Level Clustering in a One-Dimensional Finite System

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We prove rigorously that there is level clustering in the semi-classical limit of a one-dimensional Schrödinger operator which has a chain of δ-potentials.

§ 1. Introduction

Let us consider the Schrödinger operator of the type

$$H_v(\hbar)=-\hbar^2\frac{d^2}{dx^2}+v\sum_{j=1}^{n}\delta(x-x_j)\,,\quad 0\leq x\leq 1\,,$$

under the Dirichlet boundary condition at $x=0$ and $x=1$. Here $\hbar$ is the Planck's constant, $v>0$, and

$$0=x_0<x_1<\cdots<x_n<x_{n+1}=1$$

are arbitrarily fixed. In this note, we analyse rigorously statistics for the energy level distribution of $H_v(\hbar)$. (In this note, we say that κ is an *eigenvalue* of $H_v(\hbar)$ if and only if the equation $H_v(\hbar)\psi=\kappa^2\psi$ has a non-trivial solution ψ satisfying $\psi(0)=\psi(1)=0$. On the other hand, we call κ^2 itself an *energy level* of $H_v(\hbar)$.)

To be precise, define for each $\hbar>0$ and $k=0, 1, \ldots$ the set

$$\mathcal{A}_k(\hbar)=\{t\in(a_1, a_2)|(t, t+c\hbar] \text{ contains exactly } k \text{ eigenvalues of } H_v(\hbar)\}\,,$$

where the constant $c>0$ and the interval (a_1, a_2) are arbitrarily fixed. Then denoting the Lebesgue measure of a set $B\subset \boldsymbol{R}$ by $|B|$, we obtain

Theorem 1 *For each $k=0, 1, \ldots,$ the limit*

$$\pi_k(c)=\pi_k(c; x_1, \ldots, x_n)=\lim_{\hbar\downarrow 0}(a_1-a_2)^{-1}|\mathcal{A}_k(\hbar)|$$

exists, and is independent of $v>0$. In particular, when the numbers $y_j=x_{j+1}-x_j$, $j=0, 1, \ldots, n$, are rationally independent, we can compute $\pi_k(c)$ explicitly as follows:

$$\pi_k(c)=\begin{cases}\sum_{A\in\mathcal{F}(k-M_c)}\prod_{j\in A}\left\{\frac{cy_j}{\pi}\right\}\prod_{j\in A^c}\left(1-\left\{\frac{cy_j}{\pi}\right\}\right), & 0\leq k-M_c\leq n+1\\ 0 & \text{otherwise}\,,\end{cases}$$

where $\mathcal{F}(p)$ is the totality of subsets of $\{0, 1, \ldots, n\}$ of cardinality p, and

*) This research was supported by University of Tsukuba Research Projects.

$$M_c \equiv \sum_{j=0}^{n} [cy_j/\pi] .$$

Here we use the following notation: for $A \subset \{0, 1, \ldots, n\}$, A^c *denotes the complement* $\{0, \ldots, n\} \backslash A$ *of* A; $[a]$ *and* $\{a\}$ *denote respectively the integral and fractional part of the real number* a.

Corollary *Suppose* y_j, $j=0, 1, \ldots, n$ *are rationally independent. Then we can compute the probability distribution of eigenvalue spacing* $\rho(c)$ *in the sense that*

$$\rho(c) = \lim_{\hbar \downarrow 0} n_v(\hbar; a_1, a_2)^{-1} \xi_v(\hbar, c; a_1, a_2)$$
$$= \left(\prod_{j=0}^{n} \left(1 - \frac{cy_j}{\pi} \right) \right) \sum_{j=0}^{n} \frac{y_j}{1 - cy_j/\pi}$$

for $0 < c < \pi$, *where* $n_v(\hbar; a_1, a_2)$ *is the number of eigenvalues* $\kappa_j = \kappa_j(v, \hbar)$ *of* $H_v(\hbar)$ *contained in* (a_1, a_2), *and* $\xi_v(\hbar, c; a_1, a_2)$ *is the number of eigenvalues in* (a_1, a_2) *satisfying* $\kappa_j - \kappa_{j-1} > c\hbar$. *In particular,*

$$\lim_{c \downarrow 0} (-\rho'(c)) = \frac{1}{\pi} \left(1 - \frac{1}{\pi} \sum_{j=0}^{n} y_j^2 \right) > 0 .$$

In order to see what happens in typical cases, let us randomize $(x_j)_{j=1}^n$ in the following manner. Let $X_1, \ldots, X_n$ be independent random variables uniformaly distributed on $(0, 1)$, and let $x_1^{(n)} < \cdots < x_n^{(n)}$ be their arrangement according to magnitude. Finally set $x_j = x_j^{(n)}$, $j=1, \ldots, n$. Then $\pi_k(c) = \pi_k(c; x_1^{(n)}, \ldots, x_n^{(n)})$ in Theorem 1 are also random variables.

Theorem 2 *With probability one, we have*

$$\lim_{n \to \infty} \pi_k(c; x_1^{(n)}, \ldots, x_n^{(n)}) = e^{-c/\pi} \frac{1}{k!} \left(\frac{c}{\pi} \right)^k, \; k=0, 1, \ldots .$$

Namely if the system is sufficiently random, then for a typical realization of $H_v(\hbar)$, *the sequence of its eigenvalues looks like a Poisson process.*

Let us make some historical remarks. Pokrovskii[1] considered a model which, after a scaling of the space variable, is reduced to our $H_v(\hbar)$ with $\hbar \approx n^{-1}$, and argued that when the system is sufficiently random, then there is the so called "level repulsion" in high energy region. Afterwards, two contrasting comments are made on this assertion. Molchanov declared Pokrovskii's conclusion to be wrong in the introduction of a paper[2] in which he proved rigorously his "correct" result showing the Poisson character of energy level distribution. But in fact, Molchanov studied a problem somewhat different from Pokrovskii's one in the sense that Molchanov considered the limiting case where the system size expands to infinity, so that his criticism is irrelevant. This was pointed out by Berry,[3] who argued that level repulsion should naturally be observed in the semi-classical limit for one-dimensional Schrödinger operators. Our results above (especially Corollary to Theorem 1) show that even in the energy level statistics in Berry's sense, one typically observes level clustering rather than level repulsion. In fact, Pokrovskii considered energy region much higher than the semi-classical region ($\kappa \approx \hbar^{-1}$) considered here, and we shall remark at the end of § 2 that in that region, it is qualitatively obvious that we observe only level repulsion.

§ 2. Proof of Theorem 1 and its corollary

If $H_v(\hbar)$ has eigenvalues $\{\kappa_m(v, \hbar)\}_{m=1}^{\infty}$, then after scaling, $H_{v\hbar^{-2}}(1)$ has eigenvalues $\{\hbar^{-1}\kappa_m(v, \hbar)\}_{m=1}^{\infty}$. From this consideration, it is easy to see that

$$(a_2-a_1)^{-1}|\mathcal{A}_k(\hbar)|=\{(a_2-a_1)L\}^{-1}|\mathcal{B}_k{}^v(L)|\,,$$

where we have set $L=\hbar^{-1}$ and

$$\mathcal{B}_k{}^v(L)=\{t\in(a_1L, a_2L)|(t, t+c]\text{ contains exactly } k \text{ eigenvalues of } H_{vL^2}(1)\}\,.$$

Let us first assume $v=+\infty$. In this limiting case, $H_{vL^2}(1)=H_\infty(1)$ is the direct sum of $(n+1)$-Dirichlet Laplacians, each on the subinterval $[x_j, x_{j+1}]$, $j=0, 1, \ldots, n$, of $[0, 1]$. Hence the totality of its eigenvalues is precisely given by

$$\left\{\frac{\pi m}{y_j}\middle| m\geq 1, 0\leq j\leq n\right\},$$

where $y_j=x_{j+1}-x_j$, $j=0, 1, \ldots, n$.

Now if $N(\Delta)$ is the number of eigenvalues of $H_\infty(1)$ in the set Δ, then

$$N(\Delta)=\sum_{j=0}^{n}\sum_{m=1}^{\infty}\chi_\Delta\left(\frac{\pi m}{y_j}\right),$$

so that

$$N((t, t+c])=\sum_{j=0}^{n}\sum_{m=1}^{\infty}\chi_{[m-cy_j/\pi,m)}\left(\frac{y_j}{\pi}t\right),$$

χ_A being the indicator (or the characteristic function) of the set A. We can write this

$$N((t, t+c])=M_c+\sum_{j=0}^{n}\sum_{m=0}^{\infty}\chi_{[m-\{cy_j/\pi\},m)}\left(\frac{y_j}{\pi}t\right),$$

where $M_c=\sum_{j=0}^{n}[cy_j/\pi]$. This shows first of all

$$M_c\leq N((t, t+c])\leq M_c+n+1\,,$$

and that for $M_c\leq k\leq M_c+n+1$, we have $N((t, t+c])=k$ if and only if there is a subset A of $\{0, 1, \ldots, n\}$ of cardinality $k-M_c$, namely $A\in\mathcal{F}(k-M_c)$, such that

$$\begin{cases} y_jt/\pi\in[1-\{cy_j/\pi\}, 1) \quad (\text{mod } 1) \quad \text{for } j\in A \\ y_jt/\pi\in[0, 1-\{cy_j/\pi\}) \quad (\text{mod } 1) \text{ for } j\in A^c=\{0, 1, \ldots, n\}\setminus A\,. \end{cases}$$

Let us define the flow $\{S_t\}$ on $\boldsymbol{T}^{n+1}$, the $(n+1)$-dimensional torus, by

$$S_t(\theta_0, \theta_1, \ldots, \theta_n)=\left(\theta_0+\frac{y_0t}{\pi}(\text{mod } 1),\ \theta_1+\frac{y_1t}{\pi}(\text{mod } 1), \ldots, \theta_n+\frac{y_nt}{\pi}(\text{mod } 1)\right).$$

Then $N((t, t+c])=k$ if and only if

$$S_t(0, 0, \dots, 0)\in R(A)\equiv \prod_{j\in A}[1-\{cy_j/\pi\}, 1)\times \prod_{j\in A^c}[0, 1-\{cy_j/\pi\})$$

for some $A\in \mathcal{F}(k-M_c)$. Hence

$$\{(a_2-a_1)L\}^{-1}|\mathcal{B}_k^{v=\infty}(L)|=\sum_{A\in\mathcal{F}(k-M_c)}\{(a_2-a_1)L\}^{-1}\int_{a_1L}^{a_2L}\chi_{R(A)}(S_t(0, \dots, 0))dt\ .$$

Suppose for a moment that $y_0, y_1, \dots, y_n$ are rationally independent. Then as is well known,[4] the flow $\{S_t\}$ is uniquely ergodic on $\boldsymbol{T}^{n+1}$, and if f is a continuous function on $\boldsymbol{T}^{n+1}$, then for every $\theta\in \boldsymbol{T}^{n+1}$, one has

$$\lim_{L\to\infty}\{(a_2-a_1)L\}^{-1}\int_{a_1L}^{a_2L}f(S_t\theta)dt=\int_{\boldsymbol{T}^{n+1}}f(\theta')\mu(d\theta')\ ,$$

where μ is the normalized Lebesgue measure on $\boldsymbol{T}^{n+1}$. This formula is valid even if we let $f=\chi_{R(A)}$, because the boundary of the set $R(A)$, which is the set of discontinuity points of the function $\chi_{R(A)}$, has zero Lebesgue measure. Thus we arrive at

$$\begin{aligned}\pi_k(c)&\equiv\lim_{L\to\infty}\{(a_2-a_1)L\}^{-1}|\mathcal{B}_k^{v=\infty}(L)|\\ &=\sum_{A\in\mathcal{F}(k-M_c)}\mu(R(A))\\ &=\sum_{A\in\mathcal{F}(k-M_c)}\prod_{j\in A}\left\{\frac{cy_j}{\pi}\right\}\prod_{j\in A^c}\left(1-\left\{\frac{cy_j}{\pi}\right\}\right).\end{aligned}$$

If, among $y_0, y_1, \dots, y_n$, exactly $n'<n$ are rationally independent, then $\{S_t\}$ is uniquely ergodic on an n'-dimensional submanifold of $\boldsymbol{T}^{n+1}$, which is isomorphic to $\boldsymbol{T}^{n'}$, so that by essentially the same analysis as above, we can conclude the existence of the limit $\pi_k(c)$. However the exact form of $\pi_k(c)$ will be somewhat complicated. We omit the detail about it.

Let us return to the general case $0<v<+\infty$. By definition, the equation $H_v(1)\psi=\kappa^2\psi$ is equivalent to the following set of conditions:

$$\begin{cases}-\psi''(x)=\kappa^2\psi(x) \quad \text{for } x_j<x<x_{j+1},\ j=0, 1, \dots, n;\\ \psi(x) \text{ is continuous at } x=x_j,\ j=1, \dots, n;\\ \psi'(x_j+0)-\psi'(x_j-0)=v\psi(x_j),\ j=1, \dots, n\ .\end{cases}$$

Elementary calculation shows that if a solution $\psi(x)$ of $H_v(1)\psi=\kappa^2\psi$ is represented by

$$\psi(x)=\alpha_j\sin\kappa x+\beta_j\cos\kappa x\ ,$$

on each interval $x_j<x<x_{j+1}$, $j=0, 1, \dots, n$, then (α_0, β_0) and (α_n, β_n) are related by

$$\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix}=T_{\kappa,v}(x_n)T_{\kappa,v}(x_{n-1})\cdots T_{\kappa,v}(x_1)\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix},$$

where the matrices $T_{\kappa,v}(x_j)$ are defined by

$$T_{\kappa,v}(x_j)=I+\frac{v}{\kappa}V(\kappa x_j);$$

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

$$V(\kappa x)=\begin{pmatrix}\sin\kappa x\cos\kappa x & \cos^2\kappa x\\ -\sin^2\kappa x & -\sin\kappa x\cos\kappa x\end{pmatrix}.$$

Now κ is an eigenvalue of $H_v(1)$ if and only if any solution ψ of $H_v(1)\psi=\kappa^2\psi$ with $\psi(0)=0$ satisfies also $\psi(1)=0$. This is equivalent to saying that for the choice $\alpha_0=1$, $\beta_0=0$, we have

$$\alpha_n\sin\kappa+\beta_n\cos\kappa=0\,,$$

where (α_n, β_n) is defined by the above formula. It is not hard to compute

$$V(\kappa x_{j_p})V(\kappa x_{j_{p-1}})\cdots V(\kappa x_{j_1})$$
$$=\prod_{i=2}^{p}\sin\kappa(x_{j_i}-x_{j_{i-1}})$$
$$\times\begin{pmatrix}\cos\kappa x_{j_p}\sin\kappa x_{j_1} & \cos\kappa x_{j_p}\cos\kappa x_{j_1}\\ -\sin\kappa x_{j_p}\sin\kappa x_{j_1} & -\sin\kappa x_{j_p}\cos\kappa x_{j_1}\end{pmatrix}$$

for $1\leq j_1<\cdots<j_p\leq n$. Hence $\kappa\neq0$ is an eigenvalue of $H_v(1)$ if and only if it is a zero of the entire function

$$\Gamma_v(\kappa)=\sum_{p=0}^{n}\left(\frac{v}{\kappa}\right)^p\sum_{1\leq j_1<\cdots<j_p\leq n}\prod_{s=1}^{p+1}\sin\kappa(x_{j_s}-x_{j_{s-1}})\,,$$

where we set $x_{j_0}=x_0=0$ and $x_{j_{p+1}}=x_{n+1}=1$ for convenience.

At this stage, we substitute vL^2 into v above and restrict our attention to $|\kappa|\leq CL$, $C>a_2$ being a constant. Then $\kappa\neq0$ is an eigenvalue of $H_{vL^2}(1)$ if and only if

$$\Phi_{vL^2}(\kappa)\equiv\left(\frac{\kappa}{vL^2}\right)^n\Gamma_{vL^2}(\kappa)=0\,.$$

If we write

$$\Phi_{vL^2}(\kappa)=f(\kappa)+g_L(\kappa);$$

$$f(\kappa)=\prod_{j=0}^{n}\sin\kappa(x_{j+1}-x_j)=\prod_{j=0}^{n}\sin\kappa y_j\,,$$

then it is not difficult to obtain the following

Lemma i) *For $\kappa=\sigma+i\tau$ with $|\kappa|\leq CL$,*

$$|g_L(\kappa)|\leq BL^{-1}e^{|\tau|}$$

holds with a constant B depending only on n.
ii) *There are constants K_1, $K_2>0$ such that if $|\tau|\geq K_1$, then*

$$|f(\kappa)|\geq K_2e^{|\tau|}\,.$$

iii) *If $f(\sigma_1)=f(\sigma_2)=0$, $\sigma_2>\sigma_1$, then for $\sigma_1<\sigma<\sigma_2$,*

$$|f(\sigma)| \geq \sin\nu(\sigma-\sigma_1)\sin\nu(\sigma_2-\sigma)|^{n+1},$$

where $\nu = \min_{0\leq j\leq n} y_j$.
iv) If, in addition, $\sigma_2 - \sigma_1 \geq \delta > 0$, *then there is a constant* W *depending only on* δ, n, $y_1, \ldots, y_n$ *such that*

$$\left|f\left(\frac{\sigma_1+\sigma_2}{2}+i\tau\right)\right| \geq We^{|\tau|}.$$

For $k=0, 1, 2, \ldots$ let $\mathscr{B}_k^{v=\infty}(\infty)$ be the totality of $t\geq 0$ such that $(t, t+c]$ contains exactly k eigenvalues of $H_\infty(1)$. Then clearly $\mathscr{B}_k^{v=\infty}(\infty)$ is the union of mutually disjoint intervals:

$$\mathscr{B}_k^{v=\infty}(\infty) = \cup_{r\geq 1} I_r^{(k)};\ I_r^{(k)} = [a_r^{(k)}, b_r^{(k)}).$$

To each $I_r^{(k)}$, there corresponds a set of $(k+2)$ eigenvalues of $H_\infty(1)$: $\lambda_1^r < \kappa_1^r < \cdots < \kappa_k^r < \lambda_2^r$ with the following properties:
(1) $I_r^{(k)} \subset [\lambda_1^r, \lambda_2^r - c)$;
(2) $\kappa_k^r - \kappa_1^r < c$;
(3) $a_r^{(k)} = (\kappa_k^r - c)\vee\lambda_1^r$, $b_r^{(k)} = \kappa_1^r \wedge(\lambda_2^r - c)$, and hence

$$\begin{aligned}|I_r^{(k)}| &= b_r^{(k)} - a_r^{(k)} \\ &= (\kappa_1^r - \kappa_k^r + c)\wedge(\kappa_1^r - \lambda_1^r)\wedge(\lambda_2^r - \kappa_k^r)\wedge(\lambda_2^r - \lambda_1^r - c).\end{aligned}$$

Note also that $\lambda_2^r - \lambda_1^r = O(1)$ as $r\to\infty$.

Now fix an arbitrary $\delta>0$. If $I_r^{(k)} \subset [0, CL]$ and $|I_r^{(k)}| > \delta$, then $\kappa_1^r - \lambda_1^r > \delta$ and $\lambda_2^r - \kappa_k^r > \delta$. Hence noting i), ii), and iv) of Lemma and using Rouché's theorem, we can conclude that there are exactly k zeros of $\Phi_{vL^2}(\kappa) = f(\kappa) + g_L(\kappa)$, namely eigenvalues of $H_{vL^2}(1)$, in the interval

$$\left(\frac{1}{2}(\lambda_1^r + \kappa_1^r), \frac{1}{2}(\kappa_k^r + \lambda_2^r)\right).$$

(Recall that λ_i^r and κ_j^r are zeros of $f(\kappa)$.) Moreover from i) and iii) of Lemma, we see that there is $\eta(L) = O(L^{-1/(n+1)})$ such that $\Phi_{vL^2}(\kappa)$ has no zero in the intervals

$$[\lambda_1^r + \eta(L), \kappa_1^r - \eta(L)] \quad\text{and}\quad [\kappa_k^r + \eta(L), \lambda_2^r - \eta(L)].$$

To summarize, we have shown that if $I_r^{(k)} \subset (a_1L, a_2L)$ and if $|I_r^{(k)}| = b_r^{(k)} - a_r^{(k)} > \delta$, then

$$(a_r^{(k)} + \eta(L), b_r^{(k)} - \eta(L)) \subset \mathscr{B}_k^v(L),$$

as far as L is large enough. Hence

$$\begin{aligned}|\mathscr{B}_k^v(L)| &\geq \sum_{r;I_r^{(k)}\subset(a_1L, a_2L), |I_r^{(k)}|>\delta} (|I_r^{(k)}| - 2\eta(L)) \\ &\geq \sum_{I_r^{(k)}\subset(a_1L, a_2L)} |I_r^{(k)}| - (\delta + 2\eta(L))\#\{r; I_r^{(k)}\subset(a_1L, a_2L)\}.\end{aligned}$$

But since $|I_r^{(k)}| = O(1)$ and $\#\{r; I_r^{(k)}\subset(a_1L, a_2L)\} \leq DL$ for some constant $D>0$, we get

$$\liminf_{L\to\infty}\{(a_2-a_1)L\}^{-1}|\mathcal{B}_k{}^v(L)|\geq\pi_k(c)-D\delta\,.$$

Letting $\delta\downarrow 0$, we obtain

$$\liminf_{L\to\infty}\{(a_2-a_1)L\}^{-1}|\mathcal{B}_k{}^v(L)|\geq\pi_k(c)\,,\quad k=0,1,\ldots\,.$$

From this and

$$\sum_{k=0}^{\infty}|\mathcal{B}_k{}^v(L)|=(a_2-a_1)L\,;$$

$$\sum_{k=0}^{\infty}\pi_k(c)=1\,,$$

it is now easy to conclude

$$\lim_{L\to\infty}\{(a_2-a_1)L\}^{-1}|\mathcal{B}_k{}^v(L)|=\pi_k(c)\,,\quad k=0,1,\ldots\,,$$

completing the proof of Theorem 1.

Proof of corollary By scaling, our proof is reduced to computing

$$\rho(c)=\lim_{L\to\infty}n_v(L)^{-1}\xi_v(L;c)\,,$$

where $n_v(L)$ is the number of eigenvalues $\kappa_j(vL^2;1)$ of $H_{vL^2}(1)$ contained in (a_1L,a_2L), and

$$\xi_v(L;c)=\#\{\kappa_j=\kappa_j(vL^2;1)\in(a_1L,a_2L)|\kappa_j-\kappa_{j-1}>c\}\,.$$

First let us prove $n_v(L)\sim L/\pi$ as $L\to+\infty$. This is easy when $v=+\infty$. In order to treat the case $v<+\infty$, consider

$$\mathcal{B}_0{}^{v=\infty}(\infty)=\cup_{r\geq1}[a_r{}^{(0)},b_r{}^{(0)})$$

as in the proof of Theorem 1. Then for each $r\geq1$, $a_r{}^{(0)}$ and $b_r{}^{(0)}+c$ are eigevalues of $H_\infty(1)$. If we take $L=b_r{}^{(0)}$ or $L=a_{r+1}^{(0)}$ in the formula

$$\lim_{L\to\infty}L^{-1}|[0,L]\cap\mathcal{B}_0{}^{v=\infty}(\infty)|\pi_0(c)\,,$$

we see in particular that

$$a_{r+1}^{(0)}/b_r{}^{(0)}\to1,\quad r\to\infty\,.$$

Now for L given, let

$$r(L)=\max\{r|b_r{}^{(0)}\leq L\}\quad\text{and}\quad r'(L)=\min\{r|a_r{}^{(0)}\geq L\}\,.$$

Further let $c_r=(a_r{}^{(0)}+b_r{}^{(0)}+c)/2$. Then clearly

$$n_v(c_{r(L)})\leq n_v(L)\leq n_v(c_{r'(L)})\,,$$

and by the Lemma before and Rouché's theorem again, we see

$$n_v(c_{r(L)})=n_\infty(c_{r(L)})\quad\text{and}\quad n_v(c_{r'(L)})=n_\infty(c_{r'(L)})\,.$$

But as $L\to\infty$, we have

$$n_\infty(c_{r(L)}) \sim \frac{1}{\pi} c_{r(L)} \sim \frac{L}{\pi}$$

and

$$n_\infty(c_{r'(L)}) \sim \frac{1}{\pi} c_{r'(L)} \sim \frac{L}{\pi},$$

so that

$$n_v(L) \sim \frac{L}{\pi}$$

as desired.

Now denote by $N(\Delta)$ the number of eigenvalues of $H_{vL^2}(1)$ in the set Δ. Then it is not difficult to see that

$$\begin{aligned}
&|\{t \in (a_1, a_2) | N((t, t+\delta]) > 0, N((t+\delta, t+c]) = 0\}| \\
&\quad = |\{t \in (a_1, a_2) | N((t+\delta, t+c]) = 0\}| - |\{t \in (a_1, a_2) | N((t, t+c]) = 0\}| \\
&\quad \geq \delta \xi_v(L; c),
\end{aligned}$$

and

$$\begin{aligned}
&|\{t \in (a_1, a_2) | N((t, t+\delta]) > 0, N((t+\delta, t+\delta+c]) = 0\}| \\
&\quad = |\{t \in (a_1, a_2) | N((t+\delta, t+\delta+c]) = 0\}| - |\{t \in (a_1, a_2) | N((t, t+\delta+c]) = 0\}| \\
&\quad \leq \delta \xi_v(L; c).
\end{aligned}$$

Hence it follows from Theorem 1,

$$\limsup_{L\to\infty} \{(a_2 - a_1)L\}^{-1} \xi_v(l; c) \leq \frac{1}{\delta}(\pi_0(c-\delta) - \pi_0(c));$$

$$\liminf_{L\to\infty} \{(a_2 - a_1)L\}^{-1} \xi_v(L; c) \geq \frac{1}{\delta}(\pi_0(c) - \pi_0(c+\delta)),$$

and if $y_0, y_1, \ldots, y_n$ are rationally independent and if $0 < c < \pi$, we obtain by letting $\delta \downarrow 0$,

$$\begin{aligned}
\lim_{L\to\infty} \{(a_2 - a_1)L\}^{-1} &= -\frac{d}{dc}\pi_0(c) \\
&= \frac{1}{\pi}\left\{\prod_{j=0}^{n}\left(1 - \frac{cy_j}{\pi}\right)\right\}\sum_{i=0}^{n}\frac{y_i}{1 - cy_i/\pi}.
\end{aligned}$$

Combining this with $N((a_1L, a_2L)) \sim (a_2 - a_1)L/\pi$, we arrive at the desired conclusion.

Remark Instead of considering statistics among eigenvalues $\kappa \in (a_1\hbar^{-1}, a_2\hbar^{-1})$ and letting $\hbar \downarrow 0$, let us fix $\hbar = 1$ and observe the energy region $\kappa \gg 1$. Then the eigenvalues $\{\kappa_j\}_{j\geq 1}$ of $H_v(1)$ are the zeros of the entire function $\Gamma_v(\kappa) = \sin\kappa(1 + O(\kappa^{-1}))$. Hence as in Titchmarsh,[5)] we see $\kappa_j = \pi j + O(j^{-1})$. This shows that whatever randomness the system may have, the eigenvalues κ_j for large j can only fluctuate around the lattice point πj within $O(j^{-1})$. Hence there must be level repulsion.

§ 3. Proof of Theorem 2

Let $y_j^{(n)}=x_{j+1}^{(n)}-x_j^{(n)}$, $j=0, 1, \cdots, n$. As is well known, if U_j, $j=0, 1, \ldots$ are independent random variables each obeying exponential distribution of parameter 1, namely $P(U_j>t)=e^{-t}$, $j=0, 1, \ldots$, and if $s_n=\sum_{j=0}^{n} U_j$, then $(y_0^{(n)}, y_1^{(n)}, \ldots, y_n^{(n)})$ and $(U_0/s_n, U_1/s_n, \ldots, U_n/s_n)$ have the same probability distribution.

For $0<\epsilon<1$, we can compute

$$
\begin{aligned}
P(\max_{0\le j\le n} y_j^{(n)}>\epsilon) &\le \sum_{j=0}^{n} P(y_i^{(n)}>\epsilon)\\
&=(n+1)P(y_0^{(n)}>\epsilon)\\
&=(n+1)P((1-\epsilon)U_0>\epsilon\sum_{j=0}^{n} U_j)\\
&=(n+1)E\left[P\left(U_0>\frac{\epsilon}{1-\epsilon}\sum_{j=1}^{n} U_j \middle| U_1, \ldots, U_n\right)\right]\\
&=E\left[\exp\left(-\frac{\epsilon}{1-\epsilon}\sum_{j=1}^{n} U_j\right)\right]\\
&=(1-\epsilon)^n .
\end{aligned}
$$

Hence by taking $\epsilon=\epsilon_n=n^{-\eta}$, $0<n<1$, we obtain

$$
\sum_{n=1}^{\infty} P(\max_{0\le j\le n} y_j^{(n)}>n^{-\eta})<\infty ,
$$

and by Borel-Cantelli's lemma, we have with probability one and for n sufficiently large,

$$
\max_{0\le j\le n} y_j^{(n)}\le n^{-\eta} .
$$

In particular, $M_c=M_c(y_0^{(n)}, \ldots, y_n^{(n)})=0$ for n sufficiently large. Moreover with probability one, we are in the situation that $y_0^{(n)}, y_1^{(n)}, \ldots, y_n^{(n)}$ are rationally independent for all $n\ge 1$. To summarize, with probability one and for n sufficiently large, we have

$$
\begin{aligned}
\pi_k(c; y_0^{(n)}, \ldots, y_n^{(n)}) &= \sum_{A\in\mathcal{I}(k)}\prod_{j\in A}\left(\frac{cy_j^{(n)}}{\pi}\right)\prod_{j\in A^c}\left(1-\frac{cy_j^{(n)}}{\pi}\right)\\
&=\left(\prod_{i=0}^{n}\left(1-\frac{cy_j^{(n)}}{\pi}\right)\right)\left(\sum_{A\in\mathcal{I}(k)}\prod_{j\in A}\frac{cy_j^{(n)}/\pi}{1-cy_j^{(n)}/\pi}\right), \quad k=0, 1, \ldots, n .
\end{aligned}
$$

Now it is easy to see

$$
\begin{aligned}
\log\prod_{i=0}^{n}\left(1-\frac{cy_i^{(n)}}{\pi}\right) &= \sum_{i=0}^{n}\left(-\frac{c}{\pi}y_i^{(n)}+O((y_i^{(n)})^2)\right)\\
&=-\frac{c}{\pi}+O(\max_{0\le j\le n} y_j^{(n)}) .
\end{aligned}
$$

On the other hand,

$$\sum_{A\in\mathfrak{F}(k)}\prod_{j\in A}\frac{cy_j^{(n)}/\pi}{1-cy_j^{(n)}/\pi}=\left(\frac{c}{k}\right)^k\left(\sum_{0\le j_1<\cdots<j_k\le n}\left(\prod_{p=1}^{k}y_{j_p}\right)\right)\left(1+O(\max_{0\le j\le n}y_j^{(n)})\right)$$

and

$$\sum_{0\le j_1<\cdots<j_k\le n}\prod_{p=1}^{k}y_{i_p}=\frac{1}{p!}\left(\sum_{j=0}^{n}y_j^{(n)}\right)^p+O(\max_{0\le j\le n}y_j^{(n)})$$

$$=\frac{1}{p!}+o(1)\,.$$

Combining these estimates, we are lead to

$$\lim_{n\to\infty}\pi_k(c;\,y_0^{(n)},\,\ldots,\,y_n^{(n)})=e^{-c/\pi}\frac{1}{k!}\left(\frac{c}{\pi}\right)^k,\quad k=0,\,1,\,\ldots$$

completing the proof of Theorem 2.

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