On the Second Variation of Free Energies and Nonlinear Fokker-Planck Equations Involving Periodic Variables

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The stability of stationary solutions of nonlinear Fokker-Planck equations is discussed. The equivalence between Lyapunov's direct method and linear stability analysis for systems with periodic random variables is shown by means of a H-theorem based on the second variation of free energies.

§1. Introduction

Generalized Fokker-Planck equations that are nonlinear with respect to probability densities have been discussed in various contexts. For example, they have been studied to describe synchronization[1)–3)] (for a review see Ref. 4)), ferromagnetic phase transitions,[5)] nonextensive thermodynamical systems,[6),7)] and ion channels of neurons.[8)] While linear Fokker-Planck equations usually exhibit unique stationary solutions, nonlinear Fokker-Planck equations can exhibit multiple stationary solutions.[5),9)] Some of these solutions are stable in the sense that perturbations of the solutions vanish in the long time limit. Other stationary solutions may be unstable in the sense that perturbations do not decay. In literature there are two methods to investigate the stability of stationary solutions of Fokker-Planck equations: linear stability analysis and Lyapunov's direct method. So far, however, the relationship between these methods has not been explored in general. In this article, for systems with periodic random variables we will show that they are equivalent. To this end, we will introduce a H-theorem involving the second variation of free energy functionals.

§2. Linear stability analysis and Lyapunov's direct method

2.1. *Nonlinear Fokker-Planck equations and Lyaponov's direct method*

Let us consider a stochastic process described by the random vector $\boldsymbol{X}(t) = (X_1, \cdots, X_M)$ defined on the phase space Ω. Let $P(\boldsymbol{x}, t; u) = \delta(\boldsymbol{x} - \boldsymbol{X}(t))$ denote the corresponding probability density with the initial distribution u: $P(\boldsymbol{x}, t_0; u) = u(\boldsymbol{x})$. We consider systems with free energy functionals given by

$$F[P] = U[P] - QS[P] \ , \tag{2·1}$$

where $U[P]$ describes an energy measure, $Q > 0$ measures the strength of the fluctuations to which the system is subjected, and $S[P] = B[\int_\Omega s(P)\, d^M x]$ is an entropy measure. Using the concepts of linear nonequilibrium thermodynamics, the evolution

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equation

$$\frac{\partial}{\partial t}P(\boldsymbol{x},t;u) = -\mathrm{div}\boldsymbol{J}(\boldsymbol{x},t) = -\mathrm{div}[\boldsymbol{I}(\boldsymbol{x},t)P] + \frac{\partial}{\partial x_i}\left[M_{ik}(\boldsymbol{x},t)P\frac{\partial}{\partial x_k}\frac{\delta F}{\delta P}\right] \tag{2·2}$$

can be derived,[10),11)] where $\boldsymbol{I} = (I_1,\cdots,I_M)$ is a drift vector of a conservative system, M_{ik} is a semi-positive definite diffusion matrix, and $\boldsymbol{J} = (J_1,\cdots,J_M)$ can be regarded as both a probability current and a thermodynamic flux. Note that above and in what follows we use the Einstein summation convention. We require that $\boldsymbol{I}$ satisfies

$$\mathrm{div}\boldsymbol{I} = 0\ ,\ \boldsymbol{I}\cdot\nabla\frac{\delta U}{\delta P} = 0\ . \tag{2·3}$$

Then, the integral $\int_\Omega \boldsymbol{I}\cdot\boldsymbol{X}^{\mathrm{th}}P\,d^Mx$ vanishes, where $\boldsymbol{X}^{\mathrm{th}}(\boldsymbol{x},t) = -\partial\delta F/\delta P\partial x_i$ denotes a thermodynamic force vector, which means that $\boldsymbol{I}$ is indeed related to a conservative system and does not contribute to an intrinsic entropy production.[11)] This consideration holds provided that the surface integral $\mathcal{B}_1 = \oint_{\partial\Omega} L[s(P)]\boldsymbol{I}\cdot\boldsymbol{n}\,dA$ with $L[s(P)] = s - Pds/dP$ vanishes. In fact, for periodic boundary conditions we have $\mathcal{B}_1 = 0$. For natural boundary condition we need to examine $\mathcal{B}_1$ for each case separately. For example, in the linear case given by $U = \int_\Omega U(\boldsymbol{x})P\,d^Mx$ and $S = {}^{\mathrm{BGS}}S = -\int_\Omega P\ln P\,d^Mx$ one obtains $\mathcal{B}_1 = 0$.

Let us show now that $\boldsymbol{X}^{\mathrm{th}} = 0$ defines stationary probability densities of Eq. (2·2). $\boldsymbol{X}^{\mathrm{th}} = 0$ implies $\delta F/\delta P = \mu =$ constant. Consequently, stationary probability densities are defined by[11),12)]

$$P_{\mathrm{st}}(\boldsymbol{x};u) = \left[\frac{ds}{dz}\right]^{-1}\left\{\frac{\delta U/\delta P - \mu}{Q\,dB/dz}\right\} \tag{2·4}$$

with $dB(z)/dz$ at $z = \int_\Omega s(P_{\mathrm{st}})\,d^Mx$, where $[ds(z)/dz]^{-1}$ denotes the inverse of $ds(z)/dz$. Note that in general Eq. (2·4) is an implicit description for P_{st} that admits for multiple solutions (i.e., multiple stationary probability densities). In the aforementioned linear case, Eq. (2·4) describes the Boltzmann distribution of $U_0(\boldsymbol{x})$. From Eq. (2·4) it is clear that P_{st} can be written as $P_{\mathrm{st}}(\boldsymbol{x};u) = f(\delta U/\delta P)$. Consequently, we have $\mathrm{div}[\boldsymbol{I}P_{\mathrm{st}}] = \boldsymbol{I}\nabla P_{\mathrm{st}} = [f(z)/dz]\boldsymbol{I}\nabla\delta U/\delta P = 0$ and P_{st} substituted into the right hand side of Eq. (2·2) indeed yields $\partial P/\partial t = 0$. Using Eqs. (2·1) and (2·2) we obtain

$$\frac{d}{dt}F = \int_\Omega \frac{\delta F}{\delta P}\frac{\partial}{\partial t}P\,d^Mx = -\int_\Omega PM_{ik}X_i^{\mathrm{th}}X_k^{\mathrm{th}}d^Mx \le 0 \tag{2·5}$$

provided that the surface term $\mathcal{B}_2 = \oint_{\partial\Omega}[\delta F/\delta P]\,\boldsymbol{J}\cdot\boldsymbol{n}\,dA$ vanishes. Obviously, for periodic boundary conditions we have $\mathcal{B}_2 = 0$. In the case of natural boundary conditions we obtain $\mathcal{B}_2 = 0$, for example, in the linear case mentioned earlier. The inequality sign is due to the assumed semi-positivity of $\mathcal{M}$. Note that the change of entropy d_iS due to intrinsic processes[13)] is related to dF by $dF = -Qd_iS$. Therefore, Eq. (2·5) states that $d_iS \ge 0$ for all processes described by the nonlinear Fokker-Planck equation (2·2). From Eq. (2·5) we obtain the implication $P = P_{\mathrm{st}} \Rightarrow dF/dt = 0$. In addition, if M_{ik} is positive definite we have $dF/dt = 0 \Rightarrow \boldsymbol{X}^{\mathrm{th}} = 0 \Rightarrow P = P_{\mathrm{st}}$.

In many cases, we can show that F is bounded from below like $F \geq F_{\min}$. Then, the relations

$$\frac{d}{dt}F \leq 0\ , \quad F \geq F_{\min}\ , \quad \frac{d}{dt}F = 0 \Leftrightarrow P = P_{\rm st} \tag{2·6}$$

constitute a H-theorem[14)] stating that in the long time limit every transient solution converges to a stationary one. Note that in the linear case this H-theorem reduces to the conventional one involving the Kullback distance measure $K[P, P_{\rm st}] = \int_\Omega P \ln[P/P_{\rm st}]\, d^M x$.[15),16)]

Nonlinear Fokker-Planck equations of the form (2·2) that describe mean field models often exhibit multiple stationary probability densities.[5),9)] Using the free energy measure (2·1) these probability densities can be classified into stable and unstable ones. Stable ones correspond to minima of F, whereas unstable ones correspond to maxima and saddle points of F.[17),18)] In particular, the second variation of F denoted by $\delta^2 F[P_{\rm st}](\epsilon)$, where ϵ is a small deviation, can be used to determine the character of extrema of F. Consequently, by means of the expression $\delta^2 F[P_{\rm st}](\epsilon)$ and the relations (2·6), we can analyze the stability of stationary probability densities of Eq. (2·2). This analysis is based on Lyapunov's direct method. F is the Lyapunov functional. In the next section we introduce a H-theorem for linear Fokker-Planck equations based on the second variation of a free energy measure. After that we will use a modification of this H-theorem in order to show the equivalence of Lyapunov's direct method and a linear stability analysis of the stationary solutions of Eq. (2·2).

2.2. *On a H-theorem for linear Fokker-Planck equations (periodic case)*

Let $X(t)$ denote a time-dependent T-periodic random variable and $V(x)$ a potential satisfying $V(x+T) = V(x)$. We consider $X \in \Omega = [a, b]$ with $b - a = T > 0$ and assume that the probability density $P(x, t; u)$ of $X(t)$ with initial distribution $P(x, t_0; u) = u$ satisfies the linear Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{dV}{dx}P(x,t;u)\right] + Q\frac{\partial^2}{\partial x^2}P(x,t;u)\ . \tag{2·7}$$

The stationary solution of Eq. (2·7) reads $P_{\rm st}(x;u) = P_{\rm st}(x) = Z^{-1}\exp\{-V(x)/Q\} > 0$ with $Z = \int_\Omega \exp\{-V/Q\}\, dx$. Equation (2·7) can equivalently be expressed as

$$\frac{\partial}{\partial t}P(x,t;u) = Q\frac{\partial}{\partial x}\left[P_{\rm st}(x)\frac{\partial}{\partial x}\frac{P(x,t;u)}{P_{\rm st}(x)}\right]\ , \tag{2·8}$$

which can be verified by substituting $P_{\rm st}(x) = Z^{-1}\exp\{-V(x)/Q\} > 0$ into Eq. (2·8). Let us define now the functional

$$L[P, P_{\rm st}] = \frac{1}{2}\int_\Omega \frac{[P(x,t;u) - P_{\rm st}(x)]^2}{P_{\rm st}(x)}\, dx = \frac{1}{2}\left[\int_\Omega \frac{[P(x,t;u)]^2}{P_{\rm st}(x)}\, dx - 1\right]\ , \tag{2·9}$$

which satisfies $L[P, P_{\rm st}] \geq 0$ and $L[P_{\rm st}, P_{\rm st}] = 0$. Differentiation with respect to t gives us

$$\frac{d}{dt}L[P, P_{\rm st}] = -\int_\Omega P_{\rm st}\left[\frac{\partial}{\partial x}\frac{P}{P_{\rm st}}\right]^2 dx + \mathcal{B}_3 \tag{2·10}$$

with the surface term $\mathcal{B}_3$ defined by

$$\mathcal{B}_3 = P\frac{\partial}{\partial x}\frac{P}{P_{\rm st}}\bigg|_a^b = 0\ . \tag{2·11}$$

As indicated above, the expression $\mathcal{B}_3$ vanishes due to the periodic boundary conditions. Consequently, we have $dL/dt \leq 0$. Furthermore, the implications $P = P_{\rm st} \Rightarrow dL/dt = 0$ and $dL/dt = 0 \Rightarrow P = CP_{\rm st}$ hold. On account of the normalization of P and $P_{\rm st}$ we have $C = 1$. Therefore, we conclude that the following relations hold:

$$L \geq 0\ ,\quad \frac{d}{dt}L \leq 0\ ,\quad \frac{d}{dt}L = 0 \Leftrightarrow P = P_{\rm st}\ . \tag{2·12}$$

They imply that for every initial distribution u the probability density $P(x,t;u)$ converges to the stationary solution $P_{\rm st}$: $\lim_{t\to\infty} P(x,t;u) = P_{\rm st}(x)$. That is, we have obtained a new H-theorem for linear Fokker-Planck equation subjected to periodic boundary conditions. This H-theorem involves a Lyapunov functional $L[P,P_{\rm st}]$ that differs from the Kullback distance measure $K[P,P_{\rm st}]$.

$L[P,P_{\rm st}]$ can also be interpreted in terms of thermodynamic quantities. For $P(x,t;u) \approx P_{\rm st}(x)$ we introduce the small deviation $\epsilon(x,t;u) = P(x,t;u) - P_{\rm st}(x)$. Let $S = S_{\rm BGS} = -\int P \ln P\, dx$ denote the Boltzmann-Gibbs-Shannon entropy. Then, the second variation reads $\delta^2 S[P_{\rm st}](\epsilon) = -\int_\Omega [\epsilon(x,t;u)]^2/P_{\rm st}(x)\, dx$. Since $F[P] = \int_\Omega V(x) P\, dx - QS[P]$ we have $\delta^2 F[P_{\rm st}](\epsilon) = -Q\delta^2 S[P_{\rm st}](\epsilon)$. Consequently, we obtain

$$L[P,P_{\rm st}] = -\frac{1}{2}\delta^2 S[P_{\rm st}](P-P_{\rm st}) = \frac{1}{2}\delta^2 F[P_{\rm st}](P-P_{\rm st})\ . \tag{2·13}$$

For probability densities $P(x,t;u) = P_{\rm st}(x) + \epsilon(x,t;u)$ in a neighborhood of $P_{\rm st}$, the H-theorem based on Eq. (2·12) states that for $t \to \infty$ the deviation $\delta^2 F$ of the free energy vanishes. That is, for $t \to \infty$ we have $\delta^2 F = 2L \to 0$. Note that H-theorems with functionals similar to (2·9) have been previously discussed.[19),20)] The functionals in these studies, however, have not been discussed in the context of the free energy approach to linear and nonlinear Fokker-Planck equations.

2.3. *Nonlinear Fokker-Planck equations and linear stability analysis*

Let $X(t)$ describe the stochastic behavior of a system with the free energy (2·1) and the probability density $P(x,t;u) = \langle \delta(x - X(t)) \rangle$ that evolves according to Eq. (2·2). Again, we consider a T-periodic time-dependent random variable $X(t)$ on the phase space $\Omega = [a,b]$ with $b - a = T > 0$. Then, Eq. (2·2) becomes

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}M_{xx}P\frac{\partial}{\partial x}\frac{\delta F}{\delta P}\ . \tag{2·14}$$

For the sake of convenience let us put $M_{xx} = 1$. Then, we get

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}P\frac{\partial}{\partial x}\frac{\delta F}{\delta P}\ . \tag{2·15}$$

We require that the energy potential under consideration is T-periodic and that the second variation of F is symmetric:

$$\frac{\delta U}{\delta P}(x+T) = \frac{\delta U}{\delta P}(x)\ ,\quad \frac{\delta^2 F[P]}{\delta P(x)\delta P(y)} = \chi(x,y) \tag{2·16}$$

with $\chi(x,y)=\chi(x,y)$. For example, consider a system with mean field interactions characterized by an entropy measure $S=\int_\Omega s(P)dx$ and an energy functional

$$U[P]=\int_\Omega U_0(x)P\,dx+\frac{1}{2}\int_\Omega\int_\Omega U_{\rm MF}(x-y)P(x)P(y)dx\,dy \tag{2·17}$$

with $U_0(z+T)=U_0(z)$, $U_{\rm MF}(z+T)=U_{\rm MF}(z)$, and $U_{\rm MF}(z)=U_{\rm MF}(-z)$. Then, the quantities

$$\frac{\delta U}{\delta P}(x)=U_0(x)+\int_\Omega U_{\rm MF}(x-y)P(y)\,dy \tag{2·18}$$

and

$$\frac{\delta^2 F[P]}{\delta P(x)\delta P(y)}=U(x-y)-Q\,\delta(x-y)\left.\frac{d^2 s(z)}{dz^2}\right|_{z=P(x)}, \tag{2·19}$$

satisfy Eq. (2·16). As shown in §2.1, the stationary solutions $P_{\rm st}(x;u)$ of Eq. (2·14) satisfy $\delta F[P_{\rm st}]/\delta P=\mu$. Let us consider now the evolution of $P(x,t;u)=P_{\rm st}(x;u)+\epsilon(x,t;u)$, where ϵ corresponds to a small perturbation with $\int_\Omega \epsilon(x,t;u)\,dx=0$. Then, the variational expansion of $\delta F/\delta P$ at $P_{\rm st}$ reads

$$\begin{aligned}\frac{\delta F[P]}{\delta P}&=\frac{\delta F[P_{\rm st}]}{\delta P}+\int_\Omega\frac{\delta^2F[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t;u)\,dy+O(\epsilon^2)\\&=\mu+\int_\Omega\frac{\delta^2F[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t;u)\,dy+O(\epsilon^2)\end{aligned} \tag{2·20}$$

and Eq. (2·15) becomes

$$\frac{\partial}{\partial t}\epsilon(x,t;u)=\frac{\partial}{\partial x}[P_{\rm st}+\epsilon]\frac{\partial}{\partial x}\left[\mu+\int_\Omega\frac{\delta^2F[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t;u)\,dy+O(\epsilon^2)\right]. \tag{2·21}$$

Therefore, neglecting higher order terms $O(\epsilon^2)$, the evolution of ϵ is given by the linear integrodifferential equation

$$\frac{\partial}{\partial t}\epsilon(x,t;u)=\frac{\partial}{\partial x}P_{\rm st}(x)\frac{\partial}{\partial x}\int_\Omega\frac{\delta^2F[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t;u)\,dy. \tag{2·22}$$

Equation (2·22) is the linearization of Eq. (2·15) at $P_{\rm st}$. Let us study now the functional

$$L(\epsilon)=\frac{1}{2}\delta^2F[P_{\rm st}](\epsilon)=\frac{1}{2}\int_\Omega\int_\Omega\frac{\delta^2F[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t;u)\epsilon(x,t;u)\,dx\,dy\,. \tag{2·23}$$

The evolution of $L(\epsilon)$ is given by

$$\frac{d}{dt}L(\epsilon)=-\int_\Omega P_{\rm st}(x;u)\left[\frac{\partial}{\partial x}\int_\Omega\frac{\delta^2F[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t;u)\,dy\right]^2dx+\mathcal{B}_4 \tag{2·24}$$

with

$$\mathcal{B}_4=\frac{1}{2}P_{\rm st}(x;u)\frac{\partial}{\partial x}\left[\int_\Omega\frac{\delta^2F[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t;u)\,dy\right]^2\Bigg|_a^b=0\,. \tag{2·25}$$

The surface term vanishes due to periodic boundary conditions. Consequently, we have $dL/dt \leq 0$. Moreover, we get $\epsilon \equiv 0 \Rightarrow dL/dt = 0$. If $dL/dt = 0$ then $\int_\Omega [\delta^2 F[P_{\rm st}]/\delta P(x)\delta P(y)]\epsilon(y,t;u)\,dy = C$, where C denotes a constant. Multiplying this result with $\epsilon(x,t;u)$, integrating with respect to x, and taking the normalization condition $\int_\Omega \epsilon(x,t;u)\,dx = 0$ into account, we obtain

$$\frac{d}{dt}L = 0 \Rightarrow \int_\Omega \int_\Omega \frac{\delta^2 F[P_{\rm st}]}{\delta P(x)\delta P(y)} \epsilon(y,t;u)\epsilon(x,t;u)\,dx\,dy = \delta^2 F[P_{\rm st}](\epsilon) = 0\ . \quad (2\cdot 26)$$

Let us apply these results to extrema of F. Let $P_{\rm st}$ correspond to a (local or global) minimum of F with

$$\delta^2 F[P_{\rm st}](\epsilon) > 0\ \forall \epsilon \not\equiv 0\ , \quad \delta^2 F[P_{\rm st}](\epsilon) = 0 \text{ for } \epsilon \equiv 0\ . \quad (2\cdot 27)$$

Consequently, the functional $L(\epsilon)$ is semi-positive: $L(\epsilon) = 2\delta^2 F[P_{\rm st}](\epsilon) \geq 0\,\forall\epsilon$. Furthermore, if dL/dt vanish we conclude by means of Eqs. (2·26) and (2·27) that $dL/dt = 0 \Rightarrow \delta^2 F[P_{\rm st}](\epsilon) = 0 \Rightarrow \epsilon \equiv 0$. Taking Eqs. (2·24) and (2·25) into account, we arrive at the relations

$$L \geq 0\ , \quad \frac{d}{dt}L \leq 0\ , \quad \frac{d}{dt}L = 0 \Leftrightarrow \epsilon \equiv 0 \quad (2\cdot 28)$$

that constitute a H-theorem related to the perturbation ϵ. Accordingly, if $P_{\rm st}$ corresponds to a minimum of F then in the limiting case $t \to \infty$ the quantities ϵ, L and $\delta^2 F$ behave like $\epsilon \to 0$, $L(\epsilon) \to 0$, and $\delta^2 F[P_{\rm st}](P - P_{\rm st}) \to 0$. Now, let $P_{\rm st}$ describe a maximum or a saddle point of F satisfying

$$\exists \epsilon^*(x,t;u)\ :\ \delta^2 F[P_{\rm st}](\epsilon^*) < 0\ . \quad (2\cdot 29)$$

That is, there is at least one perturbation ϵ^* such that in the direction of this perturbation the free energy decreases. From Eqs. (2·24) and (2·25) it follows that $dL(\epsilon)/dt \leq 0\ \forall\epsilon$ and, in particular, $dL(\epsilon^*)/dt \leq 0$. Consequently, we have

$$\frac{d}{dt}\delta^2 F[P_{\rm st}](\epsilon^*) \leq 0\ . \quad (2\cdot 30)$$

This inequality states that a negative valued perturbation $\delta^2 F$ of the free energy F does not vanish. $|\delta^2 F|$ increases with time or is constant. In sum, by means of a linear stability analysis we have shown that if $P_{\rm st}$ corresponds to a minimum of F, then perturbations of the stationary state vanish. In contrast, if $P_{\rm st}$ corresponds to a saddle point or a maximum of F, then there are perturbations that do not vanish. Therefore, we conclude that Lyapunov's direct method based on the free energy F is consistent with a linear stability analysis based on the second variation $\delta^2 F$ of F.

Let us conclude this section with an example. We consider the 2π-periodic variable ϕ on the phase space $\Omega = [0, 2\pi]$ and study a stochastic process described by the (nonlinear) mean field Fokker-Planck equation[2)]

$$\frac{\partial}{\partial t}\tilde{P}(\phi,t;u) = -\frac{\partial}{\partial \phi}\left[\omega - \int_\Omega \frac{\partial}{\partial \phi} U_{\rm MF}(\phi - \phi')\tilde{P}(\phi',t;u)\,d\phi'\tilde{P}(\phi,t;u)\right] + Q\frac{\partial^2 \tilde{P}}{\partial \phi^2} \quad (2\cdot 31)$$

with $Q>0$. This process can conveniently be examined in a rotation frame[2] by means of the probability density $P(x,t;u)=\tilde{P}(\phi(x,t),t;u)$, where ϕ is evaluated at position x and time t like $\phi(x,t)=x-\omega t$. Then, Eq. (2·31) becomes

$$\frac{\partial}{\partial t}P(x,t;u)=\frac{\partial}{\partial x}\left[\int_\Omega \frac{\partial}{\partial x}U_{\rm MF}(x-x')P(x',t;u)\,dx'P(x,t;u)\right]+Q\frac{\partial^2 P}{\partial x^2}. \qquad (2·32)$$

This equation can be regarded as a special case of Eq. (2·15) for $S=S_{\rm BGS}$ and an energy functional (2·17) given by $U_0=0$ and $U_{\rm MF}(z)=\sum_{n=1}^\infty c_n\cos(nz)$ (or $U_{\rm MF}(z)=-\sum_{n=1}^\infty \tilde{c}_n\cos(nz)$, where $\tilde{c}_n>0$ describes an attractive coupling that tends to synchronize the system). In this case, it can be shown that F is bounded from below.[21] Consequently, the H-theorem of §2.1 applies and we have $\lim_{\rm t\to\infty}P(x,t;u)=P_{\rm st}(x;u)$. Due to the nonlinearity, there are multiple stationary probability densities. One stationary solution is given by the uniform distribution $P_{\rm st}=1/[2\pi]$. In some previous studies[21),22] the functional F has been evaluated at $P_{\rm st}=1/[2\pi]$. In particular, calculating $\delta^2F[P_{\rm st}=1/[2\pi]](\epsilon)$ for $\epsilon=\sum_{n=1}^\infty[a_n\cos(nx)+b_n\sin(nx)]$, one can show that F has a minimum at $P_{\rm st}=1/[2\pi]$ if the inequality $2Q>-c_n=\tilde{c}_n$ is satisfied for every n (weak coupling case). In this case Lyapunov's direct method tells us that the uniform distribution is a stable stationary distribution. In contrast, F has a maximum or a saddle point at $P_{\rm st}=1/[2\pi]$ if there is at least one n^* for which $\tilde{c}_{n^*}=-c_{n^*}>2Q$ holds (strong coupling case). Then, the uniform distribution is an unstable stationary distribution and $P(x,t;u)$ for $u\not\equiv 1/[2\pi]$ cannot converge to $P_{\rm st}=1/[2\pi]$. Since Eq. (2·6) holds in any case, in the strong coupling case there is at least one stable stationary distribution $P_{\rm st}(x)\not\equiv 1/[2\pi]$ and the limiting case $\lim_{t\to\infty}P(x,t;u)=P_{\rm st}(x)\not\equiv 1/[2\pi]$ holds. Therefore, it has been conjectured[21] that in the long time limit $\tilde{P}(\phi,t;u)$ converges to a rotating wave solution of the form $P_{\rm st}(\phi+\omega t;u)$ with $P_{\rm st}(x)\not\equiv 1/[2\pi]$. This result is in line with the linear stability analysis carried out by *Kuramoto*.[2] Our objective now is to reobtain the approach by *Kuramoto* from our general linearized equation (2·22). To this end, we substitute Eq. (2·19) for $S=S_{\rm BGS}\Rightarrow s(z)=-z\ln z$ and $P_{\rm st}=1/[2\pi]$ into Eq. (2·22) and thus obtain

$$\frac{\partial}{\partial t}\epsilon(x,t;u)=\frac{1}{2\pi}\frac{\partial^2}{\partial x^2}\left[\int_\Omega U_{\rm MF}(x-y)\epsilon(y,t;u)\,dy+2\pi Q\epsilon(x,t;u)\right]. \qquad (2·33)$$

Using $U_{\rm MF}(z)=\sum_{n=1}^\infty c_n\cos(nz)$ and $\epsilon(x,t;u)=\sum_{n=1}^\infty[a_n(t;u)\cos(nx)b_n(t;u)$ $\sin(nx)]$ Eq. (2·33) can be transformed into

$$\begin{aligned}&\frac{d}{dt}\sum_{n=1}^\infty[a_n(t;u)\cos(nx)+b_n(t;u)\sin(nx)]\\&=-\frac{1}{2}\sum_{n=1}^\infty n^2(c_n+2Q)[a_n(t;u)\cos(nx)+b_n(t;u)\sin(nx)],\end{aligned} \qquad (2·34)$$

which leads to

$$\frac{d}{dt}a_n(t;u)=-\frac{n^2}{2}(c_n+2Q)\,a_n(t;u)\ ,\quad \frac{d}{dt}b_n(t;u)=-\frac{n^2}{2}(c_n+2Q)\,b_n(t;u)\ . \qquad (2·35)$$

Consequently, the linear stability analysis yields the same result as Lyapunov's direct method. If $c_n + 2Q > 0$ (i.e., $\tilde{c}_n = -c_n < 2Q$) for all n then we have $a_n(t \to \infty) = b_n(t \to \infty) = 0$. In contrast, if there is at least one n^* with $-c_{n^*} > 2Q$ then we can choose an initial distribution u such that $a_{n^*}(t_0; u) \neq 0$ and $b_{n^*}(t_0; u) \neq 0$. In this case the amplitudes will increase with time.

§3. Conclusions

We have shown that for stochastic processes described by linear Fokker-Planck equations subjected to periodic boundary conditions there is a H-theorem based on the second variation $\delta^2 F$ of the free energy F of the processes. This finding is particularly striking because for this kind of processes there is also a H-theorem involving the measure F. Consequently, we deal with stochastic processes for which we can show by means of independent considerations that in the long time limit F converges to a stationary value $F_{\rm st}$ and $\delta^2 F$ converges to zero. $\delta^2 F$ also provides us with a local Lyapunov functional for nonlinear Fokker-Planck equations illustrating that perturbations ϵ decay to zero if a stationary solution describes a free energy minimum. Therefore, the situation resembles the one for deterministic systems described by first order differential equations. For $\dot{q}(t) = \lambda(q - q_0)$ with $\lambda > 0$ the function $L(q) = [q - q_0]^2/2$ is a Lyapunov function by means of which we can prove that the limiting case $q(t \to \infty) = q_0$ holds. For $\dot{q}(t) = -dV(q)/dq$ with $dV(q_0)/dq = 0$ and $d^2V(q_0)/dq^2 = \lambda > 0$ the function $L(\epsilon) = [q - q_0]^2/2$ corresponds to a local Lyapunov function for perturbations $\epsilon(t) = q(t) - q_0$ with $\epsilon \approx 0$ by means of which the limiting case $\epsilon(t \to \infty) = 0$ can be shown.

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