Instabilities of Coupled Quasi-Homoclinic Limit Cycles

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It has been recently shown that the continuous coupling of non-linear oscillators close to an Andronov homoclinic bifurcation generically leads to two types of instabilities: a self-focusing Kuramoto like instability or a finite wavelength period doubling instability. The purpose of this paper is to study simple examples where these instabilities have been observed. It includes the coupling of two pendula, the sine-Gordon chain and the coupling of dissipative non-linear oscillators.

§1. Introduction

Recent works have shown the generic instability of the homogeneous quasi-homoclinic limit cycles in spatially extended dynamical systems.[6),2),3)] The purpose of this paper is to provide an explicit study of simple models where these phenomena have been observed. In particular we generalize the previous analysis to the case of the coupling between two non-linear oscillators and to the classical problem of a sine-Gordon chain. In the first part we consider the problem of two identical pendula coupled through a linear torque. The second part is devoted to the analysis of the sine-Gordon chain. In the last part we illustrate the geometrical analysis of Risler[7)] in the case of the coupling between two identical non-linear dissipative oscillators. This problem is clearly associated with the synchronization of non-linear oscillators.[5)]

§2. Coupled pendula

2.1. *Classical pendulum*

The equation of motion of a conservative pendulum is given by

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0 \tag{2·1}$$

or equivalently

$$\left(\frac{du}{dt}\right)^2 = H^2 - \sin^2 u, \tag{2·2}$$

where $u = \frac{\theta}{2}$ and H is the total energy of the pendulum. Its solutions are given in terms of the Jacobian elliptic functions.[1)]

1. For $H < 1$, the solutions are periodic (oscillations) with a period given by

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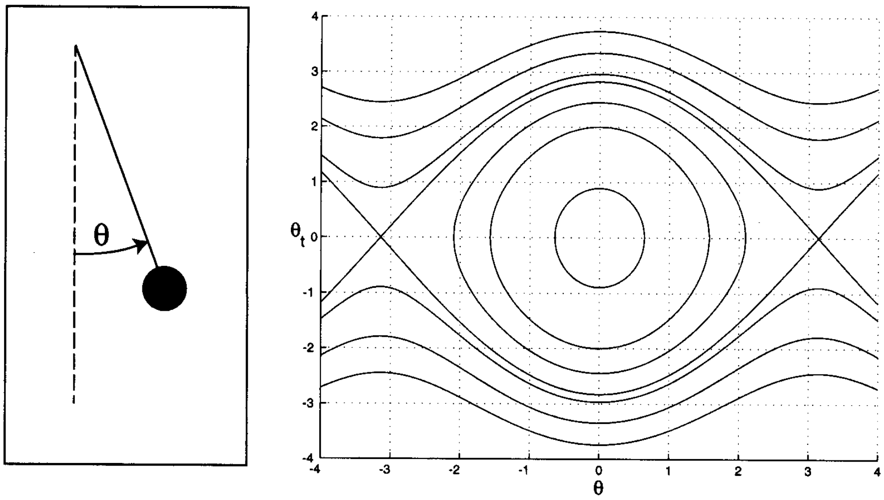

Fig. 1. Pendulum and its phase plane.

$T = 4\mathrm{K}(H^2)$ (K is the complete elliptic integral of the first kind.[1])

$$\sin u(t,H) = \frac{1}{H}\mathrm{sn}(t,H^2).$$

2. For $H=1$, the solution is aperiodic and corresponds to the separatrix (heteroclinic loop)

$$u(t) = \arctan(t).$$

3. For $H > 1$, the solutions are periodic (rotations) with a period given by $T = \frac{2}{H}\mathrm{K}(H^2)$

$$\sin u(t,H) = \mathrm{sn}\left(Ht, \frac{1}{H^2}\right).$$

2.2. *Coupled pendula*

2.2.1. Equations

The equations of two identical pendula coupled by a linear torsion are given by

$$\frac{d^2u_1}{dt^2} + \sin u_1 \cos u_1 = \gamma(u_2 - u_1), \tag{2·3a}$$

$$\frac{d^2u_2}{dt^2} + \sin u_2 \cos u_2 = \gamma(u_1 - u_2). \tag{2·3b}$$

Using the new variables $s = u_1 + u_2$ and $d = u_1 - u_2$, they read

$$\frac{d^2s}{dt^2} + \sin s \cos d = 0, \tag{2·4a}$$

$$\frac{d^2d}{dt^2} + \sin d \cos s = -2\gamma d. \tag{2·4b}$$

2.2.2. Linear stability

The linear stability of a homogeneous solution $u_1(t) = u_2(t) = u_h(t)$ is studied by introducing small deviations to the homogeneous solution $\sigma = s - 2u_h$ and $\delta = d$

and linearizing the equations

$$\frac{d^2\sigma}{dt^2} + (1 - 2\sin^2 u_h(t))\sigma = 0, \tag{2·5a}$$

$$\frac{d^2\delta}{dt^2} + (2\gamma + 1 - 2\sin^2 u_h(t))\delta = 0. \tag{2·5b}$$

These two equations are particular forms of the Hill's equation[4] $\frac{d^2y}{dt^2} + (a+\phi(t))y = 0$ where $\phi(t+T) = \phi(t)$. The stability of $y = 0$ is then given by the value of the trace of the monodromy operator $\tau = \mathrm{tr}(M)$ (see Appendix A).

Zero coupling

We note that $\sigma = 2\frac{du_h}{dt}(t)$ is a solution of Eq. (2·5a) (phase mode). Its characteristic multipliers are both equal to one. The same property is also true for Eq. (2·5b) since $\gamma = 0$.

Small coupling

Let $2\gamma = \epsilon$, where ϵ is a small parameter. A solution of Eq. (2·5b) can be expanded as a power of the coupling constant $\underline{x}(t) = \underline{x}^0(t) + \epsilon\underline{x}^1(t) + \cdots$ of Eq. (2·5b).

At leading order $\underline{x}^0(t) = S(t)\underline{a}$ with $\underline{a} = \underline{x}(t=0)$. Moreover

$$S(T) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$$

The sign of α depends on the value of H. If $H < 1$ (resp. $H > 1$), α is negative (resp. positive) and the period of oscillations increases (resp. decreases) when H increases.

At order ϵ, $\underline{x}^1(t)$ is a solution of equation $\frac{dx^1}{dt} = A(t)\underline{x}^1 + \Gamma\underline{x}^0$ with

$$\Gamma = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$$\underline{x}^1(t) = S(t)B(t)\underline{a}\ ,$$

where $B(t) = \int_0^t S^{-1}\Gamma S dt$. The stability condition is

$$\mathrm{tr}(\epsilon S(T)B(T)) = -\epsilon\alpha m^2 < 0, \tag{2·6}$$

where $m^2 = \int_0^T s_{11}^2 dt$ is a real positive coefficient.

The homogeneous periodic solutions are unstable for $H < 1$ (oscillations), even for small coupling. A simple calculation based on the amplitude equations valid for small oscillations ($\theta_{1,2} = A_0 e^{-i\frac{|A_0|^2}{4}t}$) shows that the instability occurs when $|A_0|^2 > \gamma/2$. We also notice that this instability becomes stronger when we approach the energy of the separatrix ($H = 1$).

Case $H > 1$

The case of the rotating solution ($H > 1$) is less simple to analyze. Let $\rho = \frac{1}{H}$ and $t = \frac{\rho \mathrm{K}(\rho^2)}{\pi}s$. Equation (2·5b) then becomes

$$\frac{d^2u}{ds^2} + [a_\rho(\gamma) + b_\rho p_\rho(s)]u = 0, \tag{2·7}$$

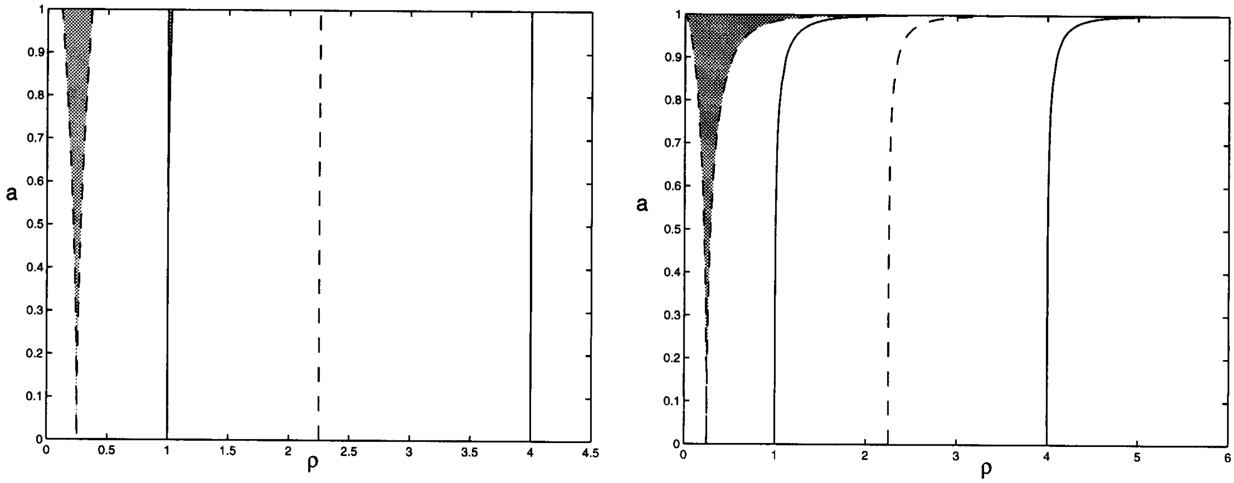

Fig. 2. Stability diagrams.

where $a_\rho(\gamma) = 2\left(\frac{\rho \mathrm{K}(\rho^2)}{\pi}\right)^2\gamma$, $b_\rho = \left(\frac{\rho \mathrm{K}(\rho^2)}{\pi}\right)^2$, $p_\rho(s) = 1 - 2\mathrm{sn}^2(\frac{\mathrm{K}(\rho^2)}{\pi}s, \rho^2)$ and $p_\rho(s)$ is 2π-periodic.

For small ρ, i.e., far from the separatrix, the elliptic function can be approximated by:

$$\begin{aligned} \mathrm{K}(\rho^2) &\sim \frac{\pi}{2} + \frac{\pi}{8}\rho^2, \\ \mathrm{sn}^2\left(\frac{\mathrm{K}(\rho)}{\pi}s, \rho^2\right) &\sim \frac{1}{2} - \frac{1}{2}\cos s. \end{aligned}$$

Equation (2·5b) then becomes the Mathieu equation

$$\frac{d^2u}{ds^2} + (\hat{a} + \hat{b}\cos s)u = 0 \tag{2·8}$$

with $\hat{a} = \frac{\rho^2\gamma}{2}$ and $\hat{b} = \frac{\rho^2}{4}$. In Fig. 2 the stability diagrams for Eq. (2·8) (left) and Eq. (2·7) (right) are shown. (Note that the parameters are a and ρ.) When the coupling increases, the first bifurcation to occur is a period doubling bifurcation (dotted lines). This instability is stronger close to the heteroclinic orbit.

2.3. *Sine-Gordon equation*

The sine-Gordon equation models the continuous limit of an infinite chain of identical pendula linearly coupled

$$\partial_{tt}\theta + \sin\theta = \partial_{xx}\theta. \tag{2·9}$$

Small perturbations with a wavenumber q are introduced in order to study the stability of the homogeneous solution

$$\begin{aligned} u(t,x) &= u_h(t) + \hat{u}(t)\sin qx, \\ v(t,x) &= v_h(t) + \hat{v}(t)\sin qx, \end{aligned}$$

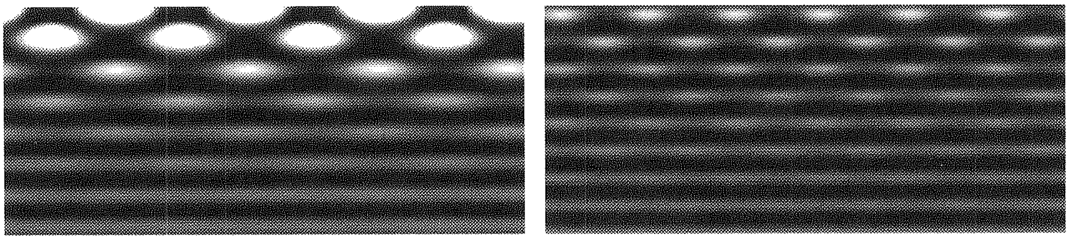

Fig. 3. Direct numerical simulation of sine-Gordon equation. x is abscissa, t is the ordinate, and the grey intensity is for v. On the left side $H = 1.25$, and on the right side $H = 1.6$.

where $u = \frac{\theta}{2}$, $v = \frac{du}{dt}$ and $u_h(t)$ is the solution of the homogeneous problem and $v_h = \frac{du_h}{dt}$. The linearized equations read

$$\frac{d^2\hat{u}}{dt^2} + (q^2 + 1 - 2\sin^2 u_h(t))\hat{u} = 0, \qquad (2\cdot10)$$

which turns out to be the equation in (2·5b).

1. For $H < 1$ (oscillations), the homogeneous solution is unstable for infinitely small wavenumber q. It is the self-focusing Benjamin-Feir instability.
2. For $H > 1$ (rotations), the homogeneous solution is unstable for many wavenumbers q (see Fig. 2). Of course there is a selected wavenumber q_0. In particular the strongest instability occurs at a finite wavelength and a period double. In Fig. 3 we show direct numerical simulation of Eq. (2·10).

§3. Instability of two coupled dissipative non-linear oscillators

We next consider the coupling of two identical dissipative non-linear oscillators, chosen to exhibit an Andronov bifurcation. Some results have already been established concerning the stability of coupled oscillators near the homoclinic bifurcation.[6),2),3)] We use here some geometrical interpretation based on the idea of Risler.[7)] A more detailed study of the neighborhood of the homoclinic bifurcation in the case of continuously coupled system can be found in Ref. 3).

3.1. *Geometrical interpretation*

We study the coupling of two oscillators governed by the following equations:

$$\frac{du}{dt} = v, \qquad (3\cdot1a)$$

$$\frac{dv}{dt} = (\mu - u)v - (u - u^2). \qquad (3\cdot1b)$$

For $0 < \mu < 0.1355$, this system admits a stable limit cycle which disappears at $\mu \simeq 0.1355$ through a homoclinic bifurcation. We consider two oscillators of this type ((u, v) and (u', v')) and we introduce a linear coupling between these two oscillators, the two equations are

$$\frac{du}{dt} = v + \gamma((u' - u) - \beta(v' - v)), \qquad (3\cdot2a)$$

$$\frac{dv}{dt} = (\mu - u)v - (u - u^2) + \gamma(\beta(u' - u) + (v' - v)) \tag{3·2b}$$

and the corresponding equation for (u', v').

We are interested in the stability of the (homogeneous) periodic solution $u = u' = u^h(t)$ and $v = v' = v^h(t)$. We use the variable $\sigma_u = u + u'$, $\sigma_v = v + v'$, $\delta_u = u - u'$ and $\delta_v = v - v'$. Let us introduce $\sigma_u = 2u_h + \widetilde{\sigma_u}$ and $\sigma_v = 2v_h + \widetilde{\sigma_v}$. The linear equations are then given by

$$\frac{d\underline{\sigma}}{dt} = L(t)\underline{\sigma}\ , \tag{3·3a}$$

$$\frac{d\underline{\delta}}{dt} = L(t)\underline{\delta} - 2\gamma C\underline{\delta} \tag{3·3b}$$

with

$$\underline{\sigma} = \begin{pmatrix} \widetilde{\sigma_u} \\ \widetilde{\sigma_v}, \end{pmatrix}, \qquad \underline{\delta} = \begin{pmatrix} \delta_u \\ \delta_v \end{pmatrix}$$

and

$$L(t) = \begin{pmatrix} 0 & 1 \\ -1 - v_h + 2u_h & \mu - u_h \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & -\beta \\ \beta & 1 \end{pmatrix}.$$

Equation (3·3a) describes the stability of the homogeneous limit cycle with respect to homogeneous perturbations. Its Floquet multipliers are 1 (phase multiplier) and λ ($|\lambda| < 1$) (amplitude multiplier). For Eq. (3·3b) the Floquet multipliers are function of γ and C. Of course the Floquet multiplier 1 is critical and may lead to an instability for infinitely small value of the coupling. It is a possible cause of instability but not the only one.

We focus on the evolution of a perturbation (x, y) in Eq. (3·3b) in the local frame $\underline{e_1}(t) = (\frac{du_h(t)}{dt}, \frac{dv_h(t)}{dt})$ and $\underline{e_2}(t) = \mathrm{Rot}_{\pi/2}(\underline{e_1}(t))$. Instead of computing the monodromy map, we will discuss its geometrical effects. The monodromy map is separated into two parts. The one associated to the homogeneous problem and one associated to the coupling matrix (see Fig. 4).

We first consider the homogeneous part (i.e. the part associated with Eq. (3·3a)). The first return map is

$$M = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

The behavior is sketched in Fig. 4. When we approach the homoclinic bifurcation a diverges[3)] and the arrows of Figs. 4(A) and (B) become parallel to the horizontal axis. Case (B) in Fig. 4 is closer to the homoclinic bifurcation than Case (A).

The coupling operator here acts as a rotation in the local frame (Figs. 4(C) and (D)). Depending on the sign of β, the rotation changes its sign. The two cases of rotation are shown in Fig. 4. The monodromy map of the homogeneous system imposes a given sign of the rotation. When one approaches the homoclinic bifurcation, it becomes singular. The coupling matrix may act in an opposite direction inducing an eigendirection with a positive eigenvalue in the local frame. This case corresponds

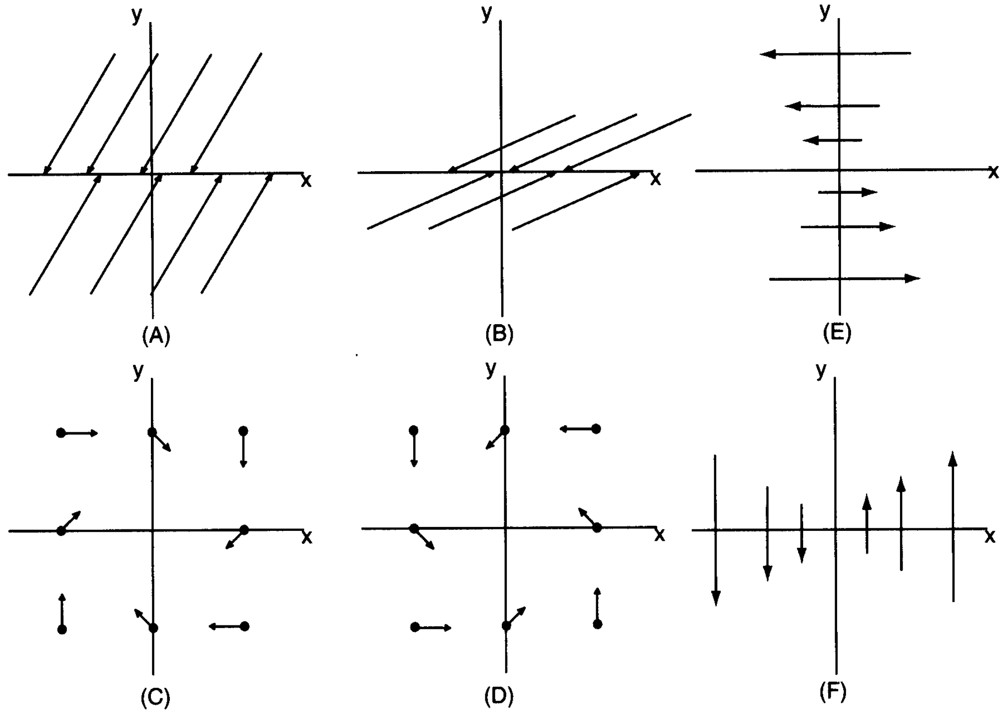

Fig. 4. Dynamics of the perturbation in the local frame. (A), (B) and (E) show the dynamics associated with the homogeneous problem. (C) and (D) show the dynamics induced by coupling.

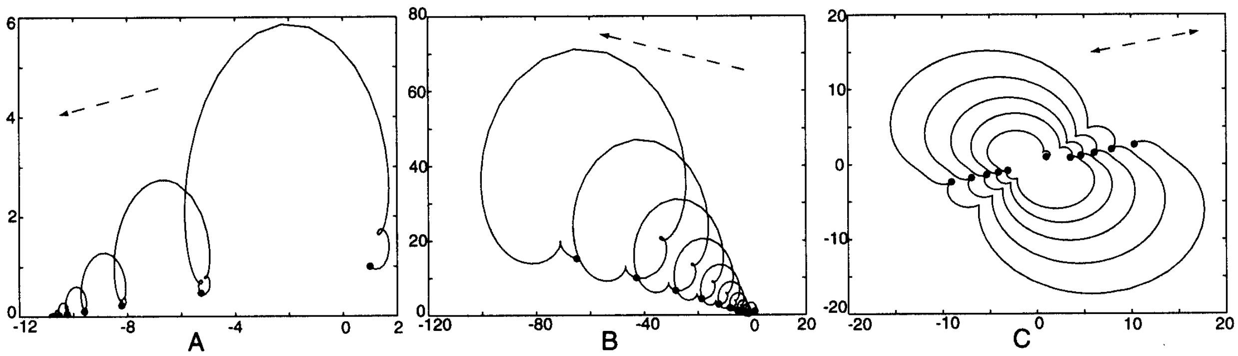

Fig. 5. Real dynamics of the perturbation in the local frame. The points are the iterates of the monodromy map. The initial point is always (1,1).

to the Kuramoto phase instability. It depends only on the rotation due to coupling and on rotation due to the local dynamics. The balance between these two effects does not depend on the intensity of coupling γ. This discussion can be extended away from the homoclinic orbit (it leads for example to the Benjamin-Feir-Kuramoto criterion for the complex Ginzburg-Landau equation [7]). For any coupling terms, the instability will always manifest as one approaches the homoclinic bifurcation.

The coupling matrix and the monodromy map of the synchronized solution may also act cooperatively. Away from the homoclinic bifurcation the coupling has to be large enough in order to induce an instability though a -1 Floquet multiplier. Even in the case of small coupling, the synchronicity will always be lost as we approach the homoclinic bifurcation.

These types of behavior have been illustrated in Fig. 5 where we show the numerical computation of the dynamics of a perturbation of Eq. (3·3b) with $\mu = 0.075$ in all cases. In Case (A), there is no coupling. In Case (B), $\beta = 1$, $\gamma = 0.04$ the Floquet multipliers are 1.51 and 0.09. In Case (C), $\beta = -1$, $\gamma = 0.08$ the Floquet multipliers are -1.14 and -0.04.

Figures 4(E) and (F) show the equivalent representation for the problem of two coupled pendula. In this case the coupling (Fig. 4(F)) is the one of the first part. Figure 4(E) shows the dynamics when $H > 1$.

3.2. *Continuously coupled oscillator*

The previous considerations also hold for a continuous diffusive coupling.[2)] When we approach the homoclinic bifurcation, one of the two instabilities which follow:

1. The Benjamin-Feir-Kuramoto phase instability.
2. A finite wavelength period doubling instability which has been interpreted as the self-parametric forcing of the limit cycle induced by the presence of the saddle fixed point.

3.3. *Non-linear behavior*

The non-linear behavior associated with these bifurcations may be very complicated. For example the system of Eq. (3·2) does exhibit chaotic behavior. There is a complex bifurcation diagram similar to the one of Ref. 6).

In the case of continuous dissipative system the equation which describes the non-linear development of the Kuramoto phase instability is given by

$$\partial_\tau \phi = \epsilon \partial_{xx}\phi - \partial_{xxxx}\phi + (\partial_x \phi)^2 \tag{3·4}$$

with $u(t,x) = u_h(t-\phi)$ while it is for the finite wavelength period doubling instability

$$\partial_\tau A = \mu A \pm |A|^2 A + \alpha A \partial_{XX}\phi + \beta(\partial_X \phi)^2 + \partial_{XX} A, \tag{3·5}$$

$$\partial_\tau \phi = \delta \partial_{XX}\phi + (\partial_X \phi)^2 + \eta |A|^2 \tag{3·6}$$

with $u(t,x) = u_h(t-\phi) + Ae^{ik_0 x}\zeta(t-\phi) + \text{c.c.} + \cdots$ and $\zeta(t)$ is the Floquet eigenvector associated to the multiplier -1.

§4. Conclusions

In this paper we have illustrated the two main instabilities that suffer synchronous quasi-homoclinic oscillations. The desynchronization instability is either the Kuramoto phase instability or a period doubling instability. In the case of a continuous system illustrated by the reversible sine-Gordon chain, the period doubling instability occurs at a finite wavelength. Our results show in particular that these instabilities can be observed in reversible dynamical systems, as for example the coupling of two pendula and the sine-Gordon chain. Of course the non-linear development of the instability depends on the reversibility property of the dynamical system considered.

Acknowledgements

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Appendix A
—— *Hill's Equation* ——

The Hill equation is given by

$$\frac{d^2y}{dt^2} + (a + \phi(t))y = 0 \tag{A·1}$$

or in vector form:

$$\frac{d\underline{x}}{dt} = A(t)\underline{x}\,, \tag{A·2}$$

where $\underline{x}(t) = \begin{pmatrix} y(t) \\ \frac{dy}{dt}(t) \end{pmatrix}$ and $A(t) = \begin{pmatrix} 0 & 1 \\ -a - \phi(t) & 0 \end{pmatrix}$. Its solution is given by $\underline{x}(t) = S(t)\underline{x}(t=0)$, where $S(t)$ is the flow associated with Eq. (A·2).

The spectrum of the monodromy map $M = S(T)$ characterizes the stability of the solution $\underline{x} = 0$. Its characteristic equation is given by

$$\lambda^2 - \mathrm{tr}(M)\lambda + 1 = 0. \tag{A·3}$$

The stability criterion is given by

$$|\mathrm{tr}(M)| \leq 2.$$

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