"Topological" Charge of $U(1)$ Instantons on Noncommutative $\boldsymbol{R}^4$

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Non-singular instantons are shown to exist on noncommutative $\boldsymbol{R}^4$ even in $U(1)$ gauge theory. Their existence is primarily due to the noncommutativity of the coordinates. The integer instanton number on the noncommutative $\boldsymbol{R}^4$ can be understood as winding number of the $U(1)$ gauge field as well as dimension of a certain projection operator acting on the representation space of the noncommutative coordinates.

§1. Introduction

Noncommutativity of space coordinates is expected to prevent singular configuration in field theory. For example, in their pioneering work, [1] Nekrasov and Schwarz constructed an explicit $U(1)$ one-instanton solution on noncommutative $\boldsymbol{R}^4$. It is a typical appearance of the effect of noncommutativity because in ordinary $U(1)$ gauge theory on commutative $\boldsymbol{R}^4$, it can be shown that non-singular instanton cannot exist. For the ordinary $U(1)$ gauge field A on commutative $\boldsymbol{R}^4$, non-trivial instanton number is incompatible with the vanishing of field strength F at infinity:

$$-\frac{1}{8\pi^2}\int_{\boldsymbol{R}^4} FF = -\frac{1}{8\pi^2}\int_{\boldsymbol{R}^4} d(AF) = -\frac{1}{8\pi^2}\int_{S^3_{\text{infinity}}} (AF) = 0. \qquad (1\cdot1)$$

Therefore if $U(1)$ gauge field has non-trivial instanton number, there must be a point where the field configuration becomes singular and gives new surface contribution, other than S^3 at infinity. Then, why $U(1)$ gauge field on noncommutative space managed to have non-trivial instanton charge in the case of Ref. 1) ? Of course it must be due to the noncommutativity of the coordinates, but the situation is not so simple. The difference between ordinary and noncommutative gauge theory is the multiplication of gauge fields (pointwise multiplication vs star product) and gauge field itself is written as function on $\boldsymbol{R}^4$. One can explicitly check that the solution is not singular. On the other hand, naively thinking, the effect of noncommutativity should be suppressed at distance much larger than the scale introduced by the noncommutativity. Therefore the noncommutativity seems irrelevant to the surface term, which is infinitely far away. So even in the noncommutative case it seems impossible to construct non-singular $U(1)$ instantons. However, explicit $U(1)$ instanton solutions do exist. What's wrong with the above arguments ?

The answer is that above naive expectation is wrong. The effect of noncommutativity does not vanish even at the long distance, in the case of $U(1)$ gauge group. The purpose of this article is to give precise explanation of this fact. Along the way we will find a beautiful relation between topological quantity and algebraic quantity on noncommutative space.

Organization of this article is as follows. In §2, as preliminaries, I give reviews on gauge fields on noncommutative $\boldsymbol{R}^4$ and on the ADHM construction of instantons. In §3, I first construct an explicit $U(2)$ one-instanton solution by the ADHM method and consider its small instanton limit. In this article, to make things concrete, I consider the case where the noncommutative parameter is anti-self-dual. Some remarks on other cases are given in the last summary section. Next, I construct a $U(1)$ one-instanton solution. It turns out that $U(1)$ one-instanton solution can be constructed in a similar way to the small instanton limit of the $U(2)$ one-instanton solution. Then I give a topological explanation to the integer instanton number in $U(1)$ gauge theory. Section 4 is devoted to the summary.

This article is based on my earlier works Refs. 2)–5).

§2. Preliminaries

2.1. *Gauge fields on noncommutative* $\boldsymbol{R}^4$

The coordinates x^μ ($\mu = 1, \cdots, 4$) of the noncommutative $\boldsymbol{R}^4$ obey the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{2·1}$$

where the noncommutative parameter $\theta^{\mu\nu}$ is a real constant matrix. By $SO(4)$ rotation in $\boldsymbol{R}^4$ one can set the components of the matrix $\theta^{\mu\nu}$ to zero except $\theta^{12} = -\theta^{21}$ and $\theta^{34} = -\theta^{43}$. I introduce the complex coordinates by

$$z_1 = \hat{x}^2 + i\hat{x}^1, \quad z_2 = \hat{x}^4 + i\hat{x}^3\,. \tag{2·2}$$

Their commutation relations become

$$\begin{aligned} &[z_1, \bar{z}_1] = \zeta_1, \quad [z_2, \bar{z}_2] = \zeta_2, \\ &[z_1, z_2] = [z_1, \bar{z}_2] = 0, \end{aligned} \tag{2·3}$$

where $\zeta_1 = -2\theta^{12}$ and $\zeta_2 = -2\theta^{34}$. In this article I study the case where $\theta^{\mu\nu}$ is anti-self-dual, i.e. $\theta^{12} + \theta^{34} = 0$. This means $\zeta_1 = -\zeta_2$. Further, I set $\zeta_1 > 0$. Then, I define

$$a_1 \equiv \sqrt{\frac{1}{\zeta_1}}z_1, \quad a_1^\dagger \equiv \sqrt{\frac{1}{\zeta_1}}\bar{z}_1\,, \tag{2·4}$$

$$a_2 \equiv \sqrt{\frac{1}{\zeta_1}}\bar{z}_2, \quad a_2^\dagger \equiv \sqrt{\frac{1}{\zeta_1}}z_2\,. \tag{2·5}$$

I realize $a^\dagger$ and a as creation and annihilation operators acting in a Fock space $\mathcal{H}$ spanned by the basis $|n_1, n_2\rangle$:

$$\begin{aligned} &a_1^\dagger\,|n_1, n_2\rangle = \sqrt{n_1+1}\,|n_1+1, n_2\rangle\,, \quad a_1\,|n_1, n_2\rangle = \sqrt{n_1}\,|n_1-1, n_2\rangle\,, \\ &a_2^\dagger\,|n_1, n_2\rangle = \sqrt{n_2+1}\,|n_1, n_2+1\rangle\,, \quad a_2\,|n_1, n_2\rangle = \sqrt{n_2}\,|n_1, n_2-1\rangle\,. \end{aligned} \tag{2·6}$$

The commutation relation (2·1) has automorphismes of the form $\hat{x}^\mu \mapsto \hat{x}^\mu + y^\mu$ (translation), where y^μ is a commuting real number. I denote the Lie algebra of this group by $\underline{\mathbf{g}}$. These automorphismes are generated by the unitary operator T_y

$$T_y \equiv \exp[y^\mu \hat{\partial}_\mu], \tag{2·7}$$

where I have introduced a **derivative operator** $\hat{\partial}_\mu$ by

$$\hat{\partial}_\mu \equiv i B_{\mu\nu} \hat{x}^\nu. \tag{2·8}$$

Here, $B_{\mu\nu}$ is an inverse matrix of $\theta^{\mu\nu}$. The derivative operator $\hat{\partial}_\mu$ satisfies the following commutation relations:

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = i B_{\mu\nu}. \tag{2·9}$$

From (2·9) we obtain

$$T_y\, \hat{x}^\mu\, T_y^\dagger = \hat{x}^\mu + y^\mu. \tag{2·10}$$

For any operator $\hat{O}$ I define **derivative of operator** $\hat{O}$ by the action of $\underline{\mathbf{g}}$:

$$\partial_\mu \hat{O} \equiv \lim_{\delta y^\mu \to 0} \frac{1}{\delta y^\mu} \left(T_{\delta y^\mu} \hat{O} T^\dagger_{\delta y^\mu} - \hat{O} \right) = [\hat{\partial}_\mu, \hat{O}]. \tag{2·11}$$

The action of the exterior derivative d on the operator $\hat{O}$ is defined by

$$d\hat{O} \equiv (\partial_\mu \hat{O})\, dx^\mu. \tag{2·12}$$

Here, the dx^μ are defined in the usual way, i.e. they commute with $\hat{x}^\mu$ and anti-commute among themselves: $dx^\mu dx^\nu = -dx^\nu dx^\mu$. The covariant derivative D is written as

$$D = d + A. \tag{2·13}$$

Here, $A = A_\mu dx^\mu$ is a $U(n)$ gauge field. A_μ is an $n \times n$ anti-Hermite operator-valued matrix. The field strength of A is given by

$$F \equiv D^2 = dA + A^2 \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu. \tag{2·14}$$

I consider the following Yang-Mills action

$$S = \frac{1}{4g^2} (\pi\zeta_1)^2\, \mathrm{Tr}_{\mathcal{H}} \mathrm{tr}_{U(n)}\, F_{\mu\nu} F^{\mu\nu}. \tag{2·15}$$

The action (2·15) is invariant under the following $U(n)$ gauge transformation:

$$A \to U dU^\dagger + U A U^\dagger. \tag{2·16}$$

Here, U is a unitary operator:

$$U U^\dagger = U^\dagger U = \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_n\,, \tag{2·17}$$

where $\mathrm{Id}_{\mathcal{H}}$ is the identity operator acting in $\mathcal{H}$ and Id_n is the $n \times n$ identity matrix. I will also simply write this kind of identity operators as "1", if this is not confusing. The gauge field A is called **anti-self-dual** if its field strength obeys the following equation:

$$F^+ \equiv \frac{1}{2}(F + *F) = 0, \tag{2·18}$$

where $*$ is the Hodge star.*) Anti-self-dual gauge fields minimize the Yang-Mills action (2·15). An instanton is an anti-self-dual gauge field with finite Yang-Mills action (2·15).

One can consider a one-to-one map from operators to ordinary c-number functions on $\boldsymbol{R}^4$. Under this map, noncommutative operator multiplication is mapped to the so-called star product. The map from operators to ordinary functions depends on an operator ordering prescription. Here, I choose the Weyl ordering.

Let us consider Weyl ordered operator of the form

$$\hat{f}(\hat{x}) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{f}_W(k)\, e^{ik\hat{x}}\,, \tag{2·19}$$

where $k\hat{x} \equiv k_\mu \hat{x}^\mu$. For the operator-valued function (2·19), the corresponding **Weyl symbol** is defined by

$$f_W(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{f}_W(k)\, e^{ikx}\,, \tag{2·20}$$

where the x^μ are commuting coordinates of $\boldsymbol{R}^4$. I define Ω_W as a map from operators to corresponding Weyl symbols:

$$\Omega_W(\hat{f}(\hat{x})) = f_W(x). \tag{2·21}$$

One can show the relation $\mathrm{Tr}_{\mathcal{H}}\Big\{\exp(ik\hat{x})\Big\} = (\pi\zeta_1)^2\, \delta^{(4)}(k)$. Then, one obtains

$$(2\pi)^2\sqrt{\det\theta}\, \mathrm{Tr}_{\mathcal{H}}\, \hat{f}(\hat{x}) = (\pi\zeta_1)^2\, \mathrm{Tr}_{\mathcal{H}}\, \hat{f}(\hat{x}) = \int d^4x\, f_W(x). \tag{2·22}$$

The **star product** of functions is defined by

$$f(x) \star g(x) \equiv \Omega_W(\Omega_W^{-1}(f(x))\Omega_W^{-1}(g(x)))\,. \tag{2·23}$$

Since

$$e^{ik\hat{x}}\, e^{ik'\hat{x}} = e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k_\nu'}\, e^{ik\hat{x}+ik'\hat{x}}, \tag{2·24}$$

the explicit form of the star product is given by

$$f(x) \star g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}} f(x)g(x')\Big|_{x'=x}\,. \tag{2·25}$$

*) In this article I only consider the case where the metric on $\boldsymbol{R}^4$ is flat: $g_{\mu\nu} = \delta_{\mu\nu}$.

From the definition (2·23), the star product is associative

$$(f(x) \star g(x)) \star h(x) = f(x) \star (g(x) \star h(x)). \tag{2·26}$$

We can rewrite (2·15) using the Weyl symbols as

$$S = \frac{1}{4g^2} \int \mathrm{tr}_{U(n)} F * F. \tag{2·27}$$

In (2·27), multiplications of the fields are understood to be the star product. The **instanton number** is defined by

$$-\frac{1}{8\pi^2} \int \mathrm{tr}_{U(n)} FF, \tag{2·28}$$

and takes an integral value.

2.2. *Review of the ADHM construction*

The ADHM construction is a way to obtain instanton solutions on $\boldsymbol{R}^4$ from solutions of some quadratic matrix equations. It was generalized to the case of noncommutative $\boldsymbol{R}^4$ in Ref. 1).*) The steps in the ADHM construction of instantons on noncommutative $\boldsymbol{R}^4$ with noncommutative parameter $\theta^{\mu\nu}$, gauge group $U(n)$ and instanton number k is as follows:

1. Matrices (entries are c-numbers):

$$\begin{aligned} B_1, B_2 &: k \times k \quad \text{complex matrices,} \\ I, J^\dagger &: k \times n \quad \text{complex matrices.} \end{aligned} \tag{2·29}$$

2. Solve the ADHM equations:

$$\mu_{\boldsymbol{R}} = \zeta, \qquad \text{(real ADHM equation)} \tag{2·30}$$
$$\mu_{\boldsymbol{C}} = 0. \qquad \text{(complex ADHM equation)} \tag{2·31}$$

Here $\zeta \equiv 2(\theta^{12} + \theta^{34})$ and $\mu_{\boldsymbol{R}}$ and $\mu_{\boldsymbol{C}}$ are defined by

$$\mu_{\boldsymbol{R}} \equiv [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \tag{2·32}$$
$$\mu_{\boldsymbol{C}} \equiv [B_1, B_2] + IJ. \tag{2·33}$$

3. Define $2k \times (2k+n)$ matrix $\mathcal{D}_z$:

$$\begin{aligned} \mathcal{D}_z &\equiv \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix}, \\ \tau_z &\equiv (\, B_2 - z_2,\, B_1 - z_1,\, I\,), \\ \sigma_z^\dagger &\equiv (\, -(B_1^\dagger - \bar{z}_1),\, B_2^\dagger - \bar{z}_2,\, J^\dagger\,). \end{aligned} \tag{2·34}$$

Here, z and $\bar{z}$ are noncommutative *operators*.

*) For more detailed explanations on the ADHM construction on noncommutative $\boldsymbol{R}^4$, see Refs. 5) and 6).

4. Look for all solutions to the equation

$$\mathcal{D}_z \Psi^{(a)} = 0, \quad (a = 1, \cdots, n) \tag{2·35}$$

where $\Psi^{(a)}$ is a $2k + n$ dimensional vector and its entries are *operators*. Here, we must impose the following normalization condition on $\Psi^{(a)}$:

$$\Psi^{(a)\dagger}\Psi^{(b)} = \delta^{ab}\mathrm{Id}_{\mathcal{H}} . \tag{2·36}$$

In the following I will call these zero-eigenvalue vectors $\Psi^{(a)}$ zero-modes.

5. Construct a gauge field by the formula

$$A_\mu^{ab} = \Psi^{(a)\dagger}\partial_\mu \Psi^{(b)}, \tag{2·37}$$

where a and b become indices of the $U(n)$ gauge group. Then, this gauge field is anti-self-dual and has instanton number k.

From the gauge field (2·37), one obtains the following expression for the field strength

$$\begin{aligned} F = &\begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger & \xi^\dagger \end{pmatrix} \\ &\begin{pmatrix} dz_1 \frac{1}{\Box_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{\Box_z} dz_2 & -dz_1 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_2 \frac{1}{\Box_z} dz_1 & 0 \\ -dz_2 \frac{1}{\Box_z} d\bar{z}_1 + d\bar{z}_1 \frac{1}{\Box_z} dz_2 & dz_2 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_1 \frac{1}{\Box_z} dz_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \\ \equiv &F^-_{\mathrm{ADHM}} , \end{aligned} \tag{2·38}$$

where I have written

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(1)} & \cdots & \Psi^{(n)} \end{pmatrix}, \qquad \begin{array}{l} \psi_1 : k \times n \text{ matrix}, \\ \psi_2 : k \times n \text{ matrix}, \\ \xi \ : n \times n \text{ matrix}. \end{array} \tag{2·39}$$

In the above I have suppressed $U(n)$ gauge indices. F^-_{ADHM} is anti-self-dual: $F_{1\bar{1}} + F_{2\bar{2}} = 0$, $F_{12} = 0$.

There is an action of $U(k)$ that does not change the gauge field constructed by the ADHM method:

$$(B_1, B_2, I, J) \mapsto (uB_1u^{-1}, uB_2u^{-1}, uI, Ju^{-1}), \qquad u \in U(k). \tag{2·40}$$

The moduli space $\mathcal{M}_\zeta(k,n)$ of instantons on noncommutative $\boldsymbol{R}^4$ with noncommutative parameter $\theta^{\mu\nu}$, gauge group $U(n)$ and instanton number k is given by

$$\mathcal{M}_\zeta(k,n) = \mu_{\boldsymbol{R}}^{-1}(\zeta) \cap \mu_{\boldsymbol{C}}^{-1}(0)/U(k). \tag{2·41}$$

Here, the action of $U(k)$ is the one given in (2·40). As stated in the previous section, in this article I consider the case where $\zeta = 0$. In this case the moduli space $\mathcal{M}_\zeta(k,n)$ has so-called small instanton singularities which appear when the size of the instanton becomes zero. When $\zeta \neq 0$, the moduli space $\mathcal{M}_\zeta(k,n)$ does not have small instanton singularities.[9)]

§3. "Topological" charge of $U(1)$ instanton on noncommutative $\boldsymbol{R}^4$

3.1. $U(2)$ *one-instanton solution and the small instanton limit*

In this section I first construct a $U(2)$ one-instanton solution by the ADHM method, as a preparation for the $U(1)$ gauge group case. In this case, B_1 and B_2 are 1×1 matrices, i.e. complex numbers. Therefore, commutators with B_1 and B_2 automatically give zero, and a solution to the ADHM equation (2·32) is given by

$$B_1 = B_2, \quad I = (\rho \; 0), \quad J^\dagger = (0 \; \rho) \tag{3·1}$$

with B_1 and B_2 being arbitrary. B_1 and B_2 are parameters that represent the position of an instanton. Due to the translational invariance on noncommutative $\boldsymbol{R}^4$, it is sufficient if we consider the $B_1 = B_2 = 0$ solution. Then, from (2·34) we obtain

$$\mathcal{D}_z = \begin{pmatrix} -z_2 & -z_1 & \rho & 0 \\ \bar{z}_1 & -\bar{z}_2 & 0 & \rho \end{pmatrix}. \tag{3·2}$$

A solution Ψ to the equation $\mathcal{D}_z \Psi = 0$ is given by

$$\Psi = \begin{pmatrix} \Psi^{(1)} \; \Psi^{(2)} \end{pmatrix},$$

$$\Psi^{(1)} = \begin{pmatrix} \rho \\ 0 \\ z_2 \\ -\bar{z}_1 \end{pmatrix} \frac{1}{\sqrt{z_1\bar{z}_1 + \bar{z}_2 z_2 + \rho^2}} = \begin{pmatrix} \rho \\ 0 \\ \sqrt{\zeta_1}a_2^\dagger \\ -\sqrt{\zeta_1}a_1^\dagger \end{pmatrix} \frac{1}{\sqrt{\zeta_1(\hat{N}+2)+\rho^2}},$$

$$\Psi^{(2)} = \begin{pmatrix} 0 \\ \rho \\ z_1 \\ \bar{z}_2 \end{pmatrix} \frac{1}{\sqrt{\bar{z}_1 z_1 + z_2\bar{z}_2 + \rho^2}} = \begin{pmatrix} 0 \\ \rho \\ \sqrt{\zeta_1}a_1 \\ \sqrt{\zeta_1}a_2 \end{pmatrix} \frac{1}{\sqrt{\zeta_1\hat{N}+\rho^2}}. \tag{3·3}$$

Here, $\hat{N} \equiv a_1^\dagger a_1 + a_2^\dagger a_2$. The zero-mode Ψ is normalized as in (2·36):

$$\Psi^\dagger \Psi = \begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & 0 \\ 0 & \mathrm{Id}_{\mathcal{H}} \end{pmatrix}. \tag{3·4}$$

The gauge field is given by (2·37):

$$A_\mu(\hat{x}) = \Psi^\dagger \partial_\mu \Psi. \tag{3·5}$$

The explicit form of the field strength can be obtained from (2·38):

$$F^-_{1\bar{1}\ \mathrm{ADHM}} = -F^-_{2\bar{2}\ \mathrm{ADHM}} = \begin{pmatrix} \frac{\rho^2}{(\zeta_1(\hat{N}+1)+\rho^2)(\zeta_1(\hat{N}+2)+\rho^2)} & 0 \\ 0 & -\frac{\rho^2}{\zeta_1(\hat{N}+\rho^2)(\zeta_1(\hat{N}+1)+\rho^2)} \end{pmatrix},$$

$$F^-_{1\bar{2}\ \mathrm{ADHM}} = -F^{-\,\dagger}_{2\bar{1}\ \mathrm{ADHM}} = \begin{pmatrix} 0 & -\frac{2\rho^2}{(\zeta_1(\hat{N}+1)+\rho^2)\sqrt{\zeta_1(\hat{N}+\rho^2)}\sqrt{\zeta_1(\hat{N}+2)+\rho^2}} \\ 0 & 0 \end{pmatrix}. \tag{3·6}$$

From (3·6) one can observe that the parameter ρ characterises the size of the instanton.*)

Now, let us consider the small instanton limit, i.e. $\rho \to 0$. The moduli space (2·41) becomes singular at $\rho = 0$. When $\rho = 0$, the zero-mode Ψ takes the following form:

$$\Psi = \begin{pmatrix} \Psi^{(1)} & \Psi^{(2)} \end{pmatrix},$$

$$\Psi^{(1)} = \begin{pmatrix} 0 \\ 0 \\ a_2^\dagger \\ -a_1^\dagger \end{pmatrix} \frac{1}{\sqrt{\hat{N}+2}}, \quad \Psi^{(2)} = \begin{pmatrix} 0 \\ |0,0\rangle\langle 0,0| \\ a_1 \frac{1}{\sqrt{\hat{N}_{\neq 0}}} \\ a_2 \frac{1}{\sqrt{\hat{N}_{\neq 0}}} \end{pmatrix}, \tag{3·7}$$

where $\frac{1}{\sqrt{\hat{N}_{\neq 0}}}$ is defined by

$$\frac{1}{\sqrt{\hat{N}_{\neq 0}}} \equiv \sum_{(n_1,n_2)\neq(0,0)} \frac{1}{\sqrt{n_1+n_2}} |n_1,n_2\rangle\langle n_1,n_2| \,. \tag{3·8}$$

Thus when $\rho = 0$, the explicit form of the gauge field is given by

$$A_\mu(\hat{x}) = U^\dagger \partial_\mu U + (1-q)\,\partial_\mu\,(1-q)\,, \tag{3·9}$$

where

$$U \equiv \frac{1}{\sqrt{\hat{N}+1}} \begin{pmatrix} a_2^\dagger & a_1 \\ -a_1^\dagger & a_2 \end{pmatrix} \tag{3·10}$$

and

$$q \equiv \begin{pmatrix} \mathrm{Id}_{\mathcal{H}} & 0 \\ 0 & \mathrm{Id}_{\mathcal{H}} - |0,0\rangle\langle 0,0| \end{pmatrix}. \tag{3·11}$$

Note that q is a projection operator: $q^2 = q$, $q^\dagger = q$. U satisfies the following equations:

$$UU^\dagger = \mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Id}_2, \quad U^\dagger U = q. \tag{3·12}$$

This means U is a partial isometry and gives a one-to-one map between $q\mathcal{H}^{\oplus 2}$ and $\mathcal{H}^{\oplus 2}$. In the $r \equiv \sqrt{x_\mu x^\mu} \to \infty$ limit the Weyl symbol of U becomes

$$\Omega_W(U) \to \frac{1}{r} \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix}. \tag{3·13}$$

*) However, on noncommutative $\boldsymbol{R}^4$ the functional form of the $\mathrm{tr}_{U(2)} FF$ depends on gauge choice. Here I have chosen a gauge where the commutative limit is smooth.

U is a generator of the winding number $\pi_3(U(2))$. Thus U gives a relation between topological quantity (winding number) and algebraic quantity (dimension of projection).

The field strength becomes

$$F_{\mu\nu}(\hat{x}) = i(1-q)B_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & |0,0\rangle\langle 0,0| \end{pmatrix} iB_{\mu\nu}. \tag{3·14}$$

Note that $\Omega_W(|0,0\rangle\langle 0,0|) = 4e^{-\frac{2}{\zeta_1}r^2}$. Thus the $\rho = 0$ corresponds to the "minimal size" instanton. The Weyl symbol of the field strength in this case is a Gaussian function concentrated at the origin, with spreading of order $\sim \sqrt{\zeta_1}$. It is explicitly non-singular. From (2·22), (2·28) and (3·14), the instanton number is equal to the dimension of the projection $1-q$.

3.2. "Topological" charge of $U(1)$ instanton on noncommutative $\boldsymbol{R}^4$

Now let us consider the case where the gauge group is $U(1)$, which is our main subject. The form of the instanton in this case can be anticipated from the small instanton solution in $U(2)$ case (3·9). We look for U which satisfy

$$UU^\dagger = \mathrm{Id}_{\mathcal{H}}, \qquad U^\dagger U = \mathrm{Id}_{\mathcal{H}} - |0,0\rangle\langle 0,0| \equiv p, \tag{3·15}$$

and construct a gauge field by

$$A_\mu = U^\dagger \hat{\partial}_\mu U + (1-p)\partial_\mu(1-p). \tag{3·16}$$

It can be shown that if U satisfies Eq. (3·15), the gauge field (3·16) is anti-self-dual (when the noncommutative parameter is anti-self-dual).[7),4)] There are infinitely many operators which satisfy (3·15), but they are all gauge equivalent. Following is a one choice for such U:

$$\begin{aligned} U = U_1 &\equiv (1 - |0\rangle\langle 0|_2) + \sum_{n_1=0}^{\infty} |n_1,0\rangle\langle n_1+1,0| \\ &= (1 - |0\rangle\langle 0|_2) + |0\rangle\langle 0|_2 \frac{1}{\sqrt{\hat{n}_1+1}} a_1, \end{aligned} \tag{3·17}$$

where $|m\rangle\langle n|_2 \equiv \sum_{n_1=0}^{\infty} |n_1,m\rangle\langle n_1,n|$ and $\hat{n}_1 \equiv a_1^\dagger a_1$. U_1 gives a one-to-one map between $p\mathcal{H}$ and $\mathcal{H}$, as shown in Fig. 1.

Now let us calculate the instanton number. It is written as a surface term in the same way as in the commutative case:

$$-\frac{1}{8\pi^2}\int_{\boldsymbol{R}^4} FF = -\frac{1}{8\pi^2}\int_{\boldsymbol{R}^4} dK = -\frac{1}{8\pi^2}\int_{\text{surface at }\infty} K, \tag{3·18}$$

where

$$K \equiv AdA + \frac{2}{3}A^3 = AF - \frac{1}{3}A^3. \tag{3·19}$$

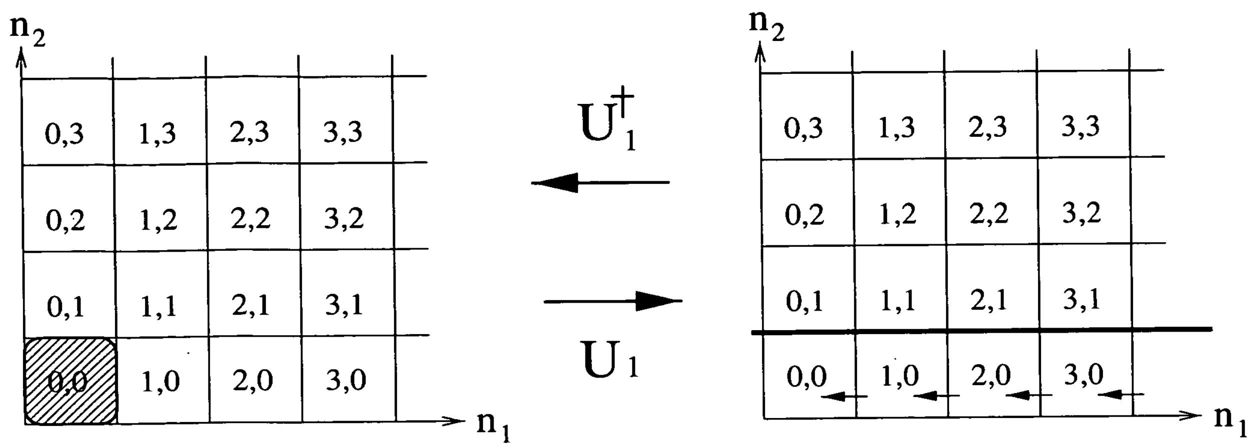

Fig. 1.

Notice that the Weyl symbol of $1-p_2 = |0\rangle\langle 0|_2$ appearing in U_1 does not vanish around the $z_2=0$ plane even if we take r to infinity:

$$\Omega_W(|0\rangle\langle 0|_2) = e^{-\frac{2}{\zeta_1}z_2\bar{z}_2}. \qquad (3\cdot 20)$$

From (3·20) we observe that the gauge field exponentially damps as $r_2 \equiv |z_2| \to \infty$. Therefore in order to calculate the surface term we can choose $r_1 \equiv |z_1| = R_1$ (const.) surface and take R_1 to ∞.

$$\int_{\boldsymbol{R}^4} dK = \int_{r_1=R_1} K = -\frac{1}{3}\int_{r_1=R_1} A^3. \quad (R_1 \to \infty) \qquad (3\cdot 21)$$

Let us consider $r_1 \to \infty$ behaviour of U_1. I introduce the polar coordinates on z_1 plane:

$$z_1 = r_1 e^{i\phi}. \qquad (3\cdot 22)$$

Then in the $r_1 \to \infty$ limit the Weyl symbol of $\frac{1}{\sqrt{\hat{n}_1+1}}a_1$ becomes

$$\Omega_W\left(\frac{1}{\sqrt{\hat{n}_1+1}}a_1\right) \to e^{i\phi}. \quad (r_1 \to \infty) \qquad (3\cdot 23)$$

Equation (3·23) essentially explains the topological origin of the instanton number. From (3·23) we obtain

$$U_1 \to p_2 + (1-p_2)e^{i\phi} = (1-p_2)(e^{i\phi}-1)-1. \quad (r_1 \to \infty) \qquad (3\cdot 24)$$

At large r_1, the gauge field becomes

$$A_{r_1} \to 0, \qquad (3\cdot 25)$$

$$A_\phi \to U_1^\dagger \partial_\phi U_1 = i(1-p_2) = i\,|0\rangle\langle 0|_2\,, \qquad (3\cdot 26)$$

$$A_{z_2} \to U_1^\dagger \partial_{z_2} U_1 = -e^{i\phi}(e^{-i\phi}-1)\sqrt{\frac{1}{\zeta_1}}\,|0\rangle\langle 1|_2\,, \qquad (3\cdot 27)$$

$$A_{\bar{z}_2} \to U_1^\dagger \partial_{\bar{z}_2} U_1 = e^{-i\phi}(e^{i\phi}-1)\sqrt{\frac{1}{\zeta_1}}\,|1\rangle\langle 0|_2\,. \qquad (r_1 \to \infty) \qquad (3\cdot 28)$$

It is convenient to use Weyl symbol for the calculation in z_1 plane and operator calculus for z_2 plane:

$$\int_{r_1=R_1} A^3 = \int_0^{2\pi} d\phi\, 2i\,(\pi\zeta_1)\,\mathrm{tr}_2 A_\phi A_{z_2} A_{\bar{z}_2} \times 3. \quad (R_1 \to \infty) \qquad (3\cdot29)$$

Notice that $\int dz_2 d\bar{z}_2 = \int 2i dx_1 dx_2 = 2i\,(\pi\zeta_1)\,\mathrm{tr}_2$, and $\mathrm{tr}_2 A_\phi A_{\bar{z}_2} A_{z_2} = 0$ (give care to the ordering). Thus

$$\begin{aligned}\int_{r_1=R_1\to\infty} A^3 &= 2\pi \int_0^{2\pi} d\phi\,(e^{i\phi}-1)(e^{-i\phi}-1) \times 3 \\ &= 24\pi^2,\end{aligned} \qquad (3\cdot30)$$

and the instanton number becomes

$$-\frac{1}{8\pi^2}\int FF = \frac{1}{8\pi^2}\int \frac{1}{3}A^3 = 1. \qquad (3\cdot31)$$

Since the gauge field only has $|0\rangle\langle 0|_2$, $|1\rangle\langle 0|_2$ and $|0\rangle\langle 1|_2$ components, the trace over n_2 essentially reduces the instanton number to the winding number of the gauge field A_ϕ around S^1 on z_1 plane (Fig. 2). Thus the instanton number is characterized by $\pi_1(U(1))$.*) Indeed in the ϕ integration, only the constant part contributes, and A_{z_2} and $A_{\bar{z}_2}$ only give appropriate numerical factor. This explains the origin of the integer instanton number of the noncommutative $U(1)$ instanton.

The above discussion is a gauge dependent description. However, notice that any operator that satisfies (3·15) are gauge equivalent and necessarily introduces surface contribution.

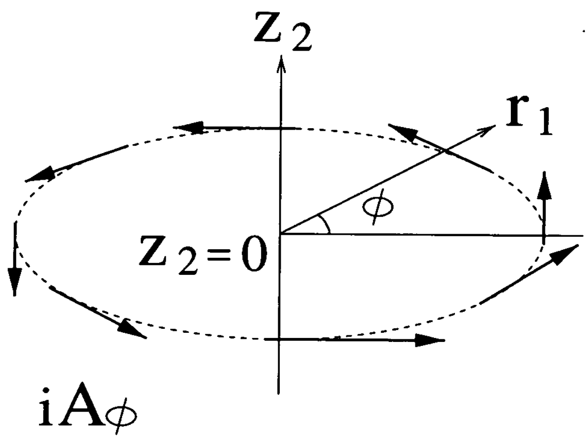

Fig. 2. At large r_1 the gauge field concentrates around $z_2 = 0$ plane. After the integration over z_2 the instanton number reduces to the winding number of gauge field around S^1 on z_1 plane.

§4. Summary

In this article I showed that the integer instanton number of the $U(1)$ instanton on noncommutative $\boldsymbol{R}^4$ can be understood as topological winding number. I first constructed a $U(2)$ one-instanton solution by the ADHM method and studied its

*) The star product has rigid structure and it is quite difficult to consider continuous deformations of the noncommutative $\boldsymbol{R}^4$. However, after fixing the gauge, the effect of the star product in the direction of the z_1 plane disappears at large r_1. Then, we may regard this $\pi_1(U(1))$ as "topological" quantity on the z_1 plane which is invariant under continuous deformations of S^1 at infinity.

small instanton limit. In this limit the solution is essentially described by the partial isometry. The partial isometry is a map from $\mathcal{H}^{\oplus 2}$ to $q\mathcal{H}^{\oplus 2}$ where q is a projection operator. On the other hand the partial isometry is a generator of the instanton winding number classified by $\pi_3(U(2))$. The instanton number can be understood as the dimension of the projection $1-q$, as well as the winding number. From the explicit $U(2)$ one-instanton solution, one could anticipate the form of the $U(1)$ solution. $U(1)$ instanton solution is similar to the small instanton limit of the $U(2)$ solution and it is also essentially described by partial isometry. The partial isometry in this case is a map from $\mathcal{H}$ to $p\mathcal{H}$ where p is a projection operator, and the instanton number is classified by $\pi_1(U(1))$. It also coincides with the dimension of the projection $1-p$. It is straightforward to extend the discussions in this article to the case of multi-instanton solutions.

In this article I studied the case where the noncommutative parameter $\theta^{\mu\nu}$ is anti-self-dual.[7),4)] However, the arguments on the topological origin of $U(1)$ instantons are essentially the same for general $\theta^{\mu\nu}$, although the form of the instanton solution is quite different for different $\theta^{\mu\nu}$. Before ending this article, I make a brief overview of the original developments of the subject studied in this article.

The explicit $U(1)$ instanton solution on noncommutative $\boldsymbol{R}^4$ was first given in Ref. 1) in the case where the noncommutative parameter is self-dual. However, this solution was written in a noncommutative analog of the singular gauge. In noncommutative case, this can be defined *without singularity* although this is an analog of the singular gauge in commutative case, but appropriate modification of the covariant derivative is necessary.[2)] In this noncommutative analog of the singular gauge, a one-to-one correspondence[*)] between projection operators and $U(1)$ instanton naturally appears[2)] and it is a noncommutative geometric description of the ideal described in Refs. 10) and 11). By extending the notion of unitary gauge equivalence to Murray-von Neumann equivalence of projections, one can consider generalized gauge transformation from singular-type gauge to usual-type gauge.[8),3),6)] The existence of the usual-type gauge means that in principle one can avoid the singular gauge which requires the modification of the covariant derivative. However, in the case of $U(1)$ gauge group, since there is no commutative counterpart of $U(1)$ instanton, it is almost unavoidable to take two steps, first construct an instanton in singular-type gauge and then transform it to usual-type gauge. Therefore the results in Refs. 2),3) are useful also in this respect. The generalized gauge transformation is described by partial isometry. The important point is that the partial isometry gives a relation between topological quantity and algebraic quantity.[3)] Although the way of the appearance of the partial isometry is slightly different for different noncommutative parameter, this essential point is the same. In Ref. 3) I discussed the case of $U(2)$ gauge group as an example. In the case of $U(1)$ gauge group, the partial isometry takes a bit unfamiliar form because there is no commutative counterparts to this case. Hence the peculiar nature of the noncommutativity typically appears in this case. The explicit form of the partial isometry which is suitable for understanding

[*)] This correspondence is slightly different from the case discussed in this article where the noncommutative parameter is anti-self-dual.

the topological origin of the $U(1)$ instantons on noncommutative $\boldsymbol{R}^4$ was given in Ref. 5), and it is essentially the same to the one discussed in this article.

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