Primordial Black Holes and Gravitational Memory

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We discuss the various ways in which primordial black holes may have formed in the early Universe and how the effects of such black holes can be used to place constraints on cosmological models. We show that such constraints may be severely modified if the value of the gravitational "constant" G varies with cosmological epoch, a possibility which arises in many scenarios for the early Universe. The nature of the modification depends upon whether the value of G near a black hole maintains the value it had at its formation epoch (corresponding to gravitational memory) or whether it tracks the background cosmological value. This is still uncertain but we discuss various approaches which might help to resolve the issue.

§1. Introduction

It is well known that primordial black holes (PBHs) could have formed in the early Universe. [1),2)] A comparison of the cosmological density at any time after the Big Bang with the density associated with a black hole shows that PBHs would have of order the particle horizon mass at their formation epoch. PBHs could thus span an enormous mass range: those formed at the Planck time (10^{-43} s) would have the Planck mass (10^{-5} g), whereas those formed at 1 s would be as large as $10^5 M_\odot$, comparable to the mass of the holes thought to reside in galactic nuclei.

PBHs could arise in various ways. [3)] Since the early Universe is unlikely to have been exactly Friedmann, they would form most naturally from initial inhomogeneities but they might also form through other mechanisms at a cosmological phase transition. We discuss these various formation mechanisms in §2. PBHs could also have various types of cosmological consequence. Those larger than 10^{15}g could contribute to the dark matter density and might even produce observable microlensing effects. Those smaller than this would have evaporated by the Hawking mechanism [4)] and could contribute to the flux of cosmic rays.

Studying the formation and cosmological consequences of PBHs is important because it enables one to place constraints on the early Universe (e.g., on the spectrum of density fluctuations and the nature of any phase transitions). Indeed PBHs serve as a probe of times much earlier than that associated with most other "relicts" of the Big Bang. While photons decoupled at 10^6 y, neutrinos at 1 s and WIMPs at 10^{-10} s, non-evaporating PBHs go back to 10^{-23} s and evaporating ones all the way back to the Planck time. Therefore even if PBHs never formed, their non-existence gives interesting information. We review and update these constraints in §2.

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The main purpose of this paper is to examine how the PBH constraints are modified if the value of the gravitational "constant" G was different at early times. As reviewed in §3, this idea has a long history and should no longer be regarded as exotic. It arises in various scalar-tensor theories of gravity and these are a natural setting for many currently popular models of the early Universe. A number of astrophysical constraints suggest the G could not have varied much since the epoch of cosmological nucleosynthesis but PBHs would have formed much earlier than this. Homogeneous cosmological solutions in such theories have been much studied and generally lead to a simple power-law dependence of G on t. However, *inhomogenous* models (such as would be associated with PBH formation) would lead to a value of G varying both temporally and spatially and are much less well understood.

Black hole formation and evaporation could be greatly modified in variable-G cosmologies, since many of their properties (e.g., their radius and Hawking temperature) depend explicitly on G. However, the nature of the modification depends upon the extent to which a PBH preserves the value of G at its formation epoch rather than following the background cosmological value. Barrow[5)] first drew attention to this problem and introduced two possibilities: "Scenario A", where G everywhere maintains the background value and "Scenario B", where the value of G stays constant near the black hole but evolves far away from it. This last possibility he termed "gravitational memory". There would be interesting modifications to the cosmological consequences of PBH formation in both cases but they would be more dramatic in Scenario B.

Barrow & Carr[6)] considered the implications of these two scenarios in detail and we will review some of their conclusions in §3. However, they did not address the issue of gravitational memory itself and we turn to this in §4. Although the problem is still not resolved, we will report on several interesting developments. These are described in more detail elsewhere[7)] and exploit two important equivalences. The first equivalence is between scalar-tensor theories and general relativity with a scalar field. This comes about because these two theories are related by a conformal transformation and this means that the problem of gravitational memory can be probed by investigating the formation and evolution of a black hole in a cosmological background containing a scalar field. This leads to four variants of the gravitational memory scenario, which we describe in §5. The second equivalence is between a scalar field and a stiff fluid (with equation of state $p = \rho$). This allows one to relate the issue of gravitational memory to the problem of black hole accretion in a Universe containing a stiff fluid. This problem has already been investigated by several people and, as discussed in §6, their findings lead to several new insights.

§2. PBH formation and constraints on the early Universe

One of the most important reasons for studying PBHs is that it enables one to place limits on the spectrum of density fluctuations in the early Universe. This is because, if the PBHs form directly from density perturbations, the fraction of regions undergoing collapse at any epoch is determined by the root-mean-square amplitude ϵ of the fluctuations entering the horizon at that epoch and the equation of state

$p = \gamma\rho$ $(0 < \gamma < 1)$. One usually expects a radiation equation of state ($\gamma = 1/3$) in the early Universe. In order to collapse against the pressure, an overdense region must be larger than the Jeans length at maximum expansion and this is just $\sqrt{\gamma}$ times the horizon size. On the other hand, it cannot be larger than the horizon size, else it would form a separate closed universe and not be part of our Universe.[8)]

This has two important implications. Firstly, PBHs forming at time t should have of order the horizon mass then:

$$M(t) \approx \frac{c^3 t}{G} \approx 10^{15} \left(\frac{t}{10^{-23}\ \mathrm{s}}\right) \mathrm{g}. \tag{2·1}$$

Secondly, for a region destined to collapse to a PBH, one requires the fractional overdensity at the horizon epoch, δ, to exceed γ. Providing the desnity fluctuations have a Gaussian distribution and are spherically symmetric, one can infer that the fraction of regions of mass M which collapse is[9)]

$$\beta(M) \sim \epsilon(M) \exp\left[-\frac{\gamma^2}{2\epsilon(M)^2}\right], \tag{2·2}$$

where $\epsilon(M)$ is the value of ϵ when the horizon mass is M. The PBHs can have an extended mass spectrum only if the fluctuations are scale-invariant (i.e., with ϵ independent of M) but this is expected in many scenarios. Recent hydrodynamical calculations for the $\gamma = 1/3$ case have refined these estimates somewhat: Niemeyer & Jedamzik[10)] find that one needs $\delta > 0.8$ rather than $\delta > 0.3$ to ensure PBH formation, and Shibata & Sasaki[11)] reach similar conclusions, but the same basic picture still applies.

Another interesting development has been the application of "critical phenomena" to PBH formation. Studies of the collapse of various types of spherically symmetric matter fields have shown that there is always a critical solution which separates those configurations which form a black hole from those which disperse to an asymptotically flat state. The configurations are described by some index p and, as the critical index p_c is approached, the black hole mass is found to scale as $(p - p_c)^\eta$ for some exponent η. This effect was first discovered for scalar fields[12)] but subsequently demonstrated for radiation[13)] and then more general fluids with equation of state $p = \gamma\rho$.[14),15)]

In all these studies the spacetime was assumed to be asymptotically flat. However, Niemeyer & Jedamzik[16)] have recently applied the same idea to study black hole formation in asymptotically Friedmann models and have found similar results. For a variety of initial density perturbation profiles, they find that the relationship between the PBH mass and the the horizon-scale density perturbation has the form

$$M = K M_H (\delta - \delta_c)^\gamma, \tag{2·3}$$

where M_H is the horizon mass and the constants are in the range $0.34 < \gamma < 0.37$, $2.4 < K < 11.9$ and $0.67 < \delta_c < 0.71$ for the various configurations. Since $M \to 0$ as $\delta \to \delta_c$, this suggests that PBHs may be much smaller than the particle horizon at formation (although it is clear that a fluid description must break down if they are too small) and it also modifies the mass spectrum.[17),18)]

These developments might be regarded as refinements of the original scenario since they all presuppose a relationship of the form (2·2). However, in some situations Eq. (2·2) would fail qualitatively. For example, PBHs would form more easily if the equation of state of the Universe were ever soft ($\gamma \ll 1$). This might apply if there was a phase transition which channelled the mass of the Universe into non-relativistic particles or which temporally reduced the pressure. In this case, only those regions which are sufficiently spherically symmetric at maximum expansion can undergo collapse; the dependence of β on ϵ would then be weaker than indicated by Eq. (2·2) but there would still be a unique relationship between the two parameters.[19]

The fluctuations required to make the PBHs may either be primordial or they may arise spontaneously at some epoch. One natural source of fluctuations would be inflation[20),21] and, in this context, $\epsilon(M)$ depends implicitly on the inflationary potential. Indeed many people have studied PBH formation in inflationary scenarios as an important way of constraining this potential.[22)-28] Recently the Gaussian assumption has been questioned in the inflationary context,[29),30] so Eq. (2·2) may not apply, but one still finds that β depends very sensitively on ϵ.

Some formation mechanisms for PBHs do not depend on having primordial fluctuations at all. For example, at any spontaneously broken symmetry epoch, PBHs might form through the collisions of bubbles of broken symmetry.[31)-33] PBHs might also form spontaneously through the collapse of cosmic strings.[34)-38] In these cases $\beta(M)$ depends, not on $\epsilon(M)$, but on other cosmological parameters, such the bubble formation rate or the string mass-per-length.

In all these scenarios, the current density parameter $\Omega_{\rm PBH}$ associated with PBHs which form at a redshift z or time t is related to β by[9]

$$\Omega_{\rm PBH} = \beta \Omega_R (1+z) \approx 10^6 \beta \left(\frac{t}{1\,{\rm s}}\right)^{-1/2} \approx 10^{18} \beta \left(\frac{M}{10^{15}\,{\rm g}}\right)^{-1/2}, \qquad (2\cdot4)$$

where $\Omega_R \approx 10^{-4}$ is the density parameter of the microwave background and we have used Eq. (2·1). The $(1+z)$ factor arises because the radiation density scales as $(1+z)^4$, whereas the PBH density scales as $(1+z)^3$. Any limit on $\Omega_{\rm PBH}$ therefore places a constraint on $\beta(M)$ and the constraints are summarized in Fig. 1. The constraint for non-evaporating mass ranges above 10^{15}g comes from requiring $\Omega_{\rm PBH} < 1$. Much stronger constraints are associated with PBHs which were smaller than this, since they would have evaporated by now. For example, the constraints below 10^6 g are based on the (not necessarily secure) assumption that evaporating PBHs leave stable Planck mass relics, in which case these relics are required to have less than the critical density.[23),39),40] Other constraints are associated with the generation of entropy, modifications to the cosmological production of light elements and the contribution to the cosmological γ-ray background.

The constraints in Fig. 1 are discussed in detail by Carr et al.[23] but we note that Kohri & Yokoyama[41] have recently improved the constraints on $\beta(10^8-10^{10}{\rm g})$ which come from cosmological nucleosynthesis considerations. Here we wish to emphasize that the strongest constraint is the cosmic ray limit associated with the 10^{15}g PBHs evaporating at the present epoch.[42)-46] Indeed the recent detection of a Galactic γ-ray background,[47] measurements of the antiproton flux,[48] and the

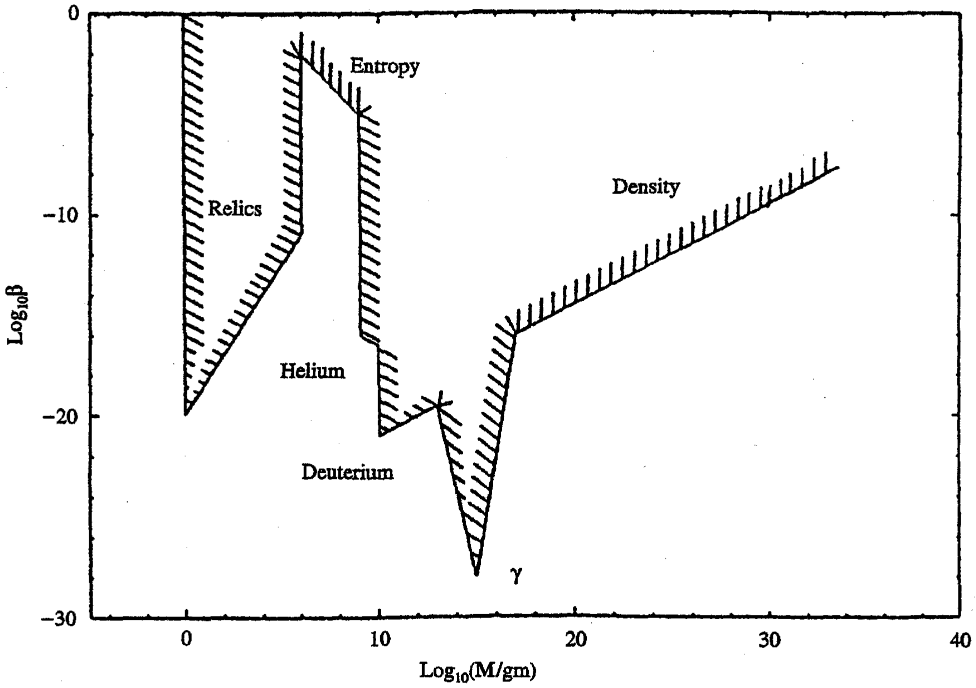

Fig. 1. Constraints on $\beta(M)$.

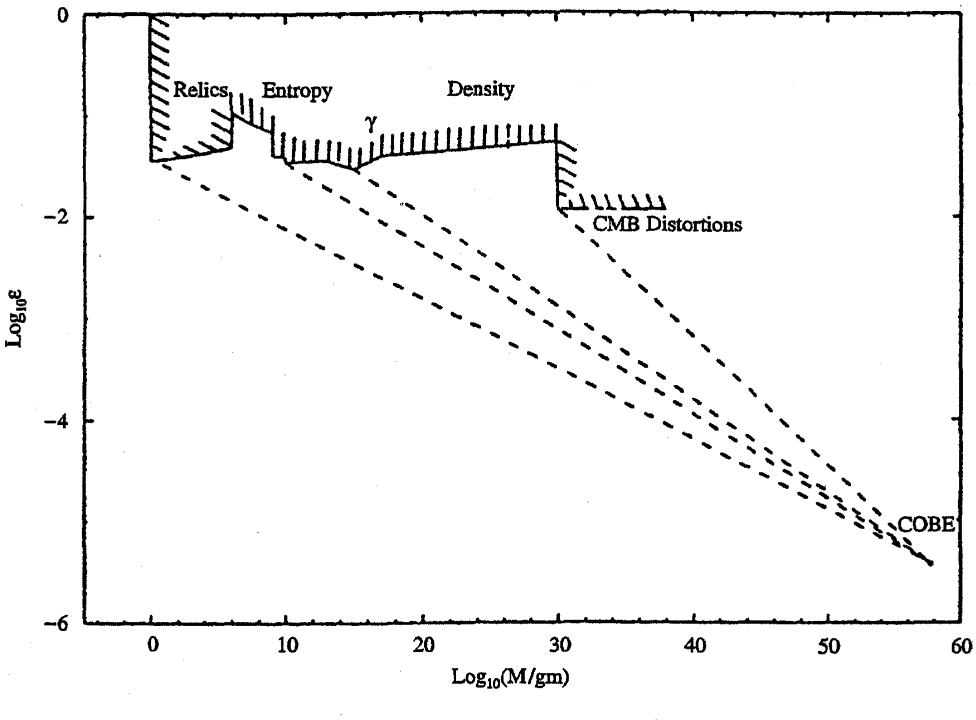

Fig. 2. Constraints on $\epsilon(M)$.

discovery of very short period γ-ray burts,[49] may even provide positive evidence for such PBHs. This is discussed in detail elsewhere.[50] The constraints on $\beta(M)$ can be converted into constraints on $\epsilon(M)$ using Eq. (2·2) and these are shown in Fig. 2. Also shown here are the (non-PBH) constraints associated with the spectral distortions in the cosmic microwave background induced by the dissipation of intermediate scale density perturbations and the COBE quadrupole measurement, as well as lines corresponding to various slopes in the $\epsilon(M)$ relationship.

Finally it should be emphasized that there has been particular interest recently in whether PBHs could have formed at the quark-hadron phase transition at 10^{-5} s. This is because the horizon mass is of order $1M_{\odot}$ then, so such PBHs would naturally have the sort of mass required to explain the MACHO microlensing results.[51] One might expect PBHs to form more easily at that epoch because of a temporary softening of the equation of state. If the QCD phase transition is assumed to be of 1st order, then hydrodynamical calculations show that the value of δ required for PBH formation is indeed reduced below the value which pertains in the radiation case.[52] This means that PBH formation will be strongly enhanced at the QCD epoch, with the mass distribution being peaked around the horizon mass then.[18]

Since the focus of this meeting is gravitational waves, it should be emphasized that one of the interesting implications of the PBH MACHO scenario is the possible existence of a halo population of *binary* black holes.[53] In this case, there could be 10^8 binaries inside 50 kpc and some of these could be coalescing due to gravitational radiation losses at the present epoch.[54] Current interferometers (LIGO, VIRGO, GEO, TAMA) could detect such coalescences within 50 Mpc, corresponding to a few events per year. Future space-borne interferometers (such as LISA or OMEGA) might detect 100 coalescences per year. If the associated gravitational waves were detected, it would provide a unique probe of the halo distribution (e.g. its density profile and core radius).[55]

§3. Cosmology in varying-G theories

Most variable-G scenarios associate the gravitational "constant" with some form of scalar field ϕ. This notion has its roots in Kaluza-Klein theory, in which a scalar field appears in the metric component g_{55} associated with the 5th dimension. Einstein-Maxwell theory then requires that this field be related to G.[56] Although this was assumed constant in the original Kaluza-Klein theory, Dirac[57] noted that the ratio of the electric to gravitational force between two protons (e^2/Gm_p^2) and the ratio of the age of the Universe to the atomic timescale (t/t_a) and the square-root of the number of particles in the Universe ($\sqrt{M/m_p}$) are all comparable and of order 10^{40}. This unlikely coincidence led him to propose that these relationships must *always* apply, which requires

$$G \propto t^{-1}, \quad GM/R \sim 1, \tag{3·1}$$

where $R \sim ct$ is the horizon scale. The first condition led Jordan[58] to propose a theory in which the scalar field in Kaluza-Klein theory is a function of *both* space and time, and this then implies that $G \sim \phi^{-1}$ has the same property. The second condition implies the Mach-type relationship $\phi \sim M/R$, which suggests[59] that ϕ is a solution of the wave equation $\Box\phi \sim \rho$. This motivated Brans-Dicke (BD) theory,[60] in which the Einstein-Hilbert Lagrangian is replaced by

$$L = \phi R - \frac{\omega}{\phi}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu} + L_m \ , \tag{3·2}$$

where L_m is the matter Langrangian and the constant ω is the BD parameter. The potential ϕ then satisfies

$$\Box\phi = \left(\frac{8\pi}{2\omega+3}\right)T, \tag{3·3}$$

where T is the trace of the matter stress-energy tensor, and this has the required Machian form. Since ϕ must have a contribution from *local* sources of the form $\Sigma_i(m_i/r_i)$, this entails a violation of the Strong Equivalence Principle. In order to test this, the PPN formalism was introduced. Applications of this test in a variety of astrophysical situations (involving the solar system, the binary pulsar and white dwarf cooling) currently require $|\omega| > 500$, which implies that the deviations from general relativity can only ever be small in BD theory.[61)]

The introduction of generalized scalar-tensor theories,[62)-64)] in which ω is itself a function of ϕ, led to a considerably broader range of variable-G theories. In particular, it permitted the possibility that ω may have been small at early times (allowing noticeable variations of G then) even if it is large today. In the last decade interest in such theories has been revitalized as a result of early Universe studies. Inflation theory[65)] has made the introduction of scalar fields almost mandatory and extended inflation specifically requires a model in which G varies.[33)] In higher dimensional Kaluza-Klein-type cosmologies, the variation in the sizes of the extra dimensions also naturally leads to a variation in G.[66)-68)] The currently popular low energy string cosmologies necessarily involve a scalar (dilaton) field[69)] and bosonic superstring theory, in particular, leads[70)] to a Lagrangian of the form (3.2) with $\omega = -1$.

The intimate connection between dilatons, inflatons and scalar-tensor theory arises because one can always transform from the (physical) Jordan frame to the Einstein frame, in which the Lagrangian has the standard Einstein-Hilbert form[71)]

$$L = \bar{R} - 2\psi_{,\mu}\psi_{,\nu}\bar{g}^{\mu\nu} + L_m. \tag{3·4}$$

Here the new scalar field ψ is defined by

$$d\psi = \left(\frac{2\omega+3}{2}\right)^{1/2}\frac{d\phi}{\phi} \tag{3·5}$$

and the barred (Einstein) metric and gravitational constant are related to the original (Jordan) ones by

$$g_{\mu\nu} = A(\phi)^2\bar{g}_{\mu\nu}, \quad G = [1+\alpha^2(\phi)]A(\phi)^2\bar{G}, \quad \alpha \equiv A'/A, \tag{3·6}$$

where the function $A(\phi)$ specifies a conformal transformation. Thus scalar-tensor theory can be interpreted as general relativity plus a scalar field.

The behaviour of homogeneous cosmological models in BD theory is well understood.[72)] Their crucial feature is that they are vacuum-dominated at early times but always tend towards the general relativistic solution during the radiation-dominated era. This is a consequence of the fact that the radiation energy-momentum tensor is trace-free [i.e. $T=0$ in Eq. (3·3)]. This means that the full radiation solution can be approximated by joining a BD vacuum solution to a general relativistic radiation solution at some time t_1, which may be regarded as a free parameter of the theory. However, when the matter density becomes greater than the radiation density at $t_e \sim 10^{11}$ s, the equation of state becomes that of dust ($p=0$) and G begins to vary again. For a $k=0$ model, one can show that in the three eras [6]

$$G = G_0(t_0/t)^n, \quad a \propto t^{(2-n)/3}, \quad (t > t_e) \tag{3·7}$$

$$G = G_e \equiv G_0(t_0/t_e)^n, \quad a \propto t^{1/2}, \quad (t_1 < t < t_e) \tag{3·8}$$

$$G = G_e(t/t_1)^{-(n+\sqrt{4n+n^2})/2}, \quad a \propto t^{(2-n-\sqrt{4n+n^2})/6}, \quad (t < t_1) \tag{3·9}$$

where G_0 is the value of G at the current time t_0, $n \equiv 2/(4+3\omega)$ and $(t_0/t_e) \approx 10^6$.

Since the BD coupling constant is constrained by $|\omega| > 500$, which implies $|n| < 0.001$, Eqs. (3·7) to (3·9) imply that the deviations from general relativity are never large if the value of n is always the same. However, for reasons explained below, it is also interesting to consider BD models in which n and ω can violate the current constraints. This allows considerably more exotic behaviour, especially in the vacuum-dominated era. In particular, we note the following features: [6] models with $\omega < -3/2$ are probably excluded because the energy density of the scalar field is negative; models with $-3/2 < \omega < -4/3$ undergo power-law inflation during the vacuum-dominated era because the exponent of t in the expression for a in Eq. (3·9) exceeds 1; models with $-4/3 < \omega < 0$ can bounce during the vacuum-dominated era because this exponent is negative.

The behaviour of cosmological models in more general scalar-tensor theories depends on the form of $\omega(\phi)$ but they still retain the feature that the general relativistic solution is a late-time attractor during the radiation era. Since one requires $G \approx G_0$ to 10% at the epoch of primordial nucleosynthesis, [72] one needs the vacuum-dominated phase to end at some time $t_v < 1$ s. The theory approaches general relativity in the weak field limit only if $\omega \to \infty$ and $\omega'/\omega^3 \to 0$ (where a prime denotes $d/d\phi$) but $\omega(\phi)$ is otherwise unconstrained. Barrow & Carr consider a toy model in which

$$2\omega + 3 = 2\beta(1 - \phi/\phi_c)^{-\alpha}, \tag{3·10}$$

where α and β are constants. This leads to

$$2\omega + 3 \propto t^{-\alpha/(2-\alpha)}, \quad \omega'/\omega^3 \propto t^{(1-2\alpha)/(1-\alpha)}, \quad (t < t_v) \tag{3·11}$$

so one requires $1/2 < \alpha < 2$ in order to have $\omega \to \infty$ and $\omega'/\omega^3 \to 0$ as $t \to \infty$. In the $\alpha = 1$ case, one finds

$$G \propto t^{-2\lambda/(3-\lambda)}, \quad a \propto t^{(1-\lambda)/(3-\lambda)}, \quad \lambda \equiv \sqrt{3/(2\beta)}. \quad (t < t_v) \tag{3·12}$$

During the vacuum-dominated era, such models can therefore be regarded as BD solutions in which ω is determined by the parameter β and unconstrained by any limits on ω at the present epoch. After t_v, G is constant and one has the standard radiation-dominated or dust-dominated behaviour.

On the assumption that a PBH of mass M has a temperature and mass-loss rate[4)]

$$T = (8\pi GM)^{-1}, \quad \dot{M} \approx -(GM)^{-2}, \tag{3·13}$$

with G having the value $G(t)$ in Scenario A and $G(M)$ in Scenario B, Barrow & Carr calculate the evaporation time τ for various values of the parameters n and t_1 in BD theory. The results are shown in Fig. 3(a) for Scenario A and Fig. 3(b) for Scenario B. Here M_* is the mass of a PBH evaporating at the present epoch, M_e is the mass of a PBH evaporating at time t_e and $M_{\rm crit}$ is the mass of a PBH evaporating at the present epoch in the standard (constant G) scenario. In Scenario A with $n < -1/2$, there is a maximum mass of a PBH which can ever evaporate and this is denoted by M_∞. The results for the scalar-tensor with $\omega(\phi)$ given by Eq. (3·10) with $\alpha = 1$ are shown in Fig. 3(c) for Scenario B with various values of the parameters λ and t_v. The corresponding modifications to the constraints on $\beta(M)$ in all three cases are shown in Fig. 3(d), which should be compared to Fig. 1.

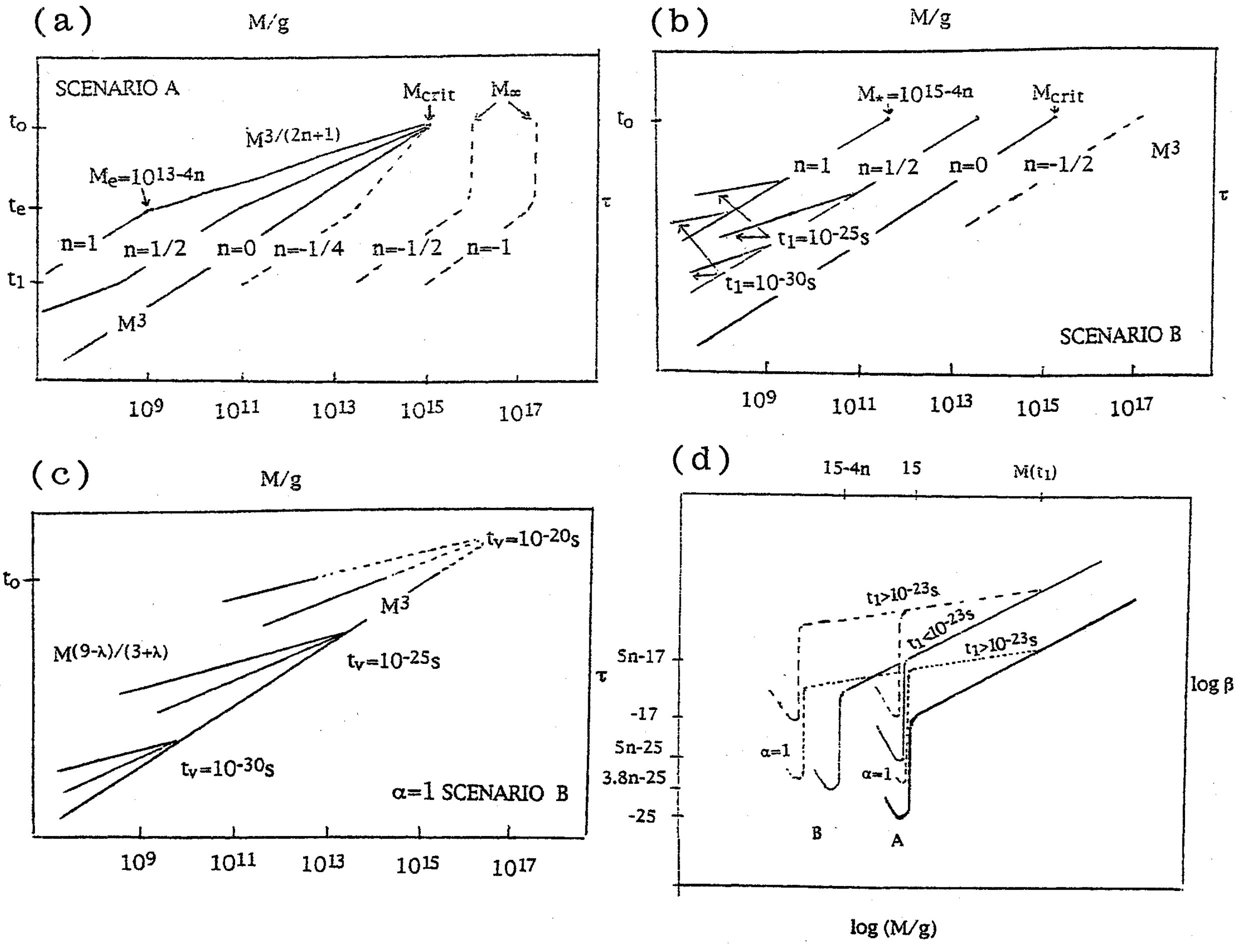

Fig. 3. Dependence of the PBH evaporation time τ on initial mass M in (a) BD theory with Scenario A, (b) BD theory with Scenario B, (c) scalar-tensor theory with $\alpha = 1$ and Scenario B. Also shown are (d) the modifications to the constraints on $\beta(M)$ in these cases.

§4. Black holes in scalar-tensor theory

In BD theory Hawking[73] showed that, providing the weak energy condition holds, the gradient of ϕ must be zero everywhere for stationary, asymptotically flat black holes. This means that such black holes are identical to those in general relativity. This result can be generalized to all scalar-tensor theories and suggests that such theories are in agreement with the "no-hair" theorem.

Numerical calculations support this theorem.[74)-76)] An Oppenhiemer-Snyder type collapse reveals outgoing scalar gravitational radiation, which radiates away the scalar mass. When all the scalar mass is lost, the black hole settles down with a constant scalar field to the Schwarzschild form. The scalar radiation generated in this way would be of great interest in itself. As shown by Harada et al.,[76] if a black hole of mass M collapses, the ratio of the amplitudes of the scalar and tensor gravitational waves is

$$\frac{h_S}{h_T} \sim 10^{-3}\left(\frac{500}{f}\right)\left(\frac{10GM}{a}\right)^{-1}\frac{e^2}{\sqrt{1-e^2}}, \quad f \sim \left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{r_s}{15\,\mathrm{km}}\right), \tag{4·1}$$

where a is the initial size of the region, e is its eccentricity and f is the frequency. There might also be a stochastic background of scalar gravitational waves generated by the formation of PBHs in the early Universe, although its density would have been diluted by now because of redshift effects.

It should be pointed out that the simulations of scalar collapse also show a rather unusual feature:[74),75)] the area of the event horizon decreases for a time, violating the area theorem, and it may also lie inside the apparent horizon during this period. This occurs for all values of ω and is due to the fact that the scalar field violates the weak energy condition. This condition holds in the Einstein frame as long as $\omega > -3/2$ but it need not be true in the physical Jordan frame. Examples have been found[77)-79)] of black holes in BD theory which have scalar hair but these arise only for certain ranges of ω: $\omega < -4/3$ in Ref. 77), $\omega < -3/2$ in Ref. 78) and $-5/2 \leq \omega < -3/2$ in Ref. 79). It is unclear that these can represent the end state of realistic gravitational collapse.

In addressing the question of which of Barrow's Scenarios A or B is most plausible, it should be stressed that the scalar no hair theorem has only been proved for asymptotically flat spacetimes, so it is not clear that it also applies in the asymptotically Friedmann case. While the no hair theorem suggests that ϕ should tend to a *locally* constant value (close to the black hole), it is not obvious that this needs to be the asymptotic cosmological value. Indeed, since the homogeneizing of ϕ is only ensured by scalar wave emission, one might infer that this can only be achieved on scales less than the particle horizon.

The only way to determine what happens is to seek a precise mathematical model for a black hole in a cosmological background. One approach is to try matching the black hole and cosmological solutions over some boundary Σ. An example of such a matching in general relativity is the Einstein-Straus or "Swiss cheese" model.[80] Here a Friedmann exterior is matched with a general spherically symmetric interior. If there is no scalar field, it turns out that the latter has to be the static Schwarzschild solution but the situation may be more complicated in the present context due to the presence of scalar gravitational radiation. In general one can show that the following continuity conditions must apply at Σ:

$$[g_{\mu\nu}] = 0, \quad [G_{\mu\nu}n^{\mu}n^{\nu}] = [G_{\mu\nu}u^{\mu}n^{\nu}] = 0, \quad [\phi] = 0, \quad [\phi_{\mu}n^{\mu}] = 0, \tag{4·2}$$

where n^{μ} and u^{μ} are 4-vectors normal and tangent to Σ, respectively.[81)]

Unfortunately, it turns out that an Einstein-Straus type solution does not exist in BD theory. This is because the only way to satisfy the junction conditions (4·2) is if ϕ is spatially and temporally constant, which is just the general relativistic case. However, Oppenheimer-Snyder collapse has been investigated[76), 82)] in which a ball of dust described by a $k = +1$ Friedmann interior and a Schwarzschild exterior collapses to a black hole. In these calculations the scalar field is taken to be constant before the collapse and its back-reaction on the metric is assumed to be always negligible. It is found that, as the collapse proceeds, a scalar gravitational wave propagates outwards before the scalar field settles down to being constant again. In principle, it should be possible to extend this approach to the present problem by attaching a $k = +1$ Friedman interior to a $k = 0$ Friedman exterior.

Jacobson[83)] has addressed the problem analytically by looking for a spherically symmetric solution which represents a black hole in an asymptotically Friedmann model in which ϕ satisfies the wave equation

$$\left(-g^{tt}\partial_t^2 + \frac{1}{\sqrt{-g}}\partial_r[\sqrt{-g}\, g^{rr}\partial_r]\right)\phi(t,r) = 0. \tag{4·3}$$

As $r \to \infty$, he assumes that ϕ asymptotes to

$$\phi_c(t) \approx \dot{\phi}_c t \tag{4·4}$$

in the Einstein frame. This presupposes that the black hole event horizon is much smaller than the particle horizon, so that the cosmological timescale is much longer than the black hole timescale, in which case $\dot{\phi}_c$ in Eq. (4·4) can be regarded as constant. He then seeks a solution which is a combination of a homogeneous part $\phi_1(t)$ and a time-independent part $\phi_2(r)$, where

$$\phi_1 = \dot{\phi}_c t, \quad \phi_2 = \ln(1 - r_H/r). \tag{4·5}$$

Both these terms diverge at the event horizon (r_H) but one can find a combination of them which is regular there:

$$\phi_3 = \phi_1 + r_H\phi_2\dot{\phi}_c = \dot{\phi}_c[v - r - r_H\ln(r/r_H)], \tag{4·6}$$

where v is advanced "tortoise" time, together with a superposition of waves falling into the black hole or dispersing to infinity. The equipotential $\phi = \phi_3$ intersects the event horizon at $v = v_H$ and infinity at $t = t_\infty$. Since $v = v_H$ and and $t = t_\infty$ themselves intersect at $r \approx 1.5GM$, he infers that there is little lag between the value of ϕ at the event horizon and particle horizon, which means that the memory can only be weak.

Although Jacobson's analysis demonstrates that a solution with little memory does exist, it must be emphasized that he has really put this feature into the solution at the outset by assuming the black hole is much smaller than the particle horizon. This assumption is inappropriate if the black hole has a size *comparable* to the particle horizon at formation (as applies for a PBH), so the field around a collapsing region need not necessarily evolve to the form (4·6). Another rather crucial feature of his analysis is that he claims that, while the PBH mass is nearly constant in the Einstein frame, it increases as

$$M = A(\phi)^{-1}\bar{M} \propto \phi^{1/2} \tag{4·7}$$

in the Jordan frame. If this were correct, it would invalidate the analysis of Barrow & Carr and hence Figs. 3. However, we would argue that it is not clear in which frame the mass should be taken to be constant.

§5. Variations of gravitational memory

We now consider the question of what happens to black holes in BD or more general scalar-tensor theories during the evolution of the Universe. In general the present value of the scalar field, $\phi(t_0)$, will be different from its value when the black hole was formed, $\phi(t_f)$. In the following discussion, we will characterize the degree of gravitational memory by comparing the value of the scalar field at the black hole event horizon ($\phi_{\rm EH}$) and the cosmological particle horizon ($\phi_{\rm PH}$). We first consider the two extreme situations described by Barrow:[5]

$$\text{Scenario A}: \qquad \phi_{\rm EH}(t) = \phi_{\rm PH}(t) \quad \text{for all } t. \tag{5·1}$$

A Schwarzschild black hole forms at time t_f with its event horizon radius being $R_f = 2G(t_f)M$. If $G(t)$ evolves with time, then the black hole adjusts quasi-statically through a sequence of Schwarzschild states approximated by $R = 2G(t)M$, see Fig. 4(a). In this scenario there are no stationary black holes when $G(t)$ is changing and no gravitational memory.

$$\text{Scenario B}: \qquad \phi_{\rm EH}(t) = \phi_{\rm EH}(t_f) \quad \text{for all } t. \tag{5·2}$$

A Schwarzschild black hole of size R_f forms at time t_f and, while $G(t)$ equals the evolving background value beyond some scale-length $R_m \geq R_f$, it remains constant within R_m, see Fig. 4(b). In this case the black hole size is determined by $G(t_f)$ even at the present epoch and this means that the region $R < R_m$ has a memory of the gravitational "constant" at the time of its formation.

Neither of these scenarios can be completely realistic since they both assume that ϕ is homogeneous almost everywhere. However, even if ϕ were homogeneous initially, one would expect it to become inhomogeneous as collapse proceeds. Indeed, if the background value is increasing (as usually applies), one would expect ϕ in the collapsing region to become first *larger* than the background value on a local dynamical timescale and then *smaller* than it on a cosmological timescale. Such behaviour would necessarily entail a variation of ϕ in space as well as time. Solutions have been found for specific ω in which ϕ is inhomogeneous,[84] so such models are clearly possible in principle. We must also allow for the possibility that ϕ may vary interior to R_m but on a slower or faster timescale than the background. We therefore propose two further scenarios:

$$\text{Scenario C}: \qquad |\dot{\phi}_{\rm EH}(t)| \geq |\dot{\phi}_{\rm PH}(t)| \quad \text{for all } t, \tag{5·3}$$

where the dot represents a time derivative. This implies that the scalar field evolves faster at the event horizon than at the particle horizon until it eventually becomes homogeneous, see Fig. 4(c). We describe this as *short-term* gravitational memory and it reduces to Scenario A as the timescale to become homogeneous tends to zero. This would apply, for example, if ϕ were to change on the dynamical timescales of the black hole since this is usually less than that of the cosmological background.

$$\text{Scenario D}: \qquad |\dot{\phi}_{\rm EH}(t)| < |\dot{\phi}_{\rm PH}(t)| \quad \text{for all } t. \tag{5·4}$$

This implies that ϕ evolves faster at the particle horizon than the event horizon, see Fig. 4(d). We describe this is as *weak* gravitational memory and it reduces to Scenario B when the left-hand side of Eq. (5·4) is zero. In this case, the evolution of ϕ is again dominated by the black hole inside some length-scale R_m. Note that, in either this scenario or the last one, the length-scale R_m need not be fixed, since it

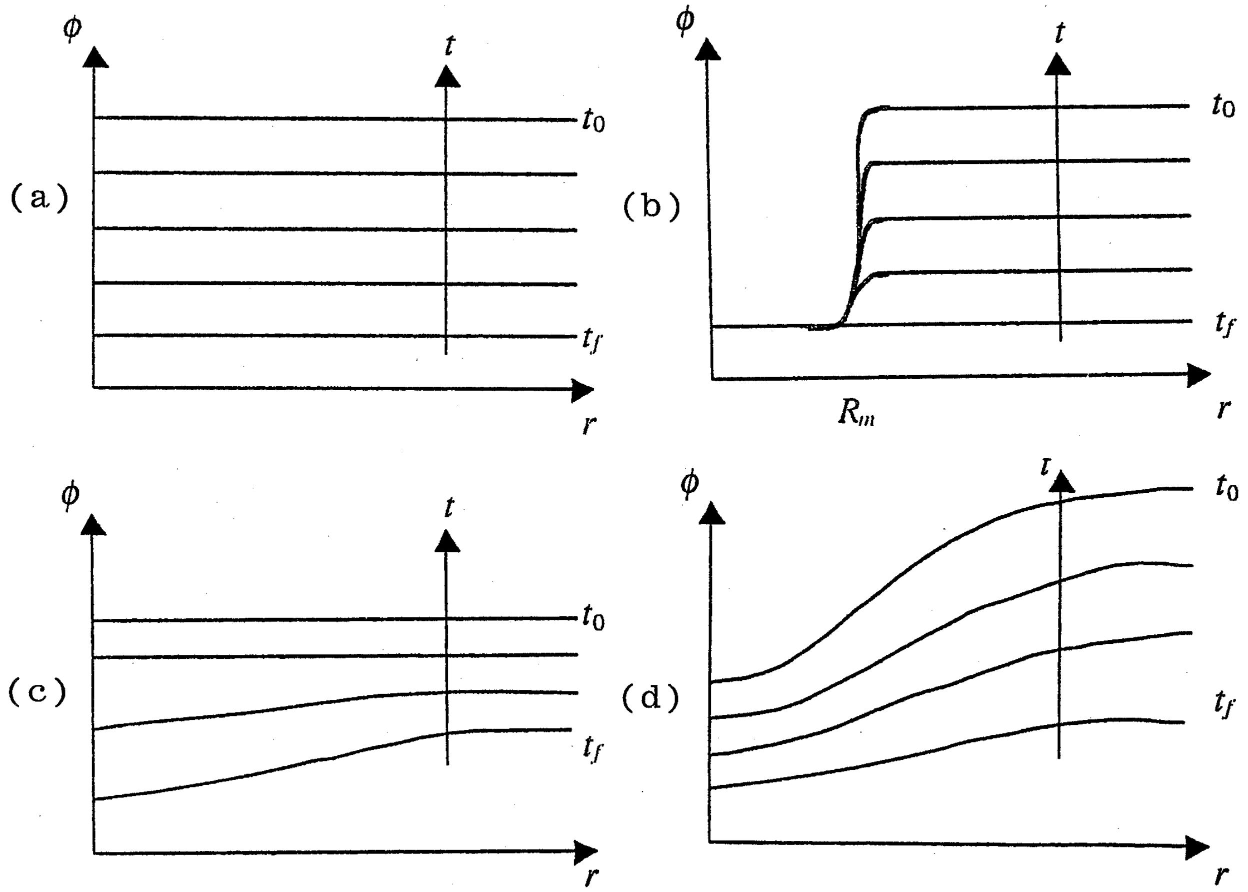

Fig. 4. Different possiible forms for the evolution of the scalar field profile $\phi(r)$ for (a) no memory, (b) strong memory, (c) short-term memory, (d) weak memory.

could either grow or shrink as scalar gravitational radiation propagates. A particular example of this, to which we return shortly, would be *self-similar* gravitational memory, in which the ratio of $\phi_{\rm EH}$ to $\phi_{\rm PH}$ always remains the same.

If ϕ, or equivalently G, is inhomogeneous, then one has to consider carefully what is meant by the Hawking temperature. In general relativity the temperature of a Schwarzschild black hole is given by Eq. (3·13). However, in a scalar-tensor theory $G \equiv G(t,r)$, so one must decide which value of G is appropriate. Since the particle creation responsible for the effect takes place near the event horizon, perhaps the local value $G_{\rm EH}$ should be used, giving

$$T_H = (8\pi G_{\rm EH} M)^{-1}. \tag{5·5}$$

On the other hand, since the radiation is only measured asymptotically, one might argue that the asymptotic value G_∞ is appropriate, giving

$$T_H = (8\pi G_\infty M)^{-1}. \tag{5·6}$$

It is not certain whether Eq. (5·5) or (5·6) should be used. If Eq. (5·5) applies, then the temperature depends crucially on whether or not gravitational memory exists. However, if Eq. (5·6) applies, the temperature is unaffected by this and just depends on the background.

Gravitational memory has also been investigated in the context of boson stars.[85] A boson star is the analogue of a neutron star, formed when a large collection of bosonic particles become gravitationally bound.[86],[87] In this case, weak gravitational memory is already known to occur. An interesting feature of this is that two boson stars of the same mass may differ in other physical properties (e.g. radius), depending on when they formed. However, the situation may be very different for a black hole since this has an event horizon.

§6. Gravitational memory and the accretion of a stiff fluid

In general relativity there is an equivalence between a scalar field and a stiff fluid and we now show how this can be exploited in studying gravitational memory. In the Einstein frame, the energy momentum tensor for a perfect fluid is

$$\bar{T}_{\mu\nu} = (\rho + p) u_\mu u_\nu + \bar{g}_{\mu\nu} p, \tag{6·1}$$

where u_μ is the velocity of the fluid. If we define a velocity field by

$$u_\mu = \frac{\bar{\phi}_\mu}{(-\bar{g}^{\rho\sigma}\bar{\phi}_\rho\bar{\phi}_\sigma)^{1/2}}\,, \tag{6·2}$$

this gives

$$\bar{T}_{\mu\nu} = -\frac{(\rho+p)\bar{\phi}_\mu\bar{\phi}_\nu}{\bar{g}^{\rho\sigma}\bar{\phi}_\rho\bar{\phi}_\sigma} + p\bar{g}_{\mu\nu}\,. \tag{6·3}$$

By comparing this to the energy-momentum tensor for a scalar field, we find that

$$p = \rho = -\frac{1}{2}\bar{g}^{\rho\sigma}\bar{\phi}_\rho\bar{\phi}_\sigma, \tag{6·4}$$

so we have a stiff fluid. This equivalence applies provided that the derivative of the scalar field is timelike. Otherwise the velocity field defined in Eq. (6·2) would be imaginary.

This is relevant to the gravitational memory problem because we can now interpret the various scenarios discussed in §5 in terms of the *accretion* of a stiff fluid. If the black hole does not accrete at all or accretes very little, this will correspond to strong or weak gravitational memory (Scenarios B and D, respectively). However, if enough accretion occurs to homogenize ϕ, this will correspond to short-term gravitational memory (Scenario C). The faster the accretion, the shorter the memory, so Scenario A corresponds to the idealization in which homogenization is instantaneous.

A simple Newtonian treatment [1] for a general fluid suggests that the accretion rate in the Einstein frame should be

$$\dot{M} = 4\pi\rho R_A^2 v_s, \tag{6·5}$$

where $R_A = GM/v_s^2$ is the accretion radius and v_s is the sound-speed in the accreted fluid. For a stiff fluid, $v_s = c$ and $R_A = GM/c^2$, while $\rho \sim 1/(Gt^2)$ in a Friedmann universe at early times, so we have

$$\frac{dM}{dt} \approx \frac{GM^2}{c^3t^2}. \tag{6·6}$$

This can be integrated to give

$$M \approx \frac{c^3t/G}{1+\frac{t}{t_f}\left(\frac{t_f}{M_f}-1\right)}, \tag{6·7}$$

where M_f is the black hole mass at the time t_f when it formed. If we assume that $M_f = \eta t_f$ with $\eta < 1$, then Eq. (6·7) implies

$$M \to M_f(1-\eta)^{-1} \quad \text{as} \quad t \to \infty. \tag{6·8}$$

If $\eta \ll 1$, the black hole could not grow very much. However, if η is close to 1, which must be the case if $v_s \approx c$, then the black hole could grow significantly. In particular, in the limit $\eta = 1$, Eq. (6·7) implies $M \sim t$, so the black hole grows at the same rate as the universe. This simple calculation suggests that a black hole surrounded by a stiff fluid can accrete enough to grow at the same rate as the Universe.

Since the above calculation neglects the effects of the cosmological expansion, one needs a relativistic calculation to check this. The Newtonian result suggests that one should look for a spherically symmetric *self-similar* solution, in which every dimensionless variable is a function of $z = r/t$, so that it is unchanged by the transformation $t \to at$, $r \to ar$ for any constant a. This problem has an interesting but rather convolved history. By looking for a black hole solution attached to an exact Friedmann solution via a sonic point, Carr & Hawking first showed that there is no such solution for a radiation fluid [8] and the argument can be extended to a general $p = \gamma\rho$ fluid with $0 < \gamma < 1$. Lin et al. [87] subsequently claimed that there is such a solution in the special case $\gamma = 1$. However, Bicknell & Henriksen [88] then showed that this solution is unphysical, in that the density gradient diverges at the event horizon. The solution can be completed only by attaching a Vaidya ingoing radiation solution interior to some surface (i.e., the scalar field has to turn into a null fluid). If one regards this as unphysical, this suggests that the black hole must soon become much smaller than the particle horizon, after which Eq. (6·7) implies there will be very little further accretion. Therefore the stiff fluid analysis suggests that there should be at least *weak* gravitational memory.

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