t}}(\bar{t}\theta_1+\gamma_R'+\lambda_R'-\bar{t}\alpha)^2$$
$$+2\pi i\left\langle\theta_1,\ \bar{\tau}\left(\lambda_L'+\frac{t}{2}\beta-\lambda_R'+\frac{\bar{t}}{2}\alpha\right)-\tau\left(\gamma_L'+\frac{t}{2}\delta-\gamma_R'+\frac{\bar{t}}{2}\alpha\right)\right\rangle$$
$$+2\pi i\left\langle\theta_2,\ \gamma_L'-\lambda_L'+\frac{t}{2}(\delta-\beta)\right\rangle+\cdots\Big). \tag{4·15}$$

The dots represent a part independent of θ_1 and θ_2. The integration of θ_2 over the region $\cup_{w\in W}w\mathcal{N}_0$ gives a Kronecker delta function $\delta_{\lambda_L'+(t/2)\beta,\gamma_L'+(t/2)\delta}$.[11] It becomes equivalent to $\delta_{\tilde{w}_L,w_L}\delta_{\lambda_L,\gamma_L}\delta_{\beta,\delta}$. Since the region of the θ_1 integral is $\mathcal{N}_0$, we cannot trivially perform it as a Gaussian integral. Fortunately, after the θ_2 integral, one finds a full expression

$$\langle\!\langle \psi'|\psi\rangle\!\rangle = C\delta_{\lambda_R,\gamma_R}\delta_{\lambda_L,\gamma_L}e^{2k\xi r}(-\bar{t})^r\left(-\frac{\tau_2}{\bar{t}}\right)^{r/2}$$
$$\times\sum_{w\in W}\sum_{\alpha\in\check{T}}\int_{\mathcal{N}_0}d^r\theta_1\exp\left(-t\pi\tau_2\left(\theta_1-\alpha-\frac{2}{t}w\gamma_L\right)^2\right). \tag{4·16}$$

According to the fact that the Cartan subalgebra is $\boldsymbol{t}=\cup_{w\in W,\epsilon\in\check{T}}(w\mathcal{N}_0+\epsilon)$, the region of the θ_1 integral can be extended to the whole region of $\boldsymbol{t}$. Thus it follows that

$$\sum_{w\in W}\sum_{\alpha\in\check{T}}\int_{\mathcal{N}_0}d^r\theta_1\exp\left(-t\pi\tau_2\left(\theta_1-\alpha-\frac{2}{t}w\gamma_L\right)^2\right)$$
$$=\int_{t}d^r\theta_1\exp\left(-t\pi\tau_2\left(\theta_1-\frac{2}{t}\gamma_L\right)^2\right)=\left(\frac{1}{t\tau_2}\right)^{r/2}. \tag{4·17}$$

Substituting (4·17) into (4·16) and using $e^{2k\xi r}=(-t/\bar{t})^{(r/2)}$, we finally achieve that

$$\langle\!\langle \psi'|\psi\rangle\!\rangle = C\delta_{\lambda_R,\gamma_R}\delta_{\lambda_L,\gamma_L}. \tag{4·18}$$

This completes the proof of the orthogonality of the physical states given by (4·1) in the $k\neq0$ case.

4.2. *Proof of the orthogonality in k=0 case*

Let us try the problem of proving the orthogonality in the $k=0$ case in which the exotic term $I_k[e,\omega]$ is absent. In this case, the level of the holomorphic theta functions and that of the anti-holomorphic ones coincide. Let us denote them by $t/2=-\bar{t}/2=l+h$ for $l\in\boldsymbol{Z}_{\geq0}$ (non-negative integers). Let us pick two physical states specified by pairs of weight vectors $(\tilde{\Lambda}_R,\tilde{\Lambda}_L)$ and (Λ_R,Λ_L),

$$\tilde{\psi}^{ll}_{\tilde{\Lambda}_R\tilde{\Lambda}_L}=\tau_2{}^{r/2}e^{-(\bar{t}/2)Q_0}\overline{\theta_{l+h,\rho+\tilde{\Lambda}_R}^-}\,\theta_{l+h,\rho+\tilde{\Lambda}_L}^-,$$
$$\tilde{\psi}^{ll}_{\Lambda_R\Lambda_L}=\tau_2{}^{r/2}e^{-(\bar{t}/2)Q_0}\overline{\theta_{l+h,\rho+\Lambda_R}^-}\,\theta_{l+h,\rho+\Lambda_L}^-. \tag{4·19}$$

It is possible to prove the orthogonality in quite the same way as in the previous subsection.

We discuss the inner product in the $k=0$ case given by

$$\langle\!\langle \psi'|\psi\rangle\!\rangle=\int_{\mathcal{M}} d\mu \bar{\tilde{\psi}}'\cdot e^{\Delta/2l}\tilde{\psi}\ . \tag{4·20}$$

Here the parameter ξ in (3·13) must be replaced by $1/4l$. The Laplacian is, in this case, of the form $\Delta=-(2\tau_2/\pi)C^{ij}(\partial/\partial u^i)(\partial/\partial \bar{u}^j)$. By means of partial integrals, we can rewrite (4·20) into

$$\langle\!\langle \psi'|\psi\rangle\!\rangle=\int_{\mathcal{M}} d\mu \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tau_2}{l\pi}\right)^n(\prod_{l=1}^{n}c^{i_l j_l})\left(\prod_{l=1}^{n}\frac{\partial}{\partial u^{i_l}}\right)\bar{\tilde{\psi}}'\cdot\left(\prod_{l=1}^{n}\frac{\partial}{\partial \bar{u}^{j_l}}\right)\tilde{\psi}\ . \tag{4·21}$$

Introducing external sources, we obtain a compact expression for (4·21):

$$\langle\!\langle \psi'|\psi\rangle\!\rangle=\exp\left(\frac{l}{4\pi\tau_2}\frac{\partial}{\partial\eta}\cdot\frac{\partial}{\partial\zeta}\right)\boldsymbol{Z}_0(\eta,\ \zeta)|_{\eta=\zeta=0}\ , \tag{4·22}$$

where

$$\boldsymbol{Z}_0(\eta,\ \zeta)=\int_{\mathcal{M}} d\mu \overline{\tilde{\psi}'\left(u,\ \bar{u}-\frac{2i\tau_2}{l}\eta\right)}\cdot\tilde{\psi}\left(u,\ \bar{u}-\frac{2i\tau_2}{l}\zeta\right). \tag{4·23}$$

The external sources η and ζ are assumed to be in the Cartan subalgebra. This assumption provides us with a well-defined Fourier expansion of $\boldsymbol{Z}_0(\eta,\ \zeta)$ which is obtained by setting $k=0$ in (4·6). The Fourier coefficient $\boldsymbol{Z}_0(p,\ q)$ is obtained by carrying out the Gaussian integrals of η and ζ. The result is

$$\begin{aligned}\boldsymbol{Z}_0(p,\ q)=&\left(\frac{l}{2\tau_2}\right)^r\sum_{\alpha,\beta,\gamma,\delta\in\check{T}}\ \sum_{\tilde{w}_R,\tilde{w}_L,w_R,w_L\in W}\epsilon(\tilde{w}_R)\epsilon(\tilde{w}_L)\epsilon(w_R)\epsilon(w_L)\\ &\times\exp\left(-\frac{\pi l}{2\tau_2}p^2-\frac{\pi l}{2\tau_2}q^2+2\pi i\left\langle p,\ \lambda'_R+l\alpha-\frac{il}{2\tau_2}(u-\bar{u})\right\rangle\right.\\ &\left.+2\pi i\left\langle q,\ \gamma'_R+l\gamma-\frac{il}{2\tau_2}(u-\bar{u})\right\rangle+\cdots\right).\end{aligned} \tag{4·24}$$

The dots represent a part independent of the momentums p and q. Substituting the Fourier expansion of $\boldsymbol{Z}_0(\eta,\ \zeta)$ into (4·22), we immediately obtain

$$\begin{aligned}\langle\!\langle \psi'|\psi\rangle\!\rangle=&\left(\frac{l}{2}\right)^r\sum_{\alpha,\beta,\gamma,\delta\in\check{T}}\ \sum_{\tilde{w}_R,\tilde{w}_L,w_R,w_L\in W}\epsilon(\tilde{w}_R)\epsilon(\tilde{w}_L)\epsilon(w_R)\epsilon(w_L)\\ &\times\int_{\mathcal{M}} d\mu\int_{t\times t} d^r p\, d^r q \exp\left(-\frac{\pi l}{2\tau_2}(p+q)^2+2\pi i\langle p,\ \lambda'_R+l\alpha-l\theta_1\rangle\right.\\ &\left.+2\pi i\langle q,\ \gamma'_R+l\gamma-l\theta_1\rangle+\cdots\right).\end{aligned} \tag{4·25}$$

To perform the integrations of p and q, it is necessary to change the momentums p and q into $p_1=(p+q)/2$ and $p_2=(p-q)/2$. The p_1-integral is a Gaussian integral and the p_2-integral gives a Kronecker delta function $\delta_{\lambda'_R+l\alpha,\gamma'_R+l\gamma}$. It becomes equivalent to $\delta_{\tilde{w}_R,w_R}\delta_{\lambda_R,\gamma_R}\delta_{\alpha,\gamma}$. Summing over the Weyl elements $\tilde{w}_R$ and the coroot lattice elements γ, we find that

$$\langle\!\langle \psi'|\psi\rangle\!\rangle = Cl^r\left(\frac{\tau_2}{l}\right)^{r/2}\delta_{\lambda_R,\gamma_R}\sum_{\alpha,\beta,\delta\in\check{T}}\ \sum_{\tilde{w}_L,w_R,w_L\in W}\epsilon(\tilde{w}_L)\epsilon(w_L)$$
$$\times\int_{\mathcal{M}}d\mu\exp\Big(-2\pi l\tau_2\Big(\theta_1-\alpha-\frac{1}{l}\lambda_R'\Big)^2$$
$$+2\pi i\langle\theta_1,\ \bar{\tau}(\lambda_L'-\lambda_R'+l(\beta-\alpha))-\tau(\gamma_L'-\gamma_R'+l(\delta-\gamma))\rangle$$
$$+2\pi i\langle\theta_2,\ \gamma_L'-\lambda_L'+l(\delta-\beta)\rangle+\cdots\Big). \qquad (4\cdot 26)$$

The dots represent a part independent of θ_1 and θ_2. By definition, the θ_2 integral must be carried out over $\cup_{w\in W}w\mathcal{N}_0$. It gives a Kronecker delta function $\delta_{\lambda_L+l\beta,\gamma_L+l\delta}$ which is equivalent to $\delta_{\tilde{w}_L,w_L}\delta_{\lambda_L,\gamma_L}\delta_{\beta,\delta}$. Summing over $\tilde{w}_L\in W$ and $\delta\in\check{T}$, we arrive at a full expression

$$\langle\!\langle \psi'|\psi\rangle\!\rangle = C\delta_{\lambda_R,\gamma_R}\delta_{\lambda_L,\gamma_L}l^r\left(\frac{\tau_2}{l}\right)^{r/2}$$
$$\times\sum_{\alpha\in\check{T}}\sum_{w\in W}\int_{\mathcal{N}_0}d^r\theta_1\exp\Big(-2\pi l\tau_2\Big(\theta_1-\alpha-\frac{1}{l}w\lambda_L\Big)^2\Big). \qquad (4\cdot 27)$$

From the same discussion as in the previous subsection, the θ_1 integral over the Weyl alcove $\mathcal{N}_0$ can be extended to that over the Cartan subalgebra $\boldsymbol{t}$. It says that

$$\sum_{\alpha\in\check{T}}\sum_{w\in W}\int_{\mathcal{N}_0}d^r\theta_1\exp\Big(-2\pi l\tau_2\Big(\theta_1-\alpha-\frac{1}{l}w\lambda_L\Big)^2\Big)$$
$$=\int_t d^r\theta_1\exp\Big(-2\pi l\tau_2\Big(\theta_1-\frac{1}{l}\lambda_L\Big)^2\Big)=\Big(\frac{1}{2l\tau_2}\Big)^{r/2}. \qquad (4\cdot 28)$$

Putting this result together with (4·27), we finally achieve that

$$\langle\!\langle \psi'|\psi\rangle\!\rangle = C\delta_{\lambda_R,\gamma_R}\delta_{\lambda_L,\gamma_L}. \qquad (4\cdot 29)$$

This completes the proof of the orthogonality of the physical states given by (4·19) on the torus in the $k=0$ case.

We have established the orthogonality of the physical states belonging to the physical state basis in both of the $k\neq 0$ and $k=0$ cases. The inner products of them are turned out to be independent of the complex structures on the torus, i.e., the parameter τ and its complex conjugate $\bar{\tau}$, as it should be. We will see, in the next section, that the orthogonality plays a crucial role in computing the topological invariants of 3-manifolds.

§ 5. Topological invariants in the G_C CSW theory

In a case when the CSW theory has a compact gauge group, the inner product of the physical states has been discussed in a framework of the gauge field theory by a few groups.[12)~14)] It provides a huge class of the topological invariants of 3-manifolds. The quantum gauge field theory helps us to have a concrete physical image of the topological invariants. They are considered to be vacuum expectation values of links and knots of Wilson loops. We attempt to verify that the inner product of the G_C CSW theory is also permitted to possess the same physical meaning. This will be explicitly examined in the case that the quantum Hilbert space is constructed on the torus.

5.1. *Physical state basis and Verlinde operators*

It has been known that a certain set of Verlinde (knot) operators[15] generates an orthonormal basis of the quantum Hilbert space in a compact gauge group case. We would like to seek an analog of them in the complex gauge group case. First, to make the discussion easy, we restrict ourselves to a case that the connections $e_\mu{}^a$ are fixed at zero in the action (2·2). The resultant action is denoted by $I_k[w]$ which describes the CSW theory with the maximal compact subgroup G with level k. As far as the CSW theory of this type is concerned, a relation between the Verlinde operator and the Wilson loop has been clarified in works of Moore and Seiberg,[12] Labastida and Llatas et al.[13] We summarize their observations below.

The physical states on the torus belonging to the physical state basis in the CSW theory with G are given by states denoted by $\chi_\Lambda{}^k$ where $\rho+\Lambda$ belongs to the fundamental chamber $\mathcal{F}_k$. They are defined by

$$\chi_\Lambda{}^k(u,\tau)=\frac{v_{k+h,\rho+\Lambda}(u,\tau)}{v_{h,\rho}(u,\tau)}, \tag{5·1}$$

where $v_{l,\gamma}=\exp((\pi/2\tau_2)lu^2)\theta_{l,\gamma}^-$. The states $\chi_\Lambda{}^k$ are nothing but the holomorphic Weyl-Kac characters. A total set of them is called the Verlinde basis on the torus. It is generated by the Verlinde (knot) operators from the vacuum state $\chi_0{}^k$. It says that

$$\chi_\Lambda{}^k=W_\Lambda{}^{(1,0)}\chi_0{}^k\,. \tag{5·2}$$

Here $W_\Lambda{}^{(1,0)}$ is a special element of the Verlinde operators for coprime integers (n,m):

$$W_\Lambda{}^{(n,m)}=\sum_{\lambda\in M_\Lambda}v_{h,\rho}^{-1}\exp\left(-\frac{\pi}{\tau_2}(n\bar{\tau}+m)u\cdot\lambda+\frac{n\tau+m}{k+h}\lambda\cdot\frac{\partial}{\partial u}\right)v_{h,\rho}\,. \tag{5·3}$$

λ runs over weight vectors belonging to the irreducible module M_Λ specified by the highest weight Λ. Physical states expressed by $W_\Lambda{}^{(n,m)}\chi_0{}^k$ (Verlinde states) can be expanded by the Verlinde basis.

To show a relation between the Verlinde operator and the Wilson loop, let us introduce quantum operators $\hat{u}$ and $\hat{\bar{u}}$ corresponding to the moduli parameters u and $\bar{u}$. They are subject to the Heisenberg algebra $\{\hat{\bar{u}}^i,\hat{u}_j\}=(\tau_2/\pi)(1/k+h)\delta_j{}^i$.[12] In the holomorphic representation of this algebra, one observes that

$$\rho(\hat{u}^i)\chi_\Lambda{}^k=u^i\chi_\Lambda{}^k\,,\quad \rho(\hat{\bar{u}}^i)\chi_\Lambda{}^k=\frac{\tau_2}{\pi(k+h)}\frac{\partial}{\partial u_i}\chi_\Lambda{}^k\,. \tag{5·4}$$

This representation enables us to rewrite the Wilson loop by a differential operator acting on the Verlinde states. For instance, one observes that

$$v_{h,\rho}^{-1}\mathrm{Tr}_\Lambda\Big(\exp\Big(i\int_{Cn,m}\rho(\bar{\omega})\Big)\Big)v_{h,\rho}\cdot\chi_0{}^k$$

$$=v_{h,\rho}^{-1}\mathrm{Tr}_\Lambda\Big(\exp\Big(-\frac{\pi}{\tau_2}\int_{Cn,m}\overline{\omega(z)}\rho(\bar{u})\cdot H+\frac{\pi}{\tau_2}\int_{Cn,m}\omega(z)\rho(\bar{\bar{u}})\cdot H\Big)\Big)v_{h,\rho}\cdot\chi_0{}^k$$

$$=v_{h,\rho}^{-1}\mathrm{Tr}_\Lambda\Big(\exp\Big(-\frac{\pi}{\tau_2}(n\bar{\tau}+m)u\cdot H+\frac{n\tau+m}{k+h}H\cdot\frac{\partial}{\partial u}\Big)\Big)v_{h,\rho}\cdot\chi_0{}^k\,. \qquad (5\cdot5)$$

The contour $C_{n,m}$ stands for a loop which winds $n(m)$ times around the $\beta(\alpha)$ cycle of the torus. One can immediately notice that the Wilson loop with the contour $C_{n,m}$ conjugated by $v_{h,\rho}$ is a differential operator acting on the Verlinde states and coincides with the Verlinde operator $W_\Lambda{}^{(n,m)}$ defined by (5·3). Thus we can interpret that the Verlinde operator $W_\Lambda{}^{(n,m)}$ generates a Wilson loop with a contour $C_{n,m}$ in the Λ representation. As a matter of fact, it can move to the interior of a solid torus ($D^2\times S^1$), which is due to the nature of the flat connections.

Let us consider a structure of the physical states of G_C CSW theory on the torus. It is useful to rewrite them by means of the Verlinde states. Let us pick the physical state $\tilde{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}$ given by (4·1). It can be rewritten as

$$\tilde{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}=H^{1/2}\psi^{l_R l_L}_{\Lambda_R\Lambda_L}=H^{1/2}\exp\Big(-\frac{k\pi}{2\tau_2}u^2-\frac{l_R\pi}{\tau_2}u\cdot\bar{u}\Big)\overline{\chi^{l_R}_{\Lambda_R}}\chi^{l_L}_{\Lambda_L}\,, \qquad (5\cdot6)$$

where we used the formula $H^{1/2}=\tau_2{}^{r/2}e^{hQ_0}|\theta_{\bar{h},\rho}|^2$.[8)] H is the determinant of the Laplacian on the torus. Here we would like to pay attention to (5·6). It indicates that any physical state on the torus belonging to the physical state basis of the G_C CSW theory is proportional to a product of the Verlinde states $\overline{\chi^{l_R}{}_{\Lambda_R}}$ and $\chi^{l_L}_{\Lambda_L}$. They belong to the quantum Hilbert spaces of the CSW theories described by the actions $I_{l_R}[\omega]$ and $I_{l_L}[\omega]$ respectively. From this observation, one can arrive at the following conclusion.

THEOREM *Let $\mathcal{H}_{G_C}(T^2)$ be the quantum Hilbert space on T^2 of the G_C CSW theory with real coupling constants t and $\bar{t}$ which consists of the parallel L_2-sections. Let $\mathcal{H}_G{}^{l_L}(T^2)(\overline{\mathcal{H}_G{}^{l_R}}(T^2))$ be the quantum Hilbert space on T^2 spanned by the holomorphic (antiholomorphic) Verlinde basis derived in the CSW theory with the maximal compact subgroup G with the level $l_L=t/2-h$ ($l_R=-\bar{t}/2-h$). There exists the following isomorphism:*

$$\mathcal{H}_{G_C}(T^2)\cong\overline{\mathcal{H}_G{}^{l_R}}(T^2)\otimes\mathcal{H}_G{}^{l_L}(T^2)\,. \qquad (5\cdot7)$$

The strict proof of this theorem must include a discussion about bundle isomorphism. Namely, we must show that the quantum line bundle over $\mathcal{M}_C$ (the gauge quotient space of the complex connections) is naturally isomorphic to the previously defined line bundle over $\mathcal{M}$ whose sections are L^2 functions on $\mathcal{M}$. This is obvious in the quantum mechanics. $\mathcal{M}_C$ is identified (in a J dependent fashion) with a cotangent bundle $T^*\mathcal{M}$. And in quantization of a cotangent bundle T^*X, the physical Hilbert space consists of L^2 functions on X. If adapt this fact to our G_C CSW theory, we arrive at the desired bundle isomorphism. Needless to say, such a physical justification never be adequate in a mathematical discussion. We wish to have a mathematically strict proof of the bundle isomorphism in the near future.

We are ready to propose the analog of the Verlinde operators through the expression (5·6). From (5·2) and its complex conjugate, we find that

$$\psi^{l_R l_L}_{\Lambda_R \Lambda_L} = \overline{\tilde{W}^{(1,0)}_{\Lambda_R}}\, \tilde{W}^{(1,0)}_{\Lambda_L} \psi^{l_R l_L}_{00} = \tilde{W}^{(1,0)}_{\Lambda_L}\, \overline{\tilde{W}^{(1,0)}_{\Lambda_R}}\, \psi^{l_R l_L}_{00}\,. \tag{5·8}$$

The differential operator acting on the vacuum $\psi^{l_R l_L}_{00}$ is a special element of the analog of the Verlinde operators $\overline{\tilde{W}^{(n_R,m_R)}_{\Lambda_R}}\, \tilde{W}^{(n_L,m_L)}_{\Lambda_L}$. Each of them is given by a product of the following operators:

$$\overline{\tilde{W}^{(n,m)}_{\Lambda_R}} = \mathrm{Tr}_{\Lambda_R} \overline{v_{h,\rho}}^{-1} \exp\left(-i\int_{Cn,m} \rho(\bar{\omega}) + \frac{1}{l_R+h}\int_{Cn,m} \overline{\omega(z)} H^i \left(\nabla_{\bar{i}} - \frac{h\pi}{\tau_2} u_i\right)\right) \overline{v_{h,\rho}} \tag{5·9}$$

and

$$\tilde{W}^{(n,m)}_{\Lambda_L} = \mathrm{Tr}_{\Lambda_L} v_{h,\rho}^{-1} \exp\left(-i\int_{Cn,m} \rho(\bar{\omega}) + \frac{1}{l_L+h}\int_{Cn,m} \omega(z) H^i \left(\nabla_{\underline{i}} - \frac{h\pi}{\tau_2} \bar{u}_i\right)\right) v_{h,\rho}\,. \tag{5·10}$$

It is easy to see that these operators are commutable, i.e.,

$$\overline{\tilde{W}^{(n_R,m_R)}_{\Lambda_R}}\, \tilde{W}^{(n_L,m_L)}_{\Lambda_L} = \tilde{W}^{(n_L,m_L)}_{\Lambda_L}\, \overline{\tilde{W}^{(n_R,m_R)}_{\Lambda_R}}\,. \tag{5·11}$$

The connections $\nabla_{\bar{i}}$ and $\nabla_{\underline{i}}$ are defined in (3·4). We can derive a physical meaning from (5·8). The operator $\overline{\tilde{W}^{(1,0)}_{\Lambda_R}}$ generates a Wilson loop with a contour $C_{1,0}$ in the Λ_R representation in the anti-holomorphic sector described by the $I[\bar{\mathcal{A}}]$ CSW theory. $\tilde{W}^{(1,0)}_{\Lambda_L}$ generates a Wilson loop with a contour $C_{1,0}$ in the Λ_L representation in the holomorphic sector described by the $I[\mathcal{A}]$ CSW theory. The commutability of these operators arises from the fact that the G_C CSW theory consists of two independent CSW theories described by the actions $I[\mathcal{A}]$ and $I[\bar{\mathcal{A}}]$. Quite a similar interpretation is applicable to any Verlinde operator in the G_C CSW theory.

5.2. *The Jones-Witten invariant in the G_C CSW theory*

We here attempt to construct topological invariants of 3-manifolds in the G_C CSW theory, i.e., the knot and link invariants following Witten.[1] He proposed a general procedure to obtain the topological invariants of a huge class of 3-manifolds in the study of the CSW theory with a simply connected and compact Lie group. In

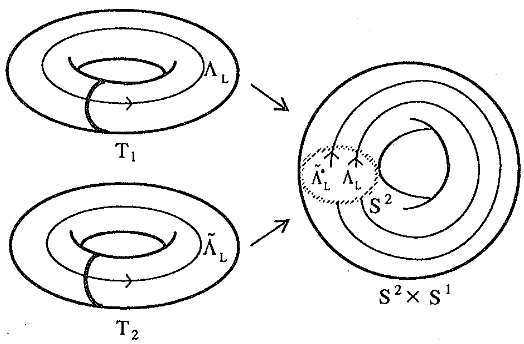

Fig. 2. $S^2\times S^1$ with two Wilson loops along S^1 made of two solid tori.

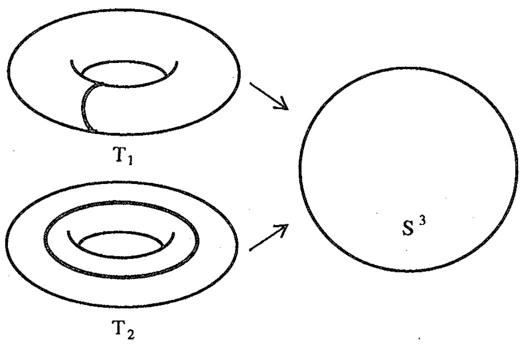

Fig. 3. S^3 made of two solid tori.

this subsection, for simplicity, let us deal only with the lens space. The lens space represents a class of 3-manifolds which admit the genus one Heegaard splitting.

Before discussing the topological invariants, let us remember that the CSW theory described by the action $I[\mathcal{A},\bar{\mathcal{A}}]$ consists of two independent CSW theories described by the actions $I[\mathcal{A}]$ and $I[\bar{\mathcal{A}}]$. So it is enough to consider procedures to construct the topological invariants in the $I[\mathcal{A}]$ CSW theory only, because procedures in the $I[\bar{\mathcal{A}}]$ CSW theory are obtained by reversing the orientation of 3-manifolds and the Wilson loops in arguments of the $I[\mathcal{A}]$-theory. For simplicity, it is convenient to assume that $C_L{}^i=C_R{}^i$ for $i=1\sim N$ where $C_L{}^i(C_R{}^i)$ represents the contour of the i-th Wilson loop in the $I[\mathcal{A}](I[\bar{\mathcal{A}}])$ CSW theory.

We start with $S^2\times S^1$ which belongs to the lens space. Let us consider two solid tori T_1 and T_2 depicted in Fig. 2. Each of the solid tori has a topology of $D^2\times S^1$. $S^2\times S^1$ is obtained after the α (β) cycle of T_1 is identified with the α (β) cycle of T_2. Let us suppose that T_1 includes a Wilson loop (a knot with a contour $C_{1,0}$) in the Λ_L representation and T_2 a Wilson loop (a knot with a contour $C_{1,0}$) in the $\tilde{\Lambda}_L$ representation. After gluing them, we obtain $S^2\times S^1$ in which two Wilson loops exist along S^1. Each of them crosses S^2 once, so S^2 has two punctures. A charge in the Λ_L representation is assigned to one of the two punctures and a charge in the $\tilde{\Lambda}_L^*$ representation is assigned to the other. Here $\tilde{\Lambda}_L^*$ is the dual representation of $\tilde{\Lambda}_L$. According to the charge conservation on S^2, the vacuum expectation value of the two Wilson loops over $S^2\times S^1$ is non-zero if and only if $\tilde{\Lambda}_L$ coincides with Λ_L and zero otherwise. This is the origin of the orthogonality of the physical states proved in § 4.

The topological invariants of S^3 (3-sphere) belonging to the lens space are the most important ones. S^3 can be made of two solid tori. In Fig. 3, there are two solid tori T_1 and T_2. S^3 is obtained by identifying the $\alpha(\beta)$ cycle of T_1 with the $\beta(\alpha)$ cycle of T_2. In such an identification, the mapping class group $PSL(2,\boldsymbol{Z})$ plays a crucial role. Since a boundary surface of the solid torus is the torus, there are two fundamental elements which generate the mapping class group acting on the Teichmüller space, i.e., the S- and T-transformations. They give rise to endomorphisms on the quantum Hilbert space, i.e., $S(T)$: $\mathcal{H}_J{}^Q\to\mathcal{H}_J{}^Q$. We are interested in matrix elements associated with the S-transformation, because it is a homeomorphism on the torus which maps from the $\alpha(\beta)$ cycle to the $\beta(\alpha)$ cycle.

The transformation properties of the physical state $\tilde{\psi}$ (the parallel section for the quantum connection $\tilde{\delta}^Q$) under the modular transformations are inconvenient for computing the topological invariants. Instead of $\tilde{\psi}$, it is necessary to employ a state

$$\hat{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}=\exp((k\pi/2\tau_2)u^2)\,\tilde{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}=H^{1/2}\exp\left(-\frac{l_R\pi}{\tau_2}u\cdot\bar{u}\right)\overline{\chi^{l_R}_{\Lambda_R}}\chi^{l_L}_{\Lambda_L}\,. \tag{5·12}$$

The transformation properties of the Weyl-Kac characters under the modular transformations are familiar:[13)]

$$T:\chi_\Lambda{}^k(u,\tau)\longrightarrow\chi_\Lambda{}^k(u,\tau+1)=\exp\left(2\pi i\left(h_\Lambda-\frac{C}{24}\right)\right)\chi_\Lambda{}^k(u,\tau)\,,$$

$$S:\chi_\Lambda{}^k(u,\tau)\longrightarrow\chi_\Lambda{}^k\left(\frac{u}{\tau},-\frac{1}{\tau}\right)=\sum_{\rho+\Lambda'\in\mathcal{I}_k}S^k_{\Lambda\Lambda'}\chi^k_{\Lambda'}(u,\tau)\,. \tag{5·13}$$

Here h_Λ is a conformal weight of a primary field in the Λ representation of the WZNW model and C stands for a central charge of it.[3] According to a fact that $u\cdot\bar{u}/\tau_2$ and H are invariant under the modular transformations, the transformation properties of the state $\hat{\psi}$ are obvious. They are given by those of the Weyl-Kac characters. Thus we immediately find that

$$S:\hat{\psi}^{l_R l_L}_{\Lambda_R \Lambda_L}(u,\bar{u},\tau,\bar{\tau})\longrightarrow \hat{\psi}^{l_R l_L}_{\Lambda_R \Lambda_L}\left(\frac{u}{\tau},\frac{\bar{u}}{\bar{\tau}},-\frac{1}{\tau},-\frac{1}{\bar{\tau}}\right)=(S\hat{\psi})^{l_R l_L}_{\Lambda_R \Lambda_L}(u,\bar{u},\tau,\bar{\tau})$$

$$=\sum_{\rho+\Lambda'_R\in\mathcal{F}_{l_R}}\ \sum_{\rho+\Lambda'_L\in\mathcal{F}_{l_L}}\overline{S^{l_R}_{\Lambda_R\Lambda'_R}}S^{l_L}_{\Lambda_L\Lambda'_L}\hat{\psi}^{l_R l_L}_{\Lambda'_R \Lambda'_L}(u,\bar{u},\tau,\bar{\tau})\,. \qquad (5\cdot14)$$

The next step to compute the topological invariants is to rewrite the inner product (3·13) by means of the states given by (5·12). The result is

$$\langle\langle\psi'|\psi\rangle\rangle=e^{2\xi kr}\int_{\mathcal{M}}d\mu\exp\left(-\frac{k\pi}{\tau_2}u\cdot\bar{u}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{c(n,m)}{n!}(4k\xi)^n\left(\frac{\tau_2}{k\pi}\right)^m$$

$$\times\left(\prod_{l=1}^{m}C^{i_l j_l}\right)\left(\prod_{l=1}^{m}\frac{\partial}{\partial u^{j_l}}\right)\overline{\hat{\psi}'^{l_R l_L}_{\Lambda_R\Lambda_L}}\cdot\left(\prod_{l=1}^{m}\frac{\partial}{\partial \bar{u}^{j_l}}\right)\hat{\psi}^{l_R l_L}_{\Lambda_R\Lambda_L}\,. \qquad (5\cdot15)$$

Since a differential operator of the form $\tau_2(\partial/\partial u^i)(\partial/\partial\bar{u}^j)$ is modular invariant, the inner product of states appropriately transformed by the modular transformations is determined by transformation matrices acting on the states $\hat{\psi}'(u,\bar{u},\tau,\bar{\tau})$ and $\hat{\psi}(u,\bar{u},\tau,\bar{\tau})$.

The first example of the topological invariants of S^3 is a trivial one, i.e., the vacuum expectation value of unity. Let us return to Fig. 3. We glue the two solid tori T_1 and T_2 after acting on the boundary of T_1 with the S-transformation which maps from the $\alpha(\beta)$ cycle to the $\beta(\alpha)$ cycle. The vacuum expectation value of the unity over S^3 in the G_C CSW theory is given by the inner product

$$Z(S^3:1)=\langle\langle\psi^{l_R l_L}_{00}|S\psi^{l_R l_L}_{00}\rangle\rangle=\text{const}\,\overline{S^{l_R}_{00}}S^{l_L}_{00}\,. \qquad (5\cdot16)$$

We used notation $Z(S^3:\mathcal{O})$ for a vacuum expectation value of an observable $\mathcal{O}$. It consists of links and knots of Wilson loops. From now on, to annihilate the dependence on the factor "const", let us employ quantities normalized as $\tau_{l_R l_L}(S^3:\mathcal{O})=Z(S^3:\mathcal{O}/Z(S^3:1)$.

A topological invariant to be considered next is a vacuum expectation value of an unknot over S^3. Let us consider two solid tori T_1 and T_2 in Fig. 4. T_1 is an empty

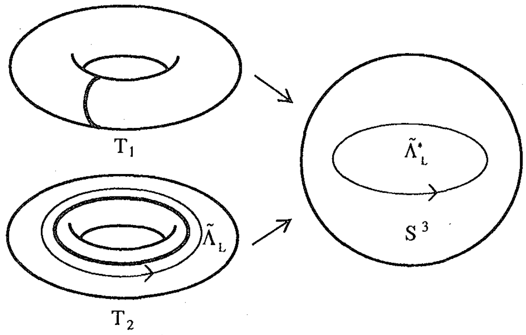

Fig. 4. An unknot in S^3.

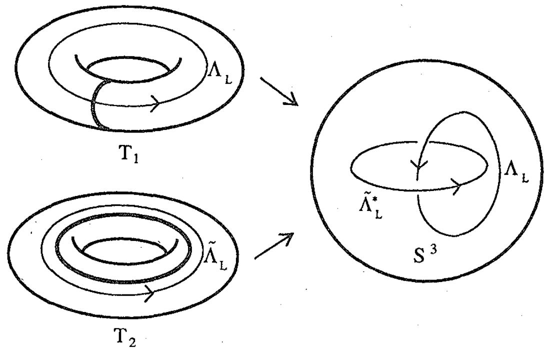

Fig. 5. A Hopf link in S^3.

solid torus (a vacuum) and T_2 includes a Wilson loop C (a knot with a contour $C_{1,0}$) in the $\tilde{\Lambda}_L$ representation. We act on the surface of T_1 with the S-transformation. Then the vacuum expectation value of the Wilson loop over S^3 is obtained by gluing the two solid tori. The result in the G_C CSW theory is

$$\tau_{l_R l_L}(S^3\colon C)=\langle\!\langle \psi^{l_R l_L}_{\tilde{\Lambda}_R \tilde{\Lambda}_L} | S\psi^{l_R l_L}_{00} \rangle\!\rangle / Z(S^3\colon 1)$$

$$=\left(\overline{\frac{S^{l_R}_{0\tilde{\Lambda}_R}}{S^{l_R}_{00}}}\right)\left(\frac{S^{l_L}_{0\tilde{\Lambda}_L}}{S^{l_L}_{00}}\right). \tag{5·17}$$

An analog of the Jones polynomials is conventionally defined by neglecting differences of framings of the links and knots.[13]

The last example is an invariant associated with the Hopf link as in Fig. 5. The solid torus T_1 includes a Wilson loop with a contour $C_{1,0}$ in the Λ_L representation and T_2 a Wilson loop with a contour $C_{1,0}$ in the $\tilde{\Lambda}_L$ representation. The vacuum expectation value of the Hopf link of the two Wilson loops over S^3 is obtained by gluing the two solid tori after acting on the surface of T_1 with the S-transformation. Let us denote the Hopf link of the Wilson loops by L. The link invariant in the G_C CSW theory is computed to be

$$\tau_{l_R l_L}(S^3\colon L)=\langle\!\langle \psi^{l_R l_L}_{\tilde{\Lambda}_R \tilde{\Lambda}_L} | S\psi^{l_R l_L}_{\Lambda_R \Lambda_L} \rangle\!\rangle / Z(S^3\colon 1)$$

$$=\left(\overline{\frac{S^{l_R}_{\Lambda_R \tilde{\Lambda}_R}}{S^{l_R}_{00}}}\right)\left(\frac{S^{l_L}_{\Lambda_L \tilde{\Lambda}_L}}{S^{l_L}_{00}}\right). \tag{5·18}$$

Let us summarize our remarks below. Each of the topological invariants in the G_C CSW theory which we have considered here is given by a product of a topological invariant in the $I[\mathcal{A}]$ CSW theory and that in the $I[\bar{\mathcal{A}}]$ CSW theory. Our construction of the topological invariants is applicable to any 3-manifold which belongs to the lens space. Let $\tau_{l_L}(\overline{\tau_{l_R}})$ be an invariant in the $I[\mathcal{A}]$ $(I[\bar{\mathcal{A}}])$ CSW theory. We can easily prove that every topological invariant of S^3 in the G_C CSW theory obtained by gluing two solid tori takes the following form:

$$\tau_{l_R l_L}(S^3\colon \mathcal{O})=\overline{\tau_{l_R}(S^3\colon \mathcal{O}_R)}\cdot \tau_{l_L}(S^3\colon \mathcal{O}_L)\,. \tag{5·19}$$

This result is consistent with a fact that every observable in the G_C CSW theory can be factorized into a product of two parts as $\mathcal{O}=\mathcal{O}_R\cdot\mathcal{O}_L$. The notation $\mathcal{O}_L(\mathcal{O}_R)$ stands for an observable in the $I[\mathcal{A}](I[\bar{\mathcal{A}}])$ CSW theory.

§ 6. Concluding remarks

In the present paper, we have investigated a relation between the G_C CSW theory and the conformal field theory (the WZNW model) on the torus. We have found that every physical state on the torus is given by a product of two Weyl-Kac characters. One is the holomorphic Weyl-Kac character of the WZNW model at the level $t/2=l_L+h$ ($l_L\in \boldsymbol{Z}_{\geq 0}$) and the other is the anti-holomorphic Weyl-Kac character at the level $-\bar{t}/2=l_R+h$ ($l_R\in \boldsymbol{Z}_{\geq 0}$). The central charges to which the left and right movers contribute are $(C_L, C_R)=\dim(G)(1-(2h/t), 1+(2h/\bar{t}))=\dim(G)((l_L/l_L+h), (l_R/l_R+h))$.

It must be remembered that the reason why l_L and l_R have been fixed at arbitrary positive integers is that the physical Hilbert space becomes finite-dimensional.

Another result which should be emphasized in the present paper is the orthogonality of the physical states belonging to the physical state basis on the torus. In the case when the parameter s is purely imaginary, the inner product of the physical states must be given by the exotic hermitian pairing. We have succeeded in proving the orthogonality of the physical states on the torus utilizing it. It seems reasonable because the quantum Hilbert space of the G_C CSW theory should be closely related to a product of quantum Hilbert spaces of the two independent CSW theories described by the actions $I[\mathcal{A}]$ and $I[\bar{\mathcal{A}}]$ defined in (2·3).

As in the G CSW theory, the orthogonality arises from the charge conservation on S^2 which is essential in computability of a vacuum expectation value of two Wilson loops over $S^2 \times S^1$. Owing to it, we are able to compute a huge class of the topological invariants of the lens space via the genus one Heegaard splitting. It also has been found in the present paper that each of them is factorized into a product of two parts. One is the invariant in the CSW theory described by the action $I[\mathcal{A}]$ (which is related to the left-handed sector of the WZNW model). The other is that in the CSW theory described by the action $I[\bar{\mathcal{A}}]$ (which is related to the right-handed sector of the WZNW model).

Let us end with a few problems remained to be investigated in the G_C CSW theory. First, in our arguments, we have dealt with the special values of the coupling constants $(t/2, -\bar{t}/2)=(l_L+h, l_R+h)$ where l_R, $l_L \in \boldsymbol{Z}_{\geq 0}$. However, t and $-\bar{t}$ can take continuous values because the parameter s may be continuous and purely imaginary. For generic real values we have not considered here, infinite-dimensionality of the quantum Hilbert space might appear. In such cases, can we find a well-defined analog of the Jones-Witten invariants? Second, we are interested in the relation between the CSW gravity and the gravity in the ADM formalism. Presumably, the quantum Hilbert space in the ADM formalism could be obtained by tensor products of that of the CSW gravity. In the Poincare gravity (in which $\Lambda=0$), the relation of them has been partially clarified by Carlip.[16] Third, the topological invariants of S^3 proposed here coincide with the Turaev-Viro invariants[17] (which are invariant under subdivisions of a PL-manifold) of S^3. This indicates an intimate relation between our continuum approach and a lattice approach (e.g., the approach of Ponzano and Regge[18]) to the quantum gravity.

Acknowledgments

The author am indebted to K. Kikkawa, K. Shizuya, T. Kubota, K. Higashijima, Y. Yamada and N. Ishibashi for encouragement. He would like to thank T. Kohno and I. Fujii for helpful comments.

References

1) E. Witten, Commun. Math. Phys. **121** (1989), 351.
2) V. F. R. Jones, Bull. AMS **12** (1985), 103; Ann. Math. **126** (1987), 335.

3) V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. **B247** (1984), 83.
4) E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. **B336** (1990), 581.
5) E. Witten, Nucl. Phys. **B311** (1988/89), 46; **B323** (1989), 113.
6) A. Hosoya and K. Nakao, Prog. Theor. Phys. **84** (1990), 739.
7) N. Woodhouse, *Geometric Quantization* (Oxford University Press, 1980).
8) S. Axelrod, S. D. Pietra and E. Witten, J. Diff. Geom. **33** (1991), 787.
9) E. Witten, Commun. Math. Phys. **137** (1991), 29.
10) M. S. Narasimhan and C. S. Seshadri, Ann. Math. **82** (1965), 540.
11) K. Gawedzki and A. Kupiainen, Nucl. Phys. **B320** (1989), 625.
12) G. Moore and N. Seiberg, Nucl. Phys. **B326** (1989), 108.
13) J. M. F. Labastida, P. M. Llatas and A. V. Rammallo, Nucl. Phys. **B349** (1991), 651.
14) M. Bos and V. P. Nair, Int. J. Mod. Phys. **A5** (1990), 959.
15) E. Verlinde, Nucl. Phys. **B300** (1988), 360.
16) S. Carlip, Phys. Rev. **D45** (1992), 3584.
17) V. G. Turaev and O. Y. Viro, Topology **32** (1992), 865.
18) G. Ponzano and T. Regge, *Spectroscopic and Group Theoretical Methods in Physics*, ed. F. Bloch (North-Holland, Amsterdam, 1968).