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# Elliptic Ruijsenaars-Schneider and Calogero-Moser Models Represented by Sklyanin Algebra and sl(n) Gaudin Algebra

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The relationship between Elliptic Ruijsenaars-Schneider (RS) and Calogero-Moser (CM) models with Sklyanin algebra is presented. Lax pair representations of the Elliptic RS and CM are reviewed. For n = 2 case, the eigenvalue and eigenfunction for Lamé equation are found by using the result of the Bethe ansatz method.

#### §1. Introduction

A general description of classical completely integrable models of n one-dimensional particles with two-body interactions  $V(q_i - q_j)$  was given in Ref. 1). To each simple Lie algora and choice of interaction, one can associate a classically completely integrable system <sup>1)-4</sup> such as a rational, hyperbolic, trigonometric or elliptic CM model.

The Lax pair representation (Lax representation) of a system is a direct method of showing its integrability, and the complete set of integrals of motion can also be constructed easily. The Lax representation and its corresponding *r*-matrix for rational, hyperbolic and trigonometric  $A_{n-1}$  CM models was constructed by Avan el al.<sup>2)</sup> The Lax representation for the elliptic CM models was constructed by Krichever<sup>5)</sup> and the corresponding *r*-matrix was given by Sklyanin<sup>6)</sup> and Braden et al.<sup>7)</sup> There exists a specific feature in that the *r*-matrices of the Lax representations for these models turn out to be dynamical (i.e., they depend on the dynamical variables) and satisfy dynamical Yang-Baxter equations.<sup>8), 7), 9), 6)</sup>

For the dynamical *r*-matrix, the fundamental Poisson algebra of the Lax operator, whose structural constants are given by a dynamical *r*-matrix, is generally no longer closed. The quantization problem and its geometrical interpretation are also difficult. Considering all of these, a non-dynamical *r*-matrix is found for these systems.<sup>10),11)</sup> The trigonometric limit of these results can be found in Ref. 42). We know the Lax representation for a completely integrable model is not unique. The different Lax representations of an integrable system are conjugate to each other (for the field system they are related by a 'gauge' transformation). The corresponding *r*-matrices are related by a 'gauge' transformation which is the classical dynamical twisting relation<sup>12</sup>) between those *r*-matrices.

The RS model is a relativistic generalization of a CM model. It describes a

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completely integrable system of n one-dimensional interacting relativistic particles. It can be related to the dynamics of solitons in some integrable relativistic field theory.<sup>8), 13), 9), 14)</sup> Its discrete-time version has been connected with the Bethe ansatz equation of the solvable statistical model.<sup>15)</sup> Recent developments have shown that it can be obtained by a Hamiltonian reduction of the cotangent bundle of some Lie group,<sup>16)</sup> and can be considered as the gauged WZW theory.<sup>17)</sup> The Lax representation and its corresponding r-matrix for rational, hyperbolic and trigonometric  $A_{n-1}$  type RS models was constructed by Avan et al.<sup>2)</sup> The Lax representation for the elliptic RS models was constructed by Reuijsenaars, <sup>18)</sup> and the corresponding r-matrix was given by Nijhoff<sup>15</sup>) and Suris.<sup>19</sup>) The main difference between the rmatrices of the relativistic (RS) and non-relativistic (CM) models is that the latter is given in terms of a linear Poisson-Lie bracket, whereas the former (RS model) is given in terms of a quadratic Poisson-Lie bracket. In contrast with the dynamical Yang-Baxter equation of the r-matrix for the CM model,  $^{16)}$  the generalized Yang-Baxter relation for the quadratic Poisson-Lie bracket (RS model) with a dynamical r-matrix is still an open problem.<sup>20)</sup> Moreover, the Poisson bracket of the Lax operator is no longer closed, and consequently the quantum version of the classical L-operator has not been constructed. However, as for the CM model, a different Lax representation which is conjugated to the original one can be found. The corresponding r-matrix changes by a 'gauge' transformation. The resulting r-matrix may be non-dynamical. Such a transformation may be called the classical dynamical twisting of the associated linear Poisson-Lie bracket. Due to the quadratic Poisson-Lie bracket of the RS model, there exist dynamical twisting relations between the r-matrices of Lax operators related by gauge transformations. Such dynamical twisting is the semi-classical limit of the quantum dynamical twisting of the *R*-matrix in Ref. 12). For recent progress in the study of CM models, see, for example, Refs. 21)-24).

The paper is organized as follows: In §2, we present some general formulae for dynamical systems. In §3, we review some results for the elliptic RS and CM models. The non-dynamical *r*-matrices for the integrable elliptic systems are then presented. Their quantization conditions correspond to the quantum Yang-Baxter relation, and the *R*-matrix is simply the  $Z_n$ -symmetric Belavin model.<sup>28)</sup> In §4, we will present the relationship between the Sklyanin algebra<sup>6), 32)</sup> and the integrable systems. In §5, we will obtain the eigenvalue and eigenfunction for the Lamé equation. The Lamé operator is equivalent to the Hamiltonian of the elliptic CM model. Section 6 has some brief summary.

### §2. The dynamical twisting of classical r-matrix

A Lax pair (L, M) consists of two functions on the phase space of the system with values in some Lie algebra g, such that the evolution equations may be written in the following form

$$\frac{dL}{dt} = [L, M],\tag{1}$$

where [,] denotes the bracket in the Lie algebra g. If we have formulated the Lax pair relation, the conserved quantities (integrals of motion) can be constructed easily. It follows that the adjoint-invariant quantities  $\operatorname{tr} L^l$   $(l = 1, \dots, n)$  are the integrals of the motion. In order to implement the Liouville theorem onto this set of possible action variables we need them to be Poisson-commuting. As shown in Ref. 25), the commutativity of the integrals  $\operatorname{tr} L^l$  follows if the Lax operator can be deduced from the fundamental Poisson bracket

$$\{L_1(u), L_2(v)\} = [r_{12}(u, v), L_1(u)] - [r_{21}(v, u), L_2(v)]$$
(2)

or quadratic form  $^{19)}$ 

$$\{L_1(u), L_2(v)\} = L_1(u)L_2(v)r_{12}^-(u, v) - r_{21}^+(v, u)L_1(u)L_2(v) + L_1(u)s^+(u, v)L_2(v) - L_2(v)s^-(u, v)L_1(u),$$
(3)

where we use the notation

$$L_1 = L \otimes 1, \quad L_2 = 1 \otimes L, \quad a_{21} = Pa_{12}P,$$
 (4)

and P is the permutation operator such that  $Px \otimes y = y \otimes x$ .

For the above relations to define a consistent Poisson bracket, one should impose some constraints on the r-matrices. The skew-symmetry of the Poisson bracket requires that

$$r_{21}^{\pm}(v,u) = -r_{12}^{\pm}(u,v), \quad s_{21}^{+}(v,u) = s_{12}^{-}(u,v), \quad (5)$$

$$r_{12}^+(u,v) - s_{12}^+(u,v) = r_{12}^-(u,v) - s_{12}^-(u,v).$$
(6)

For the numerical r-matrices  $r^{\pm}(u, v)$ ,  $s^{\pm}(u, v)$ , some constraint conditions (sufficient conditions) are imposed on the r-matrix in order to make it satisfy the Jacobi identity. <sup>26)</sup> However, generally speaking, the r-matrices  $r^{\pm}(u, v)$ ,  $s^{\pm}(u, v)$  depend on dynamical variables, and the Jacobi identity which implies an algebraic constraint for the r-matrices takes a very complicated form

$$[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cyc.perm} = 0.$$
(7)

It should be remarked that for a given integrable system, we can choose different Lax formulations. The *r*-matrices corresponding to different Lax formulations are generally different. So, in some cases, we can transform a dynamical *r*-matrix into a non-dynamical *r*-matrix.<sup>10),11)</sup> The different Lax representations of a system are conjugate to each other: if  $(\tilde{L}, \tilde{M})$  is another Lax pair of the same dynamical system conjugate to with the old one (L, M), it means that

$$\frac{d\tilde{L}}{dt} = [\tilde{L}, \tilde{M}], 
\tilde{L}(u) = g(u)L(u)g^{-1}(u), 
\tilde{M}(u) = g(u)M(u)g^{-1}(u) - \left(\frac{d}{dt}g(u)\right)g^{-1}(u),$$
(8)

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where  $g(u) \in G$  whose Lie algebra is g. Then we have **Proposition:** The Lax pair  $(\tilde{L}, \tilde{M})$  has the following *r*-matrix structure

$$\{\tilde{L}_1(u), \tilde{L}_2(v)\} = [\tilde{r}_{12}(u, v), \tilde{L}_1(u)] - [\tilde{r}_{21}(v, u), \tilde{L}_2(v)],$$
(9)

where

$$\tilde{r}_{12}(u,v) = g_1(u)g_2(v)r_{12}(u,v)g_1^{-1}(u)g_2^{-1}(v) + g_2(v)\{g_1(u), L_2(v)\}g_1^{-1}(u)g_2^{-1}(v) + \frac{1}{2}[\{g_1(u), g_2(v)\}g_1^{-1}(u)g_2^{-1}(v), g_2(v)L_2(v)g_2^{-1}(v)].$$
(10)

For a given Lax pair (L, M) and the corresponding *r*-matrix, if there exists a *g* such that

$$h_{12} = \{g_1(u)g_2(v)r_{12}(u,v)g_1^{-1}(u)g_2^{-1}(v) + g_2(v)\{g_1(u), L_2(v)\}g_1^{-1}(u)g_2^{-1}(v) + \frac{1}{2}[\{g_1(u), g_2(v)\}g_1^{-1}(u)g_2^{-1}(v), g_2(v)L_2(v)g_2^{-1}(v)]$$
(11)

and

$$\partial_{q_i} h_{12} = \partial_{p_j} h_{12} = 0 \tag{12}$$

then a non-dynamical Lax representation of the system exists.

By a straightforward calculation, we can also find that the twisted Lax pair  $(\tilde{L}, \tilde{M})$  and the corresponding *r*-matrix  $\tilde{r}_{12}$  satisfy

$$[\tilde{L}_1, [\tilde{r}_{12}, \tilde{r}_{13}] + [\tilde{r}_{12}, \tilde{r}_{23}] + [\tilde{r}_{32}, \tilde{r}_{13}] + \{\tilde{L}_2, \tilde{r}_{13}\} - \{\tilde{L}_3, \tilde{r}_{12}\}] + \text{cycl.perm} = 0.$$
(13)

Similarly, for the quadratic form, the Lax pair  $(\tilde{L}, \tilde{M})$  has the following *r*-matrix structure

$$\{\tilde{L}_{1}(u), \tilde{L}_{2}(v)\} = \tilde{L}_{1}(u)\tilde{L}_{2}(v)\tilde{r}_{12}^{-}(u,v) - \tilde{r}_{12}^{+}(u,v)\tilde{L}_{1}(u)\tilde{L}_{2}(v) + \tilde{L}_{1}(u)\tilde{s}_{12}^{+}(u,v)\tilde{L}_{2}(v) - \tilde{L}_{2}(v)\tilde{s}_{12}^{-}(u,v)\tilde{L}_{1}(u), \qquad (14)$$

where

$$\begin{split} \tilde{r}_{12}^{-}(u,v) &= g_1(u)g_2(v)r_{12}^{-}(u,v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}(u,v) + \tilde{\Delta}_{21}(v,u), \\ \tilde{r}_{12}^{+}(u,v) &= g_1(u)g_2(v)r_{12}^{+}(u,v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}^{(1)}(u,v) + \tilde{\Delta}_{21}^{(1)}(v,u), \\ \tilde{s}_{12}^{+}(u,v) &= g_1(u)g_2(v)s_{12}^{+}(u,v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{21}(v,u) - \tilde{\Delta}_{12}^{(1)}(u,v), \\ \tilde{s}_{12}^{-}(u,v) &= g_1(u)g_2(v)s_{12}^{-}(u,v)g_1^{-1}(u)g_2^{-1}(v) - \tilde{\Delta}_{12}(u,v) - \tilde{\Delta}_{21}^{(1)}(v,u), \\ \tilde{\Delta}_{12}(u,v) &= \tilde{L}_2^{-1}(v)\Delta_{12}(u,v), \quad \tilde{\Delta}_{12}^{(1)}(u,v) = \Delta_{12}(u,v)\tilde{L}_2^{-1}(v), \\ \Delta_{12}(u,v) &= g_2(v)\{g_1(u),L_2(v)\}g_1^{-1}(u)g_2^{-1}(v) \\ &\quad + \frac{1}{2}[\{g_1(u),g_2(v)\}g_1^{-1}(u)g_2^{-1}(v),g_2(v)L_2(v)g_2^{-1}(v)] \end{split}$$
(15)

There are still relations:

$$\tilde{r}_{21}^{\pm}(v,u) = -\tilde{r}_{12}^{\pm}(u,v), \qquad \tilde{s}_{21}^{+}(v,u) = \tilde{s}_{12}^{-}(u,v), \tilde{r}_{12}^{+}(u,v) - \tilde{s}_{12}^{+}(u,v) = \tilde{r}_{12}^{-}(u,v) - \tilde{s}_{12}^{-}(u,v).$$
(16)

And also we have, for given Lax pair (L, M) and the corresponding *r*-matrices, if there exists a g such that

$$g_{1}(u)g_{2}(v)s_{12}^{+}(u,v)g_{1}^{-1}(u)g_{2}^{-1}(v) - \tilde{\Delta}_{21}(v,u) - \tilde{\Delta}_{12}^{(1)}(u,v) = 0,$$
  

$$g_{1}(u)g_{2}(v)s_{12}^{-}(u,v)g_{1}^{-1}(u)g_{2}^{-1}(v) - \tilde{\Delta}_{12}(u,v) - \tilde{\Delta}_{21}^{(1)}(v,u) = 0,$$
  

$$h_{12}(u,v) = g_{1}(u)g_{2}(v)r_{12}^{-}(u,v)g_{1}^{-1}(u)g_{2}^{-1}(v) - \tilde{\Delta}_{12}(u,v) + \tilde{\Delta}_{21}(v,u)$$
  

$$= g_{1}(u)g_{2}(v)r_{12}^{+}(u,v)g_{1}^{-1}(u)g_{2}^{-1}(v) - \tilde{\Delta}_{12}^{(1)}(u,v) + \tilde{\Delta}_{21}^{(1)}(v,u)$$
(17)

and

$$\partial_{q_i} h_{12} = \partial_{p_j} h_{12} = 0, \tag{18}$$

then a non-dynamical Lax representation with Sklyanin Poisson-Lie bracket for the system exists.

#### §3. Lax pair for elliptic RS and CM models

We first define some elliptic functions:

$$\theta^{(j)}(u) = \theta \begin{bmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{bmatrix} (u, n\tau), \quad \sigma(u) = \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (u, \tau),$$
$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (u, \tau) = \sum_{n=-\infty}^{\infty} \exp\{i\pi[(m+a)^{2}\tau + 2(m+a)(z+b)]\},$$
$$\theta^{\prime(j)}(u) = \partial_{u}\{\theta^{(j)}(u)\}, \sigma^{\prime}(u) = \partial_{u}\{\sigma(u)\}, \xi(u) = \partial_{u}\{\ln\sigma(u)\}, \quad (19)$$

where  $\tau$  is a complex number with  $\text{Im}(\tau) > 0$ .

The Ruijsenaars-Schneider model is a system of n one-dimensional relativistical particles interacting by a two-body potential. In terms of the canonical variables  $p_i, q_i \ (i = 1, \dots, n)$  enjoying the canonical Poisson bracket

$$\{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \tag{20}$$

the Hamiltonian of the system is expressed as  $^{18)}$ 

$$H = mc^2 \sum_{j=1}^{n} \cosh\left(p_j \prod_{k \neq j} \left\{ \frac{\sigma(q_{jk} + \gamma)\sigma(q_{jk} - \gamma)}{\sigma^2(q_{jk})} \right\}^{\frac{1}{2}} \right), \tag{21}$$

where  $q_{jk} = q_j - q_k$ , *m* denotes the particle mass, *c* the speed of light, and  $\gamma$  is the coupling constant. The above defined Hamiltonian is known to be completely

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integrable.  $^{18), 27)}$  As we mentioned above, the Lax representation (Lax operator of the classical *L*-operator) is one of the most effective ways to show that the system is integrable. One Lax representation for the elliptic RS model was first formulated by Ruijsenaars:  $^{18)}$ 

$$L_R(u)_j^i = \frac{e^{p_j}\sigma(q_{ji}+u+\gamma)}{\sigma(\gamma+q_{ji})\sigma(u)} \prod_{k\neq j}^n \left\{ \frac{\sigma(q_{jk}+\gamma)\sigma(q_{jk}-\gamma)}{\sigma^2(q_{jk})} \right\}^{\frac{1}{2}}, \quad i,j = 1, \cdots, n.$$
(22)

Here, we use another Lax representation  $\tilde{L}_R^{(20)}$ 

$$\tilde{L}_R(u)_j^i = \frac{e^{p_j}\sigma(u+q_{ji}+\gamma)}{\sigma(u)\sigma(q_{ji}+\gamma)} \prod_{k\neq j} \frac{\sigma(\gamma+q_{jk})}{\sigma(q_{jk})}.$$
(23)

 $\tilde{L}_R$  can be obtained from the standard Ruijsenaars'  $L_R(u)$  by using a Poisson map

$$q_i \longrightarrow q_i, \quad p_i \longrightarrow p_i + \frac{1}{2} \ln \prod_{k \neq i} \frac{\sigma(q_{ik} + \gamma)}{\sigma(q_{ik} - \gamma)}.$$
 (24)

Following the work of Nijhoff et al.,<sup>20)</sup> the fundamental Poisson bracket of the Lax operator  $\tilde{L}_R(u)$  can be given in the following quadratic *r*-matrix form with dynamical *r*-matrices

$$\{\tilde{L}_{R}(u)_{1}, \tilde{L}_{R}(v)_{2}\} = \tilde{L}_{R}(u)_{1}\tilde{L}_{R}(v)_{2}r_{12}^{-}(u,v) - r_{12}^{+}(u,v)\tilde{L}_{R}(u)_{1}\tilde{L}_{R}(v)_{2} + \tilde{L}_{R}(u)_{1}s_{12}^{+}(u,v)\tilde{L}_{R}(v)_{2} - \tilde{L}_{R}(v)_{2}s_{12}^{-}(u,v)\tilde{L}_{R}(u)_{1}, (25)$$

where

$$r_{12}^{-}(u,v) = a_{12}(u,v) - s_{12}(u) + s_{21}(v), \quad r_{12}^{+}(u,v) = a_{12}(u,v) + u_{12}^{+} + u_{12}^{-}, s_{12}^{+}(u,v) = s_{12}(u) + u_{12}^{+}, \quad s_{12}^{-}(u,v) = s_{21}(v) - u_{12}^{-}$$
(26)

and

$$u_{12}^{\pm} = \sum_{ij} \xi(q_{ji} \pm \gamma) e_{ii} \otimes e_{jj},$$
  

$$a_{12}(u,v) = r_{12}^{0}(u,v) + \sum_{i=1} \xi(u-v) e_{ii} \otimes e_{ii} + \sum_{i \neq j} \xi(q_{ij}) e_{ii} \otimes e_{jj},$$
  

$$r_{12}^{0}(u,v) = \sum_{i \neq j} \frac{\sigma(q_{ij} + u - v)}{\sigma(q_{ij})\sigma(u-v)} e_{ij} \otimes e_{ji}, \quad s_{12}(u) = \sum_{i,j} \left(\tilde{L}_{R}(u)\partial_{\gamma}\tilde{L}_{R}(u)\right)_{j}^{i} e_{ij} \otimes e_{jj}.$$
(27)

The following properties are satisfied:

$$r_{21}^{\pm}(v,u) = -r_{12}^{\pm}(u,v), \quad s_{21}^{+}(v,u) = s_{12}^{-}(u,v), r_{12}^{+}(u,v) - s_{12}^{+}(u,v) = r_{12}^{-}(u,v) - s_{12}^{-}(u,v).$$
(28)

Here we would like to reformulate the Lax formulation for the RS model. Define an  $n \otimes n$  matrix A(u;q) as:

$$A(u;q)_{j}^{i} \equiv A(u,q_{1},\cdots,q_{n})_{j}^{i} = \theta^{(i)} \left( u + nq_{j} - \sum_{k=1}^{n} q_{k} + \frac{n-1}{2} \right).$$
(29)

Here we should point out that  $A(u;q)_j^i$  corresponds to the intertwiner function  $\phi_j^{(i)}$ between the  $Z_n$ -symmetric Belavin *R*-matrix<sup>28), 29)</sup> and the  $A_{n-1}^{(1)}$  face model.<sup>30), 31)</sup>

Define

$$g(u) = A(u;q)\Lambda(q), \quad \Lambda(q)_j^i = h_i(q)\delta_j^i,$$
  
$$h_i(q) \equiv h_i(q_1,\cdots,q_n) = \frac{1}{\prod_{l \neq i} \sigma(q_{il})}.$$
 (30)

We can construct the new Lax operator L(u) as

$$L(u) = g(u)\tilde{L}_R(u)g^{-1}(u).$$
(31)

More explicitly, it can be written as:

$$L(u)_{j}^{i} = \sum_{k=1}^{n} \frac{1}{\sigma(\gamma)} A(u+n\gamma;q)_{k}^{i} A^{-1}(u;q)_{j}^{k} e^{p_{k}}, \quad i,j = 1, 2, \cdots, n.$$
(32)

It can be proved that the fundamental Poisson bracket of L(u) can be given in the quadratic Poisson-Lie form with a nondynamical *r*-matrix:

$$\{L_1(u), L_2(v)\} = [r_{12}(u-v), L_1(u)L_2(v)].$$
(33)

Here the numerical r-matrix is the classical  $Z_n$ -symmetric r-matrix.<sup>32)</sup> It takes the form

$$\begin{cases} r_{ij}^{lk}(v) = \\ \left\{ \begin{array}{c} (1 - \delta_i^l) \frac{\theta'^{(0)}(0)\theta^{(i-j)}(v)}{\theta^{(l-j)}(v)\theta^{(i-l)}(0)} + \delta_i^l \delta_j^k \left( \frac{\theta'^{(i-j)}(v)}{\theta^{(i-j)}}(v) - \frac{\sigma'(v)}{\sigma(v)} \right) & \text{if } i+j = l+l \mod n \\ 0 & \text{otherwise} \end{cases} \end{cases}$$
(34)

We know the  $Z_n$  symmetric *r*-matrix satisfies the nondynamical classical Yang-Baxter equation

$$[r_{12}(v_1 - v_2), r_{13}(v_1 - v_3)] + [r_{12}(v_1 - v_2), r_{23}(v_2 - v_3)] + [r_{13}(v_1 - v_3), r_{23}(v_2 - v_3)] = 0.$$
(35)

We also know that this r-matrix is antisymmetric and  $Z_n$  symmetric:

Antisymmetry : 
$$-r_{21}(-v) = r_{12}(v),$$
  
 $Z_n \otimes Z_n$ Symmetry :  $r_{12}(v) = (a \otimes a)r_{12}(v)(a \otimes a)^{-1},$  (36)

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where a = g, h, and g, h are  $n \otimes n$  matrices defined as:

$$h_{ij} = \delta_{i+1,j \bmod n}, \quad g_{ij} = \omega^i \delta_{i,j}. \tag{37}$$

For convenience, we also define another  $n \otimes n$  matrix

$$I_{\alpha} \equiv I_{\alpha_1,\alpha_2} \equiv g^{\alpha_2} h^{\alpha_1}, \tag{38}$$

where  $\alpha_1, \alpha_2 \in Z_n$  and  $\omega = \exp(2\pi i/n)$ .

Next, we will consider the non-relativistic limit of the Lax operator L(u). First rescale the momenta  $\{p_i\}$ , the coupling constant  $\gamma$  and the Lax operator as follows: <sup>20)</sup>

$$p_i := -\beta p'_i, \quad n\gamma := \beta s, \quad L(u) := \sigma\left(\frac{\beta s}{n}\right) L'(u).$$
 (39)

Here notation L' is introduced. The non-relativistic limit is then obtained by taking the limit  $\beta \to 0$ . We have the following asymptotic properties

$$L'(u)_{j}^{i} = \delta_{j}^{i} - \beta \left( \sum_{k} \{A(u;q)_{k}^{i} A(u;q)_{j}^{k} p_{k}' - s \partial_{s} (A(u;q)_{k}^{i}) A^{-1}(u;q)_{j}^{k} \} \right) + O(\beta^{2}).$$

$$(40)$$

If we make the canonical transformation

$$p'_i \to p'_i - \frac{s}{n} \frac{\partial}{\partial q_i} \ln M(q), \quad M(q) = \prod_{i < j} \sigma(q_{ij}),$$
(41)

we finally obtain the Lax operator of the elliptic  $A_{n-1}$  CM model<sup>10</sup>

$$L_{\rm CM}(u)_j^i = -\lim_{\beta \to 0} \left. \frac{L'(u)_j^i - \delta_j^i}{\beta} \right|_{p_i' \to p_i' - \frac{s}{n} \frac{\partial}{\partial q_i} \ln M(q)}.$$
(42)

Here we have

$$\{L_{\rm CM}(u)_1, L_{\rm CM}(v)_2\} = [r_{12}(u-v), L_{\rm CM}(u)_1 + L_{\rm CM}(v)_2].$$
(43)

For the newly constructed Lax representation L(u), the quantization becomes no longer difficult. Define the  $Z_n$ -symmetric Belavin's *R*-matrix as:

$$R_{ij}^{lk}(u) = \begin{cases} \frac{\theta'^{(0)}(0)\sigma(u)\sigma(\sqrt{-1}\hbar)}{\sigma'(0)\theta^{(0)}(v)\sigma(v+\sqrt{-1}\hbar)} \frac{\theta^{(0)}(v)\theta^{(i-j)}(v+\sqrt{-1}\hbar)}{\theta^{(i-l)}(\sqrt{-1}\hbar)\theta^{(l-j)}(v)} & \text{if } i+j=l+k \mod n\\ 0 & \text{otherwise.} \end{cases}$$

$$(44)$$

We know this *R*-matrix satisfies the quantum Yang-Baxter equation

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2).$$
(45)

The *R*-matrix is  $Z_n$  symmetric in the sense that

$$R_{12}(u) = (a \otimes a)R_{12}(u)(a \otimes a)^{-1}, \quad a = g, h.$$
(46)

By taking the special limit  $\hbar \to 0$ , we can obtain the classical  $Z_n$  symmetric *r*-matrix

$$R_{12}(u)|_{\hbar \to 0} = 1 \otimes 1 + \sqrt{-1}\hbar r_{12}(u) + o(\hbar^2).$$
(47)

Now let us study the quantum L operator, using the usual canonical quantization procedure

$$p_j \to \hat{p}_j = -\sqrt{-1}\hbar \frac{\partial}{\partial q_j}, \quad q_j \to q_j, \quad j = 1, \cdots, n.$$
 (48)

The corresponding quantum *L*-operator can be formulated as:

$$\hat{L}(u)_{l}^{m} = \frac{1}{\sigma(\gamma)} \sum_{k=1}^{n} A(u+n\gamma;q)_{k}^{m} A^{-1}(u;q)_{l}^{k} e^{\hat{p}_{k}}$$
$$= \frac{1}{\sigma(\gamma)} \sum_{k=1}^{n} A(u+n\gamma;q)_{k}^{m} A^{-1}(u;q)_{l}^{k} e^{-\sqrt{-1}\hbar \frac{\partial}{\partial q_{k}}}.$$
(49)

It should be remarked that this quantum L-operator is simply the factorised difference representation for the elliptic L-operator.<sup>31), 33)</sup> The above defined quantum L-operator satisfies the quantum Yang-Baxter relation

$$R_{12}(u-v)\hat{L}_1(u)\hat{L}_2(v) = \hat{L}_2(v)\hat{L}_1(u)R_{12}(u-v).$$
(50)

The proof can be found in Refs. 31, 34, 33) and 35).

## §4. RS and CM models related with Sklyanin algebra

We introduce here some notation for elliptic functions:

$$\sigma_{\alpha}(u) = \theta \begin{bmatrix} \frac{1}{2} + \frac{\alpha_{1}}{n} \\ \frac{1}{2} + \frac{\alpha_{2}}{n} \end{bmatrix} (u, \tau),$$

$$W_{\alpha}(u) = \frac{\sigma_{\alpha}(u + \sqrt{-1}\hbar)}{\sigma_{\alpha}(\sqrt{-1}\hbar)} \frac{\sigma_{0}(\sqrt{-1}\hbar)}{\sigma_{0}(u + \sqrt{-1}\hbar)}.$$
(51)

The above mentioned quantum *R*-matrix can be rewritten as following up to a scale:

$$R(u) = \sum_{\alpha} W_{\alpha}(u) I_{\alpha} \otimes I_{\alpha}^{-1}, \qquad (52)$$

as before  $\alpha \equiv \alpha_1, \alpha_2$  and  $\alpha_i \in Z_n, i = 1, 2$ .

The quantum *L*-operator  $\hat{L}$  obtained in the last section can be represented by the generators of Sklyanin algebra  $S_{\alpha}$ :

$$\hat{L}(u) = \sum_{\alpha} V_{\alpha}(u) I_{\alpha} S_{\alpha}, \qquad (53)$$

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where

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$$V_{\alpha}(u) = \frac{\sigma_{\alpha}(u' + \sqrt{-1}\hbar\xi)}{n\sigma_0(u')\sigma_{\alpha}(\sqrt{-1}\hbar)}, \quad u' = u + n\sqrt{-1}\hbar\delta - \frac{n-1}{2}, \tag{54}$$

where  $\delta$  is a constant.

The quantum Yang-Baxter relation (49) gives the defining relations of the Sklyanin algebra:  $^{32), 6)}$ 

$$\sum_{\gamma} \frac{\omega^{\gamma_1 - \alpha_1 + (\beta_1 - \gamma_1)(\gamma_1 - \alpha_2)} \sigma_{\alpha + \beta - 2\gamma}(0) \sigma_{\beta}(2\sqrt{-1}\hbar)}{\sigma_{\gamma - \alpha}(\sqrt{-1}\hbar) \sigma_{\alpha + \beta - \gamma}(\sqrt{-1}\hbar) \sigma_{\gamma}(\sqrt{-1}\hbar) \sigma_{\beta - \gamma}(\sqrt{-1}\hbar)} S_{\alpha + \beta - \gamma} S_{\gamma} = 0, (55)$$

with  $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_n, i = 1, 2.$ 

We can give a realization of the generators of Sklyanin algebra as:

$$S_{\alpha} = \sum_{j} S_{j\alpha} e^{-n\sqrt{-1}\hbar \frac{\partial}{\partial q_{j}}}.$$
(56)

Here we introduce the symbol  $S_{j\alpha}$  for the elliptic function

$$S_{j\alpha} = (-1)^{\alpha_1} \sigma_\alpha(\sqrt{-1}\hbar) \prod_{k \neq j} \frac{\sigma_\alpha(\sqrt{-1}\hbar\xi + q_{jk})}{\sigma_0(q_{jk})}.$$
 (57)

Next, we will consider the classical limit of the above defining relations. Letting  $\hbar \to 0$ , the quantum *R*-matrix become the classical *r*-matrix, and we explicitly have the elements of the *r*-matrix in (34) presented in the last section, here we use another notation

$$R(u) = 1 + \sqrt{-1}\hbar r(u) + O(\hbar^2).$$
(58)

The classical r-matrix is written as:

$$r(u) = \sum_{\alpha} w_{\alpha}(u) I_{\alpha} \otimes I_{\alpha}^{-1},$$
(59)

where

$$w_0(u) = 0,$$
  

$$w_{\alpha}(u) = \frac{\sigma_{\alpha}(u)\sigma_0'(0)}{\sigma_{\alpha}(0)\sigma_0(u)}, \quad \alpha \neq 0.$$
(60)

In order to consider the classical limit of  $\hat{L}$ , we first present the classical limit of  $V_{\alpha}(u)$ :

$$V_0(u) = \frac{\sigma_0(u') + \sqrt{-1}\hbar\xi\sigma'_0(u') + O(\hbar^2)}{n\sigma_0(u')\sigma_0(\sqrt{-1}\hbar)},$$
(61)

$$V_{\alpha}(u) = \frac{\sigma_{\alpha}(u')}{n\sigma_{\alpha}(0)\sigma_{0}(u')} + \frac{\sqrt{-1}\hbar}{n\sigma_{0}(u')} \left[ \frac{\xi \sigma_{\alpha}'(u')\sigma_{\alpha}(0) - \sigma_{\alpha}(u')\sigma_{\alpha}'(0)}{\sigma_{\alpha}^{2}(0)} \right] + O(\hbar^{2}),$$
  

$$\alpha \neq 0.$$
(62)

From the definition of the operator  $S_{\alpha}$ , we easily have

$$S_{\alpha} = \sum_{j} S_{j\alpha} \left( 1 - n\sqrt{-1}\hbar \frac{\partial}{\partial q_{j}} + O(\hbar^{2}) \right).$$
(63)

In the limit  $\hbar \to 0$ , the elliptic functions  $S_{j\alpha}$  take the forms:

$$S_{j0} = \sigma(\sqrt{-1}\hbar) \left[ 1 + \sqrt{-1}\hbar\xi \sum_{k \neq j} \frac{\sigma'_0(q_{jk})}{\sigma_0(q_{jk})} + O(\hbar^2) \right],$$
(64)

$$S_{j\alpha} = (-1)^{\alpha_1} \sigma_{\alpha}(0) \prod_{k \neq j} \frac{\sigma_{\alpha}(q_{jk})}{\sigma_0(q_{jk})} \left[ 1 + \sqrt{-1}\hbar \left( \frac{\sigma_{\alpha}'(0)}{\sigma_{\alpha}(0)} + \xi \sum_{k \neq j} \frac{\sigma_{\alpha}'(q_{jk})}{\sigma_{\alpha}(q_{jk})} \right) + O(\hbar^2) \right],$$
  
$$\alpha \neq 0.$$
(65)

So, we have

$$V_{0}(u)S_{0} = 1 + \sqrt{-1}\hbar \left[\xi \frac{\sigma_{0}'(u')}{\sigma_{0}(u')}\right] + \frac{\sqrt{-1}\hbar}{n} \sum_{j} \left[\xi \sum_{k \neq j} \frac{\sigma_{0}'(q_{jk})}{\sigma_{0}(q_{jk})} - n \frac{\partial}{\partial q_{j}}\right] + O(\hbar^{2}),$$

$$= 1 + \sqrt{-1}\hbar \left[\xi \frac{\sigma_{0}'(u')}{\sigma_{0}(u')}\right] + \frac{\sqrt{-1}\hbar}{n} \sum_{j} \left[-n \frac{\partial}{\partial q_{j}}\right] + O(\hbar^{2}), \quad (66)$$

$$V_{\alpha}(u)S_{\alpha} = \sqrt{-1}\hbar(-1)^{\alpha_1}\frac{\sigma_{\alpha}(u')}{n\sigma_0(u')}\sum_{j}\prod_{k\neq j}\frac{\sigma_{\alpha}(q_{jk})}{\sigma_0(q_{jk})}\left[\xi\sum_{k\neq j}\frac{\sigma_{\alpha}'(q_{jk})}{\sigma_{\alpha}(q_{jk})} - n\frac{\partial}{\partial q_j}\right] + O(\hbar^2),$$
  
$$\alpha \neq 0.$$
(67)

here we have  $\sum_{j} \sum_{k \neq j} \frac{\sigma'_0(q_{jk})}{\sigma_0(q_{jk})} = 0$ , because  $\frac{\sigma'_0(q_{jk})}{\sigma_0(q_{jk})}$  is an odd function.

We can finally expand the quantum  $\hat{L}$  operator in the order of  $\hbar$  when we take a limit  $\hbar \to 0$ . However, we first introduce some notation

$$\hat{L}(u) = \sum_{\alpha} V_{\alpha}(u) S_{\alpha} I_{\alpha} = 1 + \sqrt{-1}\hbar l(u) + O(\hbar^2),$$
(68)

where l(u) is the classical l operator. We may represent l(u) in terms of generators of the "classical" Sklyanin algebra  $s_{\alpha}$ :

$$l(u) = \frac{\xi}{n} \frac{\sigma'_0(u')}{\sigma_0(u')} - \sum_{\alpha} v_{\alpha}(u) s_{\alpha}.$$
 (69)

The function  $v_{\alpha}(u)$  is defined as:

$$v_0(u) = \frac{1}{n},\tag{70}$$

$$v_{\alpha}(u) = \frac{1}{n} \frac{\sigma_{\alpha}(u')\sigma_{\alpha}(0)}{\sigma_{0}(u')}, \quad \alpha \neq 0.$$
(71)

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From the above obtained results, we can realize the generators of the "classical" Sklyanin algebra in the following forms:

$$s_0 = \sum_j n \frac{\partial}{\partial q_j},\tag{72}$$

$$s_{\alpha} = (-1)^{\alpha_1} \sigma_{\alpha}(0) \sum_{j} \prod_{k \neq j} \frac{\sigma_{\alpha}(q_{jk})}{\sigma_0(q_{jk})} \left[ n \frac{\partial}{\partial q_j} - \xi \sum_{k \neq j} \frac{\sigma_{\alpha}'(q_{jk})}{\sigma_{\alpha}(q_{jk})} \right], \quad \alpha \neq 0.$$
(73)

On the other hand, here we can say we give a definition of the generators of the "classical" Sklyanin algebra.

In the classical limit, the quantum Yang-Baxter relation becomes the following

$$[l_1(u), l_2(v)] = [r_{12}(u-v), l_1(u) + l_2(v)].$$
(74)

Substitute the *l*-operator with the generators of the "classical" Sklyanin algebra, and through tedious calculation, we have

$$[s_{\alpha}, s_{\gamma}] = \left(\omega^{\alpha_1 \gamma_2} - \omega^{\alpha_2 \gamma_1}\right) \left(\frac{\sigma'_0(0)}{n}\right) s_{\alpha+\gamma}.$$
(75)

On the other hand, we find that  $\{I_{\alpha}\}$  satisfy a similar relation

$$[I_{\alpha}, I_{\gamma}] = (\omega^{\alpha_1 \gamma_2} - \omega^{\alpha_2 \gamma_1}) I_{\alpha + \gamma}.$$
(76)

So, after rescaling  $s_{\alpha}$ , we find  $\{s_{\alpha}\}$  and  $\{I_{\alpha}\}$  satisfy the same algebra.

Finally we should point out that if we substitute  $\frac{\partial}{\partial q_k}$  by the corresponding canonical variable  $p_k$ , the *l*-operator will become a *T*-operator, and the commutative bracket on the left-hand side of the above relation (74) will change to the standard Poisson-Lie bracket. Here we rewrite as:

$$\{T_1(u), T_2(v)\} = [r_{12}(u-v), T_1(u) + T_2(v)].$$
(77)

#### §5. CM model, Gaudin model, Lamé-equation and the Bethe ansatz

For the difference factorized operator  $\hat{L}$ , we can find some commuting families which are related to conserved operators. By using the fusion procedure, the commuting family take the form

$$D_m = \operatorname{tr}[\hat{L}(u) \otimes \cdots \otimes \hat{L}(u)P_-^m],$$

there are  $m \hat{L}$ 's above,  $P_{-}^m$  is the anti-symmetric projector. In the classical limit, we also have a similar commuting family

$$a_m(u) = \sum_{\alpha_i \neq 0} v_{\alpha_1(u)} \cdots v_{\alpha_m}(u) s_{\alpha_1} \cdots s_{\alpha_m} \operatorname{tr}[I_{\alpha_1} \otimes \cdots I_{\alpha_m} P_-^m], \qquad (78)$$

where  $\alpha_i \in Z_n^2$ ,  $i = 1, \dots, m$ . Let u' = 0, so  $u = u_0 = \frac{n-1}{2} - n\hbar\xi$ , and after rescaling  $a_l(u)$ , we have

$$a_m(u_0) = \sum_{\alpha_i \neq 0} s_{\alpha_1} \cdots s_{\alpha_m} \operatorname{tr}[I_{\alpha_1} \otimes \cdots \otimes I_{\alpha_m} P_-^m].$$
(79)

We will discuss a special case n = 2, explicitly we have

$$s_{01} = 2\sigma_{01}(0) \left\{ \xi \frac{\sigma'_{01}(q_{12})}{\sigma_0(q_{12})} - \frac{\sigma_{01}(q_{12})}{\sigma_0(q_{12})} \left( \frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2} \right) \right\},\$$

$$s_{10} = 2\sigma_{10}(0) \left\{ \xi \frac{\sigma'_{10}(q_{12})}{\sigma_0(q_{12})} - \frac{\sigma_{10}(q_{12})}{\sigma_0(q_{12})} \left( \frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2} \right) \right\},\$$

$$s_{11} = 2\sigma_{11}(0) \left\{ \xi \frac{\sigma'_{11}(q_{12})}{\sigma_0(q_{12})} - \frac{\sigma_{11}(q_{12})}{\sigma_0(q_{12})} \left( \frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2} \right) \right\}.$$
(80)

We will calculate the non-trivial conserved operator

$$4a_2(u) = \sum_{\alpha \neq 0} \frac{\sigma_\alpha(u')\sigma_{-\alpha}(u')}{\sigma_\alpha(0)\sigma_{-\alpha}(0)} s_\alpha s_{-\alpha} \omega^{-\alpha_1 \alpha_2}.$$
(81)

After some tedious calculations, we have

$$a_{2} = -\xi^{2} \frac{\sigma_{0}''(q)}{\sigma_{0}(q)} + 2\xi \frac{\sigma_{0}'(q)}{\sigma_{0}(q)} \left(\frac{\partial}{\partial q_{1}} - \frac{\partial}{\partial q_{2}}\right) + 4 \frac{\partial^{2}}{\partial q_{1} \partial q_{2}} - \left(\frac{\partial}{\partial q_{1}} + \frac{\partial}{\partial q_{2}}\right)^{2} - (\xi^{2} + 2\xi) \left[\frac{\sigma_{0}'(u')^{2}}{\sigma_{0}(u')^{2}} - \frac{\sigma_{0}''(u')}{\sigma_{0}(u')}\right],$$
(82)

where  $q = q_1 - q_2$ . This relation is just the same as that obtained by Hasegawa.<sup>36)</sup> Since 1 and  $\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}$  are also conserved quantities defined above, after some tedious calculations we have another conserved operator

$$H = \frac{\partial^2}{\partial q^2} - \xi \frac{\sigma_0'(q)}{\sigma_0(q)} \frac{\partial}{\partial q} + \frac{\xi^2}{4} \frac{\sigma_0''(q)}{\sigma_0(q)}.$$
(83)

We can change it to a more familiar form. Let  $\xi = 2\beta$ , and suppose  $\psi$  and  $\Lambda$  are an eigenfunction and eigenvalue of the above Hamiltonian

$$H\psi = \Lambda\psi. \tag{84}$$

At the same time, we introduce a transformation of this eigenfunction  $\psi = \tilde{\psi}[\sigma_0(q)]^{\beta}$ , we thus have the following relations:

$$H\tilde{\psi}[\sigma_0(q)]^{\beta} = \left[\frac{d^2}{dq^2} - 2\beta \frac{\sigma_0'(q)}{\sigma_0(q)} \frac{d}{dq} + \beta^2 \frac{\sigma_0''(q)}{\sigma_0(q)}\right] \tilde{\psi}[\sigma_0^{\beta}(q)]^{\beta} = \Lambda \tilde{\psi}[\sigma_0^{\beta}(q)]^{\beta}.$$
(85)

This means:

$$H'\tilde{\psi} \equiv \left[\frac{d^2}{dq^2} + \beta(\beta+1)\frac{d^2}{dq^2}\ln\sigma_0(q)\right]\tilde{\psi} = \Lambda\tilde{\psi}.$$
(86)

One finds that H' is simply the Hamiltonian of the CM model, see, for example, Refs. 3),4),37) and 38). It is also connected to the Lamé operator, see Ref. 39) and the references therein.

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Next, we will calculate the eigenfunction and eigenvalue of the above defined Lamé operator H. Here we first review some of the results obtained by Felder and Varchenko.<sup>39)</sup> The difference operator L which is equivalent to  $S_0$ , one generator of the Sklyanin algebra when n = 2, is given by

$$L\psi(q) = \frac{\sigma_0(q+2\hbar\beta)}{\sigma_0(q)}\psi(q-2\hbar) + \frac{\sigma_0(q-2\hbar\beta)}{\sigma_0(q)}\psi(q+2\hbar).$$
(87)

This difference operator is also called the q-deformed Lamé operator. In the framework of the quantum inverse scattering method, there is a result as follows:<sup>39)</sup>

Let  $(t_1, \dots, t_m, c)$  be a solution of the Bethe ansatz equations:

$$\frac{\sigma_0(t_i - 2\hbar\beta)}{\sigma_0(t_i + 2\hbar2\beta)} \prod_{j \neq i} \frac{\sigma_0(t_j - t_i - 2\hbar)}{\sigma_0(t_j - t_i + 2\hbar)} = e^{4\hbar c}, \quad i = 1, \cdots, \beta,$$
(88)

such that  $t_i \neq t_j \mod Z + \tau Z$  if  $i \neq j$ . Then

$$\psi(q) = e^{cq} \prod_{j}^{\beta} \sigma_0(q+t_j), \tag{89}$$

is a solution of the q-deformed Lamé equation  $L\psi = \epsilon\psi$  with eigenvalue

$$\epsilon = e^{-2\hbar c} \frac{\sigma_0(4\hbar\beta)}{\sigma_0(2\hbar\beta)} \prod_{j=1}^{\beta} \frac{\sigma_0(t_j + (2\beta - 2)\hbar)}{\sigma_0(t_j + 2\beta\hbar)}.$$
(90)

By taking the special limit  $\hbar \to 0$ , the difference operator becomes:

$$L = 2 + 4\hbar^2 \left[ \frac{d^2}{dq^2} - 2\beta \frac{\sigma_0'(q)}{\sigma_0(q)} \frac{d}{dq} + \beta^2 \frac{\sigma_0''(q)}{\sigma_0(q)} \right] + O(\hbar^4).$$
(91)

We find the term with order  $\hbar^2$  is exactly the Hamiltonian presented in relation (85)

$$H = \frac{d^2}{dq^2} - 2\beta \frac{\sigma_0'(q)}{\sigma_0(q)} \frac{d}{dq} + \beta^2 \frac{\sigma_0''(q)}{\sigma_0(q)}.$$
 (92)

Since we already know the eigenfunction of the difference operator L is  $\psi(q) = e^{cq} \prod_{j=1}^{\beta} \sigma_0(q+t_j)$  which does not depend on  $\hbar$ , we need only expand the eigenvalue of L in the order of  $\hbar$ . We can obtain the eigenvalue of the Lamé operator H

$$\epsilon = 2 - 4\hbar \left[ c + \sum_{j=1}^{\beta} \frac{\sigma'_{0}(t_{j})}{\sigma_{0}(t_{j})} \right] + 8\hbar^{2}c \left[ c + \sum_{j=1}^{\beta} \frac{\sigma'_{0}(t_{j})}{\sigma_{0}(t_{j})} \right] -4\hbar^{2}c^{2} + 4\beta^{2}\hbar^{2}\frac{\sigma'''_{0}(0)}{\sigma'_{0}(0)} + 4\hbar^{2} \left\{ \sum_{j=1}^{\beta} \left[ \frac{\sigma'_{0}(t_{j})}{\sigma_{0}(t_{j})} \right]^{2} + 2\sum_{i>j} \frac{\sigma'_{0}(t_{i})\sigma'_{0}(t_{j})}{\sigma_{0}(t_{i})\sigma_{0}(t_{j})} \right\} + 4\hbar^{2}(1 - 2\beta) \sum_{j=1}^{\beta} \left[ \frac{\sigma''_{0}(t_{j})}{\sigma_{0}(t_{j})} - \left( \frac{\sigma'_{0}(t_{j})}{\sigma_{0}(t_{j})} \right)^{2} \right] + O(\hbar^{4}).$$
(93)

At the same time we take the limit  $\hbar \to 0$  for the Bethe ansatz equation, obtaining

$$c + \beta \frac{\sigma'_0(t_i)}{\sigma_0(t_i)} - \sum_{j,j \neq i} \frac{\sigma'_0(t_i - t_j)}{\sigma_0(t_i - t_j)} = 0.$$
(94)

Considering this Bethe ansatz equation, we can finally find the eigenvalue of the Lamé operator  $\Lambda$ ,

$$\Lambda = (1 - 2\beta) \left[ \sum_{j=1}^{\beta} \frac{\sigma'_0(t_j)}{\sigma_0(t_j)} \right]' + \beta^2 \frac{\sigma''_0(0)}{\sigma'_0(0)}.$$
 (95)

Here we have the results:

Let  $(t_1, \dots, t_m, c)$  be a solution of the Bethe ansatz equations:

$$c + \beta \frac{\sigma'_0(t_i)}{\sigma_0(t_i)} - \sum_{j,j \neq i} \frac{\sigma'_0(t_i - t_j)}{\sigma_0(t_i - t_j)} = 0, \qquad i = 1, \cdots, \beta,$$
(96)

such that  $t_i \neq t_j \mod Z + \tau Z$  if  $i \neq j$ . Then

$$\psi(q) = e^{cq} \prod_{j}^{\beta} \sigma_0(q+t_j) \tag{97}$$

is a solution of the equation

$$H\psi(q) = \left[\frac{d^2}{dq^2} - 2\beta \frac{\sigma_0'(q)}{\sigma_0(q)} \frac{d}{dq} + \beta^2 \frac{\sigma_0''(q)}{\sigma_0(q)}\right] \psi(q) = \Lambda \psi(q), \tag{98}$$

with eigenvalue

$$\Lambda = (1 - 2\beta) \left[ \sum_{j=1}^{\beta} \frac{\sigma'_0(t_j)}{\sigma_0(t_j)} \right]' + \beta^2 \frac{\sigma'''(0)}{\sigma'_0(0)}.$$
(99)

We can also obtain these results by directly using the algebraic Bethe ansatz method for the  $A_1^{(1)}$  Gaudin model,<sup>40)</sup> just like the algebraic Bethe ansatz for the XYZ Gaudin model.<sup>41)</sup>

#### §6. Summary

We review some developments concerning the non-dynamical structure of the elliptic RS and CM models. We also give a solution to the Lamé equation. The eigenfunction and eigenvalue for the Lamé operator are found through the results of the Bethe ansatz.

The results of the last sections are only for the n = 2 case. For general n, we can also obtain the eigenvalue and eigenfunction for the generalized Lamé operator by using the algebraic Bethe ansatz method for the sl(n) elliptic Gaudin model. The conserved quantities also correspond to the Hamiltonian of the elliptic CM model.

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#### References

- 1) M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71 (1981), 313.
- 2) J. Avan and T. Talon, Phys. Lett. B303 (1993), 33.
- 3) F. Calogero, Lett. Nuovo Cim. 13 (1975), 411; 16 (1976), 77.
- 4) J. Moser, Adv. Math. 16 (1975), 1.
- 5) I. M. Krichever, Func. Anal. Appl. 14 (1980), 282.
- 6) E. K. Sklyanin, Func. Anal. Appl. 16 (1982), 263; 17 (1983), 320; Commun. Math. Phys. 150 (1992), 181; hep-th/9308060.
- 7) H. W. Braden and T. Suzuki, Lett. Math. Phys. 30 (1994), 147.
- 8) O. Babelon and D. Bernard, Phys. Lett. **B317** (1993), 363.
- 9) H. W. Braden, Andrew and N. W. Hone, Phys. Lett. B380 (1996), 296.
- 10) B. Y. Hou and W. L. Yang, Lett. Math. Phys. 44 (1998), 35; J. of Phys. A32 (1999), 1475.
- 11) B. Y. Hou and W. L. Yang, IMPNWU-971219; math.QA/9802104.
- 12) J. Avan, O. Babelon and E. Billey, Commun. Math. Phys. 178 (1996), 281.
- 13) H. W. Braden and R. Sasaki, Prog. Theor. Phys. 97 (1997), 1003.
- 14) H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, Nucl. Phys. B338 (1990), 689;
   B356 (1991), 469.
- 15) F. W. Nijhoff, O. Ragnisco and V. B. Kuznetsov, Commun. Math. Phys. 176 (1996), 681.
- 16) G. E. Arutyunov, S. A. Frolov and P. B. Medredev, J. of Phys. A30 (1997), 5051; J. Math. Phys. 38 (1997), 5682.
- 17) A. Gorsky and N. Nekrosov, Nucl. Phys. B414 (1994), 213; B436 (1995), 582.
- 18) S. N. M. Ruijsenaars, Commun. Math. Phys. 115 (1988), 127.
- 19) Yuri B. Suris, hep-th/9603011.
- 20) F. W. Nijhoff, V. B. Kuznetrov, E. K. Sklyanin and O. Ragnisco, J. of Phys. A29 (1996), L333.
- 21) A. J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 100 (1998), 1107.
- 22) A. J. Bordner, R. Sasaki and K. Takasaki, Prog. Theor. Phys. 101 (1999), 487.
- 23) A. J. Bordner and R. Sasaki, Prog. Theor. Phys. 101 (1999), 799.
- 24) A. J. Bordner, E. Corrigan and R. Sasaki, Prog. Theor. Phys. 102 (1999), 499; hep-th/9905011.
- 25) O. Babelon and D. Bernard, Phys. Lett. B317 (1993), 363.
- 26) L. Freidel and J. M. Maillet, Phys. Lett. B262 (1991), 278.
- 27) S. N. M. Ruijsenaars and H. Schneider, Ann. of Phys. 170 (1986), 370.
- 28) A. A. Belavin, Nucl. Phys. B180 (1980), 109.
- 29) M. P. Richey and C. A. Tracy, J. Stat. Phys. 42 (1986), 311.
- 30) M. Jimbo, T. Miwa and M. Okado, Nucl. Phys. B300 (1988), 74.
- 31) B. Y. Hou, K. J. Shi and W. L. Yang, J. of Phys. A26 (1993), 4951.
- 32) B. Y. Hou and H. Wei, J. Math. Phys. 30 (1989), 2750.
- 33) K. Hasegawa, J. of Phys. A26 (1993), 3211; J. Math. Phys. 35 (1994), 6158.
- 34) Y. H. Quano and A. Fujii, Mod. Phys. Lett. A6 (1991), 3635.
- 35) B. Y. Hou, K. J. Shi, W. L. Yang and Z. X. Yang, Phys. Lett. A178 (1993), 73.
- 36) K. Hasegawa, q-alg/9512029; Commun. Math. Phys. 187 (1997), 289.
- 37) E. D'Hoker and D. H. Phong, hep-th/9808156.
- 38) K. Hikami and Y. Komori, J. Phys. Soc. Jpn. 67 (1998), 4037.
- 39) G. Felder and A. Varchenko, q-alg/9605024; Nucl. Phys. B480 (1996), 485.

- 40) R. H. Yue et al., Preprint (NWU-IMP9906).
  41) E. K. Sklyanin and T. Takebe, Phys. Lett. A219 (1996), 217; solv-int/9807008.
- 42) K. Chen, B. Y. Hou, W. L. Yang and Y. Zhen, Chinese. Phys. Lett. 16 (1999), 1.