# An Introduction to Coupled Heteroclinic Cycles

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Coupled heteroclinic cycles are introduced as a new class of coupled oscillator systems. As an example, we study a square lattice distribution of heteroclinic cycles and show the emergence of non-chaotic disordered patterns. A short analysis of them is also reported.

## §1. Introduction

Coupled element systems in which each element has autonomous dynamics and interact each other are useful classes of models to understand the dynamics of the self-organized systems. For example, coupled limit cycles or chaotic maps are frequently studied as the simplest cases.[1)–3)] Emerging phenomena in the whole system reflect the dynamical properties of each element. When we introduce a new type of dynamics as the elements, the whole system will show new phenomena.

As a new type of dynamical systems, we note systems which have heteroclinic cycles. Heteroclinic cycles are constructed with some saddle fixed points and heteroclinic orbits that connect the fixed points cyclically. If a system has a heteroclinic cycle as an attractor, an orbit stays for long periods in neighborhoods of fixed points and moves to the next quickly along with the heteroclinic orbits. The staying periods grow exponentially with respect to the number of visiting times. Therefore, the period of oscillation along with the heteroclinic cycle also grows exponentially, and such a system has no characteristic time scale. However, this infinite growth of the period can occur when the system is perfectly isolated from the environment. It is not realistic as a model of a natural system, and every system gets some noise or perturbations from the environment. The perturbations on a dynamical system force an orbit keep away from the attractor (i.e. the heteroclinic cycle in this case). Therefore, the system with the heteroclinic cycle can resume characteristic time scale by means of perturbations. In other words, a time scale of such a system depend on the perturbations from the exterior of the system. As the exterior, we introduce many other systems with heteroclinic cycles, and the interactions affect the systems as the perturbations. Though some basic properties of heteroclinic cycles have been reported,[4)–7)] collective phenomena of heteroclinic cycles have not been studied. Thus, we introduce coupled heteroclinic cycles and show a novel phenomena.

## §2. Replicator equation and robust heteroclinic cycle

First, we review a class of dynamical systems which has robust heteroclinic cycles against small changes of parameters. Since all heteroclinic cycles are structurally unstable,[8)] such a system must have certain constraints. With adequate constraints,

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the phase space have some invariant sets, and a robust heteroclinic cycle can exist in the invariant sets. As an example of such systems, we adopt the replicator equation.[7] The replicator equation describes dynamics of populations that multiply by replication and interact each other. A set of self-catalyzing molecules that make a chemical reaction network are an example of such populations. Additionally, the Lotka-Volterra equation is proved to be equivalent to the replicator equation. Thus, it has been also analyzed as the basic model of ecological systems in a lot of studies.[7] The replicator equation is written as

$$\frac{dx_i}{dt} = f_i^n(\boldsymbol{x}) \equiv x_i\left(\sum_{j=1}^{n} a_{ij}x_j - \sum_{j,k=1}^{n} a_{jk}x_jx_k\right), \qquad (i = 1,\cdots,n) \tag{1}$$

with the constraints

$$\sum_{i=1}^{n} x_i = 1, \quad 0 \le x_i \le 1, \qquad (i = 1,\cdots,n) \tag{2}$$

where $x_i$ denotes the relative abundance of species $i$ and matrix $(a_{ij})$ describes the interaction effects between species. Because the variables represent proportions, the constraints (2) are fulfilled by itself. Therefore, the phase space becomes compact set

$$S_n = \left\{\boldsymbol{x} \in \mathbb{R}^n | 0 \le x_i \le 1, \sum_{i=1}^{n} x_i = 1\right\}, \tag{3}$$

called simplex and the boundary of the simplex becomes the invariant set.

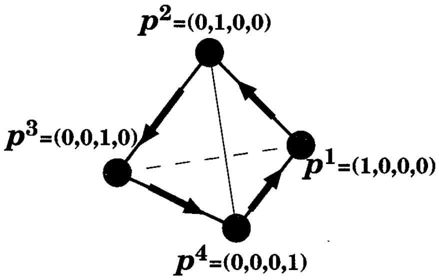


Fig. 1. A phase space of replicator equation (1) with four components ($S_4$) and a robust heteroclinic cycle. The filled circles represent saddle fixed points, and the thick solid lines with arrows represent heteroclinic orbits.

To describe the robust heteroclinic cycle, we note a four-component replicator system, which is adopted in our study reported below, with parameter matrix

$$(a_{ij}) = \begin{pmatrix} 0 & -2 & -1 & 1 \\ 1 & 0 & -2 & -1 \\ -1 & 1 & 0 & -2 \\ -2 & -1 & 1 & 0 \end{pmatrix}. \tag{4}$$

Phase space ($S_4$) is surface and interior of a tetrahedron with vertices $\boldsymbol{p}^j = (\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j})$ (Fig. 1). These vertices correspond to the ecological systems which are occupied with only one species. They are always fixed points with any parameter matrix. Moreover, edges which are segments $\{\boldsymbol{x}|x_i+x_j = 1,\ x_k = x_l = 0\}$ that connects two vertices are also invariant. In the invariant edge $\{\boldsymbol{x}|x_1+x_2 = 1,\ x_3 = x_4 = 0\}$, the fixed point ($\boldsymbol{p}^1$) at one end becomes a source and the fixed point ($\boldsymbol{p}^2$) at the other end becomes a sink with the matrix (4). Therefore, the edge becomes a heteroclinic orbit which connects the two fixed points. Because the edges are invariant, the heteroclinic orbit is robust against small changes of parameters. In this case, the four edges $\{\boldsymbol{x}|x_i + x_{i+1} = 1,\ x_{i+2} = x_{i+3} = 0\}$ are designed in the same way. Therefore, the four fixed points and the four edges compose a robust heteroclinic cycle attractor (Fig. 1).

## §3. Coupled heteroclinic cycles

In this paper, we study coupled replicators in which the replicators with four components illustrated above are distributed on a two dimensional square lattice, and nearest neighbors are coupled diffusively,

$$\frac{dx_i{}^{(u,v)}}{dt} = f_i^4(\boldsymbol{x}^{(u,v)}) + D\sum_{u'}^{u\pm1}\sum_{v'}^{v\pm1}(x_i^{(u',v')} - x_i^{(u,v)}). \qquad (i = 1, \cdots, 4) \tag{5}$$

Where $(u, v)$ is a site index that represents the location of an replicator, and $D$ is the coupling constant between adjoining sites. The corresponding situations can occur in an ecological system on dotted spots with diffusion between the spots (e.g. ecosystem on trees in an orchard) or dynamics of reaction networks of self-catalyzing molecules in cells.

If the system has a certain large coupling constant ($D \simeq 0.1$), we observe rotating spiral patterns of a well-known type. However, if we choose the coupling constant small ($D < 0.01$), a quite different pattern arises (Fig. 2). The snapshot of the pattern looks disordered in space, but it is exactly periodic in time and not chaotic (Fig. 3). The period of the pattern is the same as the period of oscillation of each replicator (Fig. 3). Thus, the frequencies of all replicators are entrained each other, while phases of their oscillations keep differences. There are many attracting patterns which divide the ambient phase space into their basins. With further investigations, we discover a local rule of the patterns: relation on the phases of oscillations between two neighboring replicators never become the anti-phase.

As mentioned above, the coupling effect keeps orbitpoint $[\boldsymbol{x}^{(u,v)}(t)]$ of each replicator from approaching the heteroclinic cycle which exists in the phase space of each replicator. Thus, smaller coupling constant makes orbitpoint approach closer to the

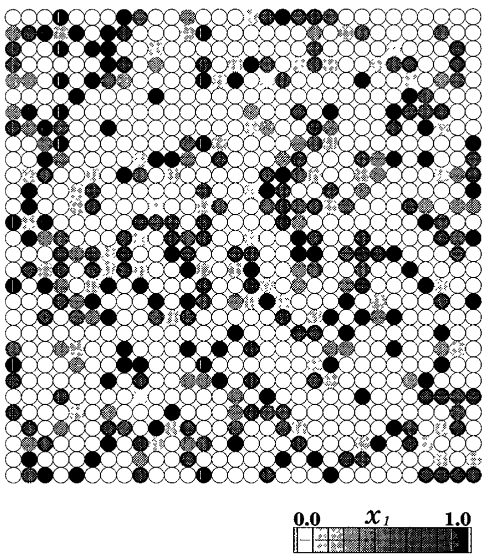


Fig. 2. A snapshot of spatial distributions of $x_1^{(u,v)}$ on a gray scale with $30 \times 30$ sites and $D = 10^{-4}$. Disordered pattern is observed.

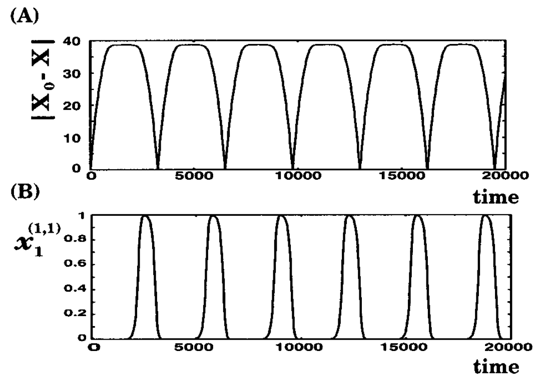


Fig. 3. Time series data of Fig. 2. (A): the distance between $\boldsymbol{X}_0 \in \mathbb{R}^{3600}$ (a point in the trajectory at a time $t_0$ in the ambient phase space) and $\boldsymbol{X}$ (the trajectory after the time). (B): the oscillation of $x_1^{(1,1)}$. The exact returns to $\boldsymbol{X}_0$ in (A) indicate that the disordered pattern forms a limit cycle in the ambient phase space. The both oscillations show the same period.

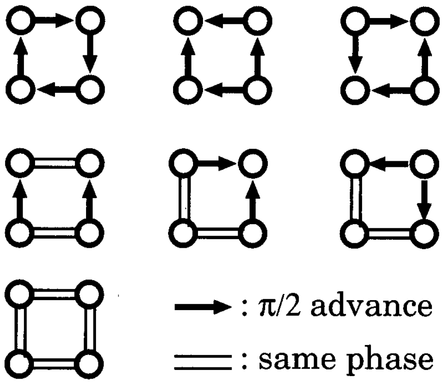


Fig. 4. Seven types of phase arrangements for four replicators that form unit square in the lattice. The circle represents a replicator system. The arrow denotes the $\pi/2$ advance of the phase of the replicator which is pointed by it. The parallel lines denote the two replicators have the same phase.

heteroclinic cycle. As a result, the orbitpoint stays long in the neighborhoods of fixed points and moves infrequently along with heteroclinic orbits. With a quite small coupling constant, all orbitpoints move infrequently and one can get the snapshots of patterns in which all orbitpoints stay in the neighborhoods of fixed points. In such states, the orbitpoints can be roughly described as four phases $(0, \pi/2, \pi, 3\pi/2\ )$ of oscillations which represent the four fixed points $(\boldsymbol{p}^1, \boldsymbol{p}^2, \boldsymbol{p}^3, \boldsymbol{p}^4)$ respectively. Therefore, difference of states between neighboring replicators are described as the deference of phases, which can take the value of $0, \pm\pi/2$ or $\pm\pi$. Furthermore, the local rule mentioned above put another restriction, and the phase difference only takes 0 or $\pm\pi/2$. With this simplification of the description, the patterns can be decomposed into seven types of units and their rotated and reflected ones (Fig. 4). Moreover, we can compose all the possible patterns with the combinations of the types.

## §4. Conclusion

In this paper, we introduced coupled heteroclinic cycles and reported a novel type of oscillating patterns. We also reported the short analysis of them. For more detailed investigation, see Ref. 9). In coupled heteroclinic cycles, the coupling effect is assumed as a perturbation for the single system mentioned above. Therefore, the differences of states between elements decide the time scales of the dynamics of the elements. Such flexibility of time scales on each element can bring such complex states as non-chaotic attractors. Another example of non-chaotic complex states will be reported in the forthcoming paper.[10)]

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