Lattice Phonon Operators and Selfsimilar Superpotentials*)

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We introduce two different deformations into quantum mechanics by which we obtain the effect of a bounded set of eigenvalues in the case of the harmonic oscillator. The methods we use are based on quantumsymmetries that are provided by a q-Heisenberg algebra. The spectra of the respective discretized quantum oscillators are derived. An unexpected fact occurs: The quantum models based on the used quantumsymmetries show the same bounded spectrum that several dilatative and selfsimilar continuum supermodels do. Thus there is some evidence for a partial equivalence principle between quantumsymmetries and supersymmetries.

§1. Heim-Lorek operators and introduction to the topic

In several investigations concerning the spectrum of quantum harmonic or anharmonic oscillators, it is a well-known fact that the unboundedness of these spectra causes various perturbation theoretical difficulties. However when it comes to discretizing the related Hamilton operators, one can often overcome these difficulties to a certain extent. In this article we will focus on a quantum mechanical toy model by which the conventional phase space is discretized. The used ideas concerning phase space discretizations essentially go back to original approaches in the sense of quantumsymmetries (q-deformations) by Lorek and Wess[9),10)] and by Connes[4)] in the sense of noncommutative geometry. From the mathematical viewpoint, their involved representations are related to $l^2(Z)$-Jacobi operators. In Lorek's thesis from 1995 there were given — for the first time — continuous representations for these discrete operators.[8)] Unconventional ideas by Heim[6)] could thus be mathematically realized in an effective way within a concrete functional analytic model. We refer to the mentioned large class of bilateral Jacobi operators by the name **Heim-Lorek operators**. From the physical viewpoint they are phonon operators A, A^+ in a lattice quantum mechanical theory that allow representations by the conventional continuous Heisenberg variables $\frac{d}{dx}, X$. The discretization itself is provided by the quantumsymmetries that are related to q-Heisenberg algebras. We will deal with a sort of operators in which two different deformations occur. What is finally remarkable is that we end up with an oscillator spectrum being equal to the one of a q-dilatative and selfsimilar supermodel, see Ref. 7). In §2 we briefly prepare the concept of Schrödinger equations based on quantum symmetries and introduce the Heim-Lorek operators in §3. Section 4 is finally devoted to linking the operators of **Heim-Lorek type** to selfsimilar and dilatative quantum mechanical supermodels.

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§2. Discrete Schrödinger equations based on quantumsymmetries

We discretize the Schrödinger equation by means of an exponential lattice which is given by the following geometric progression

$$R_q := \{+q^n, -q^n | n \in Z\}, \qquad 0 < q < 1. \tag{2·1}$$

From now on and in the sequel, N will refer to the natural numbers, Z will refer to integers, C to complex numbers.

Note that the lattice R_q has one more symmetry operation when comparing it to the equidistant lattice. Its symmetry relations are given by

$$\forall x \in R_q : \quad Rx := qx, \quad Sx := -x, \quad Jx := x^{-1}, \quad Cx := \overline{x}, \tag{2·2}$$

the overline symbol denoting complex conjugation. We clearly find invariance of the lattice under these symmetry operations:

$$R(R_q) = R_q, \quad S(R_q) = R_q, \quad J(R_q) = R_q, \quad C(R_q) = R_q. \tag{2·3}$$

In the sequel we will make use of the Hilbert space

$$\mathcal{L}^2(R_q) := \left\{ f : R_q \to C | (1-q) \sum_{n=-\infty}^{+\infty} q^n (f(q^n)\overline{f(q^n)} + f(-q^n)\overline{f(-q^n)}) < \infty \right\} \tag{2·4}$$

being established with a Euclidean scalar product as follows:

$$(f, g) := (1-q) \sum_{n=-\infty}^{+\infty} q^n (f(q^n)\overline{g(q^n)} + f(-q^n)\overline{g(-q^n)}). \tag{2·5}$$

Clearly, a basis of $\mathcal{L}^2(R_q)$ is provided by

$$e_n^\sigma(\tau q^m) := q^{-\frac{n}{2}} (1-q)^{-\frac{1}{2}} \delta_{mn} \delta_{\sigma\tau}, \tag{2·6}$$

where $m, n \in Z, \sigma, \tau \in \{+1, -1\}$, $\delta_{mn}, \delta_{\sigma\tau}$ denoting the Kronecker δ-symbol.

The discrete analogue of the multiplication operator is given by

$$X : D(X) \subseteq \mathcal{L}^2(R_q) \to \mathcal{L}^2(R_q), \quad \varphi \mapsto X\varphi, \quad \forall x \in R_q : \quad X\varphi(x) := x\varphi(x), \tag{2·7}$$

where X shall be densely defined by $D(X)$.

Our aim is roughly spoken: construct symmetric operators

$$H = H(X, R, S, J, C) : D(H) \subseteq \mathcal{L}^2(R_q) \to \mathcal{L}^2(R_q) \tag{2·8}$$

that become the oscillator Hamiltonian $-\frac{d^2}{dx^2} + x^2 + c$ (c is a real number) when the lattice R_q reduces to the set of real numbers while performing the limit $q \to 1$.

We shall basically be concerned with the stationary Schrödinger equation

$$H(X, R, S, J, C)\psi_E = E\psi_E, \qquad \psi_E \in \mathcal{L}^2(R_q), \tag{2·9}$$

where E is canonically denoting the energy eigenvalue.

§3. Constructing the Heim-Lorek operators

Let throughout the following be $0 < q < 1$. For $x \in R_q$ and a given function $\varphi : R_q \to C$ we define

$$R\varphi(x) := \varphi(qx), \qquad L\varphi(x) := \varphi(q^{-1}x), \tag{3·1}$$

$$D\varphi(x) := \frac{\varphi(qx) - \varphi(x)}{qx - x}, \quad X\varphi(x) := x\varphi(x), \quad P := -iL^{\frac{1}{2}}D, \quad u := L^{\frac{1}{2}} \tag{3·2}$$

and formally obtain a q-Heisenberg algebra, being similar to the one in Ref. 5):

$$PX - q^{\frac{1}{2}}XP = -iu, \quad Pu = q^{-\frac{1}{2}}uP, \quad Xu = q^{\frac{1}{2}}uX. \tag{3·3}$$

We easily calculate the following adjointness relations $(m, n \in Z, \sigma, \tau \in \{+1, -1\})$:

$$(De_n^\sigma, e_m^\tau) = (e_n^\sigma, -q^{-1}LDe_m^\tau), \tag{3·4}$$

$$(Xe_n^\sigma, e_m^\tau) = (e_n^\sigma, Xe_m^\tau), \quad (Pe_n^\sigma, e_m^\tau) = (e_n^\sigma, Pe_m^\tau). \tag{3·5}$$

Moreover we introduce

$$(M_q\varphi)(x) := \frac{\varphi(q^{-1}x) - \varphi(-q^{-1}x)}{x}, \qquad (M_q^+\varphi)(x) := \frac{\varphi(qx) + \varphi(-qx)}{x}, \tag{3·6}$$

which yield altogether

$$(Re_n^\sigma, e_m^\tau) = (e_n^\sigma, q^{-1}Le_m^\tau), \qquad (M_qe_n^\sigma, e_m^\tau) = (e_n^\sigma, M_q^+e_m^\tau). \tag{3·7}$$

These definitions were basically given in a recent article by Berg and Ruffing[2)] where the analysis behind these objects is investigated in detail. In the current article we focus more on a physical application in the sense of mathematically modelling quantum structures and especially on an unexpected connection with selfsimilar supermodels.

We address the following problem: Find a Hamiltonian $H = H(X, R, S) = A^+A$ that reduces to the Schrödinger oscillator $-\frac{d^2}{dx^2} + x^2 + c$ in the continuum limit $q \to 1$. Here A^+ is expected to satisfy the following adjointness relations

$$(Ae_i^\sigma, e_j^\tau) = (e_i^\sigma, A^+e_j^\tau), \qquad i, j \in Z, \sigma, \tau \in \{+1, -1\}. \tag{3·8}$$

A tedious investigation by trial and error has shown that the following ansatz for the required **Heim-Lorek operators** is successful in fulfilling the required properties and also appealing in many other respects:

$$A_\gamma := q^{-1}(LD + \gamma LM_q^+ + Lf(X)), \tag{3·9}$$

$$A_\gamma^+ := -D + \gamma M_qR + f(X)R. \tag{3·10}$$

Here, γ denotes a real number and f a real valued function that maps the lattice R_q to R. The operators A_γ and A_γ^+ are actually adjoint with respect to their action on the basis vectors e_i^σ:

$$(A_\gamma e_n^\sigma, e_m^\tau) = (e_n^\sigma, A_\gamma^+ e_m^\tau). \tag{3·11}$$

We assume that the definition ranges of these two operators are chosen as maximal in $\mathcal{L}^2(R_q)$. The following point shall be of particular interest:

Does there exist a $\psi_\gamma \in \mathcal{L}^2(R_q)$ such that the following two equations have a common solution for all $x \in R_q$ and a fixed $\lambda > 0$ (A, A^+ being defined below)?

$$(A_\gamma \psi_\gamma)(x) = 0, \tag{3·12}$$

$$(A_\gamma^+ \psi_\gamma)(x) = \lambda x \psi_\gamma(x), \tag{3·13}$$

$$A_\gamma := q^{-1}(LD + \gamma L M_q^+ + Lf(X)), \tag{3·14}$$

$$A_\gamma^+ := -D + \gamma M_q R + f(X)R. \tag{3·15}$$

Equations (3·12) and (3·13) together with the ansatz (3·14) and (3·15) clearly imitate the situation of the continuum quantum mechanical oscillator,

$$A\varphi(x) = 0 \Leftrightarrow \left(\frac{d}{dx} + x\right) e^{-\frac{1}{2}x^2} = 0, \tag{3·16}$$

$$A^+\varphi(x) = \lambda x \varphi(x) \Leftrightarrow \left(-\frac{d}{dx} + x\right) e^{-\frac{1}{2}x^2} = \sqrt{2} x e^{-\frac{1}{2}x^2}. \tag{3·17}$$

The question is how to construct the functions $f(X)$ and ψ_γ in the case of the stated discretization. Solving the difference equations (3·12) and (3·13) one ends up with

Theorem 1

A solution, up to a constant factor, to the problem (3·12) and (3·13) for the operators in (3·14) and (3·15) is fixed by the difference equation for ψ_γ

$$\forall x \in R_q : \quad \frac{\psi_\gamma^2(qx) - \psi_\gamma^2(x)}{qx - x} = -\lambda x \psi_\gamma^2(x) - 2\gamma x^{-1} \psi_\gamma^2(qx) \tag{3·18}$$

as well as by the expression for the function f:

$$\forall x \in R_q : \quad f(x) = \frac{1 + (2\gamma - 2\gamma q - 1)\sqrt{\frac{1+(1-q)\lambda x^2}{1-(1-q)2\gamma}}}{qx - x}. \tag{3·19}$$

We have to restrict to $\gamma < \frac{1}{2}, \lambda > 0$ and $0 < q < 1$ to guarantee that $\psi_\gamma \in \mathcal{L}^2(R_q)$.

The question now arises how to interpret this result from a physical viewpoint. Especially three limits are of main interest:

1. Let $\lambda = 2$ and $\gamma = 0$. Send $q \to 1$. We then end up with the conventional quantum harmonic oscillator:

$$A = \frac{d}{dx} + x, \qquad A^+ = -\frac{d}{dx} + x, \tag{3·20}$$

$$f(x) = x, \qquad \psi(x) = e^{-\frac{1}{2}x^2}. \tag{3·21}$$

2. Let $\lambda = 2$ and $\gamma = 0$ but q fix and different from 1. This means that we are confronted with the q-harmonic oscillator.

3. Let $\lambda = 2$ and $\gamma \neq 0$ but fix as well as $q \to 1$. This yields a harmonic oscillator in which the energy levels belonging to odd parity are shifted by a constant amount. Going straightforward, one derives by direct calculations the following:

Theorem 2

For any $n \in N_0$ there exist real numbers α_n such that

$$(A^+)^{n+1}\psi_\gamma(x) - \lambda q^n x (A^+)^n \psi_\gamma(x) + \alpha_n (A^+)^{n-1}\psi_\gamma(x) = 0, \tag{3·22}$$

namely

$$\alpha_{2n} = \lambda(1 + 2\gamma(q-1))\frac{q^{2n}-1}{q-1}, \qquad (n \in N_0) \tag{3·23}$$

$$\alpha_{2n+1} = \lambda(1-2\gamma) + \lambda q \frac{q^{2n}-1}{q-1}, \qquad (n \in N_0) \tag{3·24}$$

with the initial conditions

$$\alpha_0 = 0, \qquad \alpha_1 = \lambda - 2\lambda\gamma. \tag{3·25}$$

Using techniques that are similar to the continuum case of conventional quantum mechanics, one derives:[2]

Lemma 1

The functions $\psi_n := (A^+)^n \psi_\gamma$ are in $\mathcal{L}^2(R_q)$ for every $n \in N_0$.

Lemma 2

There exists a real number β such that the functions

$$\varphi_n := q^{\frac{n}{2}} \frac{1}{\beta\sqrt{\prod_{j=1}^{n}\alpha_j}}\psi_n \tag{3·26}$$

are orthonormal with respect to the scalar product in $\mathcal{L}^2(R_q)$. Moreover we have

$$A^+\varphi_n = q^{-\frac{1}{2}}\sqrt{\alpha_{n+1}}\varphi_{n+1}, \tag{3·27}$$

$$A\varphi_n = q^{-\frac{1}{2}}\sqrt{\alpha_n}\varphi_{n-1}, \tag{3·28}$$

$$A^+A\varphi_n = q^{-1}\alpha_n\varphi_n. \tag{3·29}$$

Constructing the important continuous representation of the Heim-Lorek operators

$$A_\gamma = A_\gamma\left(X, \frac{d}{dx}\right), \qquad A_\gamma^+ = A_\gamma^+\left(X, \frac{d}{dx}\right) \tag{3·30}$$

in the sense of Lorek[8] is a fascinating topic that shall be considered in a further article.

In our context, we see in detail that the undeformed limit $\lambda = 2, \gamma \to 0$ as well as $q \to 1$ yields up to a normalization the well-known conventional eigenvalues:

$$\forall n \in N_0 : \lim_{\gamma\to 0}\lim_{q\to 1}\alpha_n = 2n. \tag{3·31}$$

Let us summarize all observations in the following list:

- $q \neq 1, \gamma \neq 0$

$$A_\gamma := q^{-1}(LD + \gamma L M_q^+ + Lf(X)), \tag{3·32}$$

$$A_\gamma^+ := -D + \gamma M_q R + f(X)R, \tag{3·33}$$

$$\forall x \in R_q: \quad f(x) = \frac{1 + (2\gamma - 2\gamma q - 1)\sqrt{\frac{1+(1-q)\lambda x^2}{1-(1-q)2\gamma}}}{qx - x}. \tag{3·34}$$

- $q \to 1, \quad \gamma \to 0$

$$A = \frac{d}{dx} + x, \quad A^+ = -\frac{d}{dx} + x, \quad f(x) = x, \quad E_n = n\lambda. \quad (n \in N_0) \tag{3·35}$$

- $q \neq 1, \quad \gamma \to 0$

$$A = q^{-1}(LD + Lf(X)), \qquad A^+ = -D + f(X)R. \tag{3·36}$$

- $q \to 1, \quad \gamma \neq 0$

$$A_\gamma = \frac{d}{dx} + \gamma M^+ - \frac{2\gamma}{x}, \qquad A_\gamma^+ = -\frac{d}{dx} + \gamma M - \frac{2\gamma}{x}, \tag{3·37}$$

$$(M\varphi)(x) := \frac{\varphi(x) - \varphi(-x)}{x}, \qquad (M^+\varphi)(x) := \frac{\varphi(x) + \varphi(-x)}{x}, \tag{3·38}$$

$$f(x) = \frac{2\gamma}{x}, \qquad E_{2n} = 2n\lambda, \qquad E_{2n+1} = \lambda(1 - 2\gamma) + 2n\lambda. \qquad (n \in N_0) \tag{3·39}$$

Looking at the case $q \neq 1$ and $\gamma = 0$ but fix, one does not only find the q-harmonic oscillator but makes a further observation: Its spectrum is exactly the same as that being related to several selfsimilar supermodels. What however this analogy means is not clear and shall be discussed in some more detail in a forthcoming paper. We state here the results concerning the selfsimilar supermodels. This is going to happen in the very next paragraph.

§4. Selfsimilar superpotentials coming up with Heim-Lorek operators

For the convenience of the reader we now briefly revise the super quantum mechanical formalism by looking at two Hamilton operators

$$H_1 = -\frac{h^2}{2m}\frac{d^2}{dx^2} + V_1(x), \qquad H_2 = -\frac{h^2}{2m}\frac{d^2}{dx^2} + V_2(x), \tag{4·1}$$

and their partner potentials

$$V_1(x) = \frac{1}{2}\left(W^2 - \frac{h}{\sqrt{m}}W'(x)\right), \tag{4·2}$$

$$V_2(x) = \frac{1}{2}\left(W^2 + \frac{h}{\sqrt{m}}W'(x)\right). \tag{4·3}$$

Determination of the superpotential W yields

$$(H_1\varphi)(x) = 0 \Leftrightarrow \tag{4·4}$$

$$-\frac{h^2}{2m}\varphi''(x) + \frac{1}{2}\left(W^2(x) - \frac{h}{\sqrt{m}}W'(x)\right)\varphi(x) = 0 \tag{4·5}$$

and thus $W(x) = -\frac{h}{\sqrt{m}}\frac{d}{dx}\ln(\varphi(x))$.

We obtain generalized phonon operators

$$H_1 = B^+B, \qquad H_2 = BB^+, \tag{4·6}$$

$$B := \frac{1}{\sqrt{2}}\left(W(x) + \frac{h}{\sqrt{m}}\frac{d}{dx}\right), \qquad B^+ := \frac{1}{\sqrt{2}}\left(W(x) - \frac{h}{\sqrt{m}}\frac{d}{dx}\right) \tag{4·7}$$

and in the case of the harmonic oscillator we see

$$V_1(x) = \frac{m\omega^2}{2}x^2 - \frac{h\omega}{2}, \qquad V_2(x) = \frac{m\omega^2}{2}x^2 + \frac{h\omega}{2}, \tag{4·8}$$

$$W(x) = \sqrt{m}\omega x. \tag{4·9}$$

Generalizing the results from above (compare with those in Ref. 7)) we call two potentials **form invariant** if

$$V_2(x, a_1) = V_1(x, a_2) + R(a_1). \tag{4·10}$$

For a long time one has believed that all form invariant potentials are fixed by a **parameter translation**, i.e., by a transformation of the following type (c is a real parameter):

$$a_2 = a_1 + c. \tag{4·11}$$

However there is — and this has been the surprise — a new class of form invariant potentials which is generated by a **parameter dilatation**. In this context, q is acting as a deformation parameter, see Ref. 7):

$$a_2 = qa_1, \qquad 0 < q < 1. \tag{4·12}$$

We illustrate what happens in this case now. To do so, we expand the superpotential as follows

$$W(x, a_1) = \sum_{j=0}^{\infty} g_j(x)a_1^j. \tag{4·13}$$

This time we also give an **analytic ansatz** for the expression $R(a_1)$, namely

$$R(a_1) = \sum_{j=0}^{\infty} R_j a_1^j. \tag{4·14}$$

In the case of dilatative supersymmetry, we are still concerned with the formula

$$R(a_1) = V_2(x, a_1) - V_1(x, a_2), \tag{4·15}$$

which yields now

$$\begin{aligned} R(a_1) &= V_2(x, a_1) - V_1(x, qa_1) \\ &= W^2(x, a_1) - W^2(x, qa_1) + \frac{h}{\sqrt{m}}(W'(x, a_1) + W'(x, qa_1)). \end{aligned} \tag{4·16}$$

Inserting the expansions for the superpotential from above, we find

$$\begin{aligned} &\sum_{n=0}^{\infty} 2R_n a_1^n \\ &\quad = \left(\sum_{n=0}^{\infty}(1+q^n)g_n a_1^n\right)\left(\sum_{n=0}^{\infty}(1-q^n)g_n a_1^n\right) + \frac{h}{\sqrt{m}}\sum_{n=0}^{\infty}(1+q^n)g_n' a_1^n. \end{aligned} \tag{4·17}$$

Comparing the coefficients yields

$$R_0 = \frac{h}{\sqrt{m}} g_0', \tag{4·18}$$

$$R_n = \frac{1}{2}\sum_{i=1}^{\infty}(1+q^{n-i})(1-q^i)g_i g_{n-i} + \frac{h}{2\sqrt{m}}(1+q^n)g_n', \qquad n \in N. \tag{4·19}$$

Defining the abbreviations

$$r_n := \frac{R_n}{1-q^n}, \qquad d_n := \frac{1-q^n}{1+q^n} \tag{4·20}$$

finally leads to a **nonlinear integral equation**, namely

$$g_n = \frac{\sqrt{m}}{h} d_n \int \left[2r_n - \sum_{i=1}^{\infty} g_i(x) g_{n-i}(x)\right] dx. \tag{4·21}$$

The last equation allows a lot of freedom. We restrict ourselves to

$$R_0 = 0, \qquad g_0(x) = 0, \qquad r_n = z\delta_{n,1}, \quad z > 0. \tag{4·22}$$

In this case, we arrive at

$$R(a_1) = R_1 a_1 =: R. \tag{4·23}$$

The **nonlinear integral equation** now leads to the following results

$$g_1(x) = \beta_1 \frac{\sqrt{m}}{h} x, \qquad \beta_1 = 2d_1 r_1 = \frac{2R_1}{1+q} \tag{4·24}$$

and we find for all other functions g_n

$$g_n(x) = \beta_n \left(\frac{\sqrt{m}}{h} x\right)^{2n-1}, \tag{4·25}$$

where the coefficients β_n are fixed by

$$\beta_n = -\frac{d_n}{2n-1}\sum_{i=1}^{n-1}\beta_i\beta_{n-i}. \tag{4·26}$$

The **superpotential** now reads

$$W(x,a_1) = \sum_{j=1}^{\infty}\beta_j a_1^j\left(\frac{\sqrt{m}}{h}x\right)^{2j-1}. \tag{4·27}$$

The **groundstate** belonging to V_1 is given by the formula

$$\psi_0(x,a_1) = Ce^{\sum_{j=1}^{\infty}\frac{\beta_j}{2j}a_1^j(\frac{\sqrt{m}}{h}x)^{2j}}x, \tag{4·28}$$

which is obviously a symmetric function in x.

By direct calculation, one finds for the superpotential:

$$W(x,a_2) = \sqrt{q}W(\sqrt{q}x,a_1). \tag{4·29}$$

It is clearly **selfsimilar** — a remarkable fact in the context of supersymmetric quantum mechanics. The **energies** of the operator H_1 are finally given by

$$E_n \;=\; R\sum_{j=1}^{n-1}q^j \;=\; R\;\frac{1-q^n}{1-q}. \qquad (n\in N_0) \tag{4·30}$$

This is in accordance with the eigenvalues we had found in case of the **Heim-Lorek operators** of type (3·14) and (3·15), when setting $R=1$ and sending $\gamma\to 0$.

Apparently we have established a close analogy between the spectrum of the discretized harmonic oscillator by means of quantumsymmetries and the spectrum of the q-continuum oscillator being constructed by means of supersymmetries. A lot of work still has to be done for a better understanding of the stated facts, also with respect to moment problems of the associated eigenfunctions of A^+A and B^+B. Note that the principal role of moment studies in physics is in general becoming a more and more essential tool. (See, for instance, Ref. 15).)

§5. Conclusions

In this article we have created a link between generalized discretizations of Schrödinger operators, referred to by the name Heim-Lorek operators, and several dilatative and selfsimilar supermodels that were found some years ago. From a physical viewpoint this gives some indication that both, quantumsymmetries which provide the stated discretizations, and supersymmetries, can to a certain extent be considered as two different aspects of the same physical situation. Of course this article can only contribute a further modest step towards investigating these analogies.

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