Duality of the Random Model and the Quantum Toric Code

Koujin TAKEDA and Hidetoshi NISHIMORI

Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan

We study the properties under duality transformations of the general random spin systems with bimodal randomness to derive conjecture which relates the two multicritical points of dual random model pairs. The validity of the conjecture is checked for several cases by numerical results. We also discuss that the duality and our conjecture are useful for investigating the accuracy threshold of the toric codes in various dimensions.

§1. Introduction

Critical phenomena of random spin systems have been one of the main topics of statistical physics, especially in the field of spin glass. The structure of their phase diagrams has been a subject of active investigations in this field. Recently it was pointed out that the value of the accuracy threshold of the toric code,[1] which is one of the quantum error correcting codes, can be estimated from the location of the multicritical point of the random bond Ising model (RBIM) or the Z_2 random plaquette gauge model (RPGM).[2] Therefore it is very important to know the phase structure of random systems also from the viewpoint of quantum information theory.

It is a very difficult problem to analytically determine the phase diagram of the random systems. In the present paper we make a generalization of the duality of random systems[3),4)] proposed in Ref. 5) especially for the study of the $2D$ RBIM on the Nishimori line (NL).[6] This is also generalization of the duality of the general nonrandom spin systems by Wegner[7] to random cases. With this generalization, we can derive a conjecture which describes the relation between the multicritical points of two random spin systems, which are mapped each other by the duality transformation.

§2. Duality of random spin models

Before treating random cases, we review the duality of general non-random two-component spin systems.[7] Let us prepare the d-dimensional lattice and assign Z_2 variables (or Ising spins) on $p-1$ dimensional elements $\boldsymbol{x}$ on the lattice, which we denote $S_{\boldsymbol{x}}$. Then we define the model on the lattice by the following Hamiltonian,

$$H = -J \sum_{C} \prod_{\boldsymbol{x} \in \partial C} S_{\boldsymbol{x}}, \qquad (2\cdot1)$$

where C is the p dimensional element on the lattice and ∂C is its boundary whose dimension is $p-1$. We also consider the dual system and find that the dual Hamiltonian can be expressed by the same form as Eq. (2·1).

We give some examples of dual model pairs.

- ($d=2, p=1$) 2D Ising on square lattice $\leftrightarrow$ 2D Ising on square lattice (self-dual)

- ($d=2, p=1$) $2D$ Ising on triangular lattice $\leftrightarrow$ $2D$ Ising on hexagonal lattice
- ($d=4, p=2$) $4D$ Z_2 gauge model on hypercubic lattice $\to$ $4D$ Z_2 gauge model on hypercubic lattice (self-dual)
- ($d=3, p=1$ or 2) $3D$ Ising on cubic lattice $\leftrightarrow$ $3D$ Z_2 gauge model on cubic lattice

Next we introduce randomness. To analyze the critical point of the random system by duality, we use the technique in Ref. 5) with some modifications.

The Hamiltonian of the general random models with Z_2 spins and bimodal randomness can be written,

$$H = -J \sum_{C} \tau_C \prod_{\boldsymbol{x} \in \partial C} S_{\boldsymbol{x}}, \tag{2·2}$$

where the τ_C are the random variables which are dependent on each element C. τ_C takes the value -1 with probability p and 1 with $1-p$.

We use the replica technique for averaging over random variables and define the "averaged" Boltzmann factors for a given probability p for this purpose. We consider the n-replicated system and define the averaged Boltzmann factor x_m for the specified element $\boldsymbol{x}$. The x_m corresponds to the configuration $\prod_{\boldsymbol{x}\in\partial C} S_x = 1$ in $n-m$ replicas and -1 in m replicas. The explicit form of x_m is (see Ref. 5)),

$$x_m(p,K) = pe^{(n-2m)K} + (1-p)e^{-(n-2m)K}, \tag{2·3}$$

where $K=\beta J$. Note that the case of $n=1$ and $p=1$ corresponds to the nonrandom system discussed above.

The averaged n-replicated partition function is a function of these Boltzmann factors,

$$[Z^n]_{\mathrm{av}} \equiv Z_n\{x_0(p,K), x_1(p,K), \cdots, x_n(p,K)\}, \tag{2·4}$$

where $[\ ]_{\mathrm{av}}$ represents random average.

On the dual lattice, we can also define the dual averaged Boltzmann factor $x_m^*(p,K)$. The explicit forms are obtained by the two-component Fourier transformation,[5),8)]

$$\begin{aligned} x_{2m}^*(p,K) &= 2^{-n/2}(e^K + e^{-K})^{n-2m}(e^K - e^{-K})^{2m}, \\ x_{2m+1}^*(p,K) &= 2^{-n/2}(2p-1)(e^K + e^{-K})^{n-2m-1}(e^K - e^{-K})^{2m+1}. \end{aligned} \tag{2·5}$$

We consider the case that the system is self-dual when we remove randomness (e.g. $2D$ RBIM). Then we can express the self-duality of the n-replicated partition function using x_m and x_m^*,

$$Z_n(p,K) = Z_n\{x_0, x_1, \cdots, x_n\} = Z_n\{x_0^*, x_1^*, \cdots, x_n^*\}, \tag{2·6}$$

up to an overall constant. Self-duality is recognized by the fact that $Z_n(p,K)$ is invariant if we exchange $x_m(p,K) \leftrightarrow x_m^*(p,K)$ for all m simultaneously.

The n-replicated partition function Z_n is a complicated function of the averaged Boltzmann factor and we make conjecture[5)] for the multicritical point as

$$x_0(p_c, K_c) = x_0^*(p_c, K_c). \tag{2·7}$$

From this equation and the condition of the NL, we obtain the location of multicritical point for self-dual cases.

This argument cannot be applied to the non-self-dual systems. To make the argument applicable to such cases, we consider the product of two replicated partition functions and its dual expressions

$$\begin{aligned} &Z_n^O\{x_0(p^o,K^o),\cdots,x_n(p^o,K^o)\}Z_n^D\{x_0(p^d,K^d),\cdots,x_n(p^d,K^d)\} \\ &= Z_n^D\{x_0^*(p^o,K^o),\cdots,x_n^*(p^o,K^o)\}Z_n^O\{x_0^*(p^d,K^d),\cdots,x_n^*(p^d,K^d)\}. \end{aligned} \qquad (2\cdot8)$$

Here Z_n^O and Z_n^D represent the replicated partition functions of the original and the dual models, which are dual if we remove randomness. p^o, K^o and p^d, K^d are parameters of the original and the dual models respectively. The product of the partition functions is invariant under the simultaneous exchange $x_m(p^o,K^o) \leftrightarrow x_m^*(p^d,K^d)$ and $x_m(p^d,K^d) \leftrightarrow x_m^*(p^o,K^o)$ for all m. Considering this duality, we modify the conjecture for multicritical points as

$$x_0(p_c^o,K_c^o)x_0(p_c^d,K_c^d) = x_0^*(p_c^o,K_c^o)x_0^*(p_c^d,K_c^d), \qquad (2\cdot9)$$

and expect that the two multicritical points are related by this equation.

We can check that Eq. (2·7) gives the exact location of a unique multicritical point and Eq. (2·9) gives the exact relation between two multicritical points in the case of $n = 1, 2$ and $n \to \infty$ using the fact that the replicated systems are equivalent to the well-known nonrandom spin models.

If we take the $n \to 0$ limit of Eq. (2·9) combined with the condition of the NL, we obtain the following relation for the multicritical points from $O(n^1)$ terms,

$$H(p_c^o) + H(p_c^d) = 1, \qquad (2\cdot10)$$

where $H(p) \equiv (-p\log p - (1-p)\log(1-p))/\log 2$.

We can check Eq. (2·10) numerically. If the system is self-dual, p_c must satisfy $H(p_c) = \frac{1}{2}$ which gives $p_c = 0.889972\cdots$,[5] and this value is supported by several numerical studies for the $2D$ RBIM.[9] This also yields the result that the multicritical point of the $4D$ RPGM locates at the same point as the $2D$ RBIM,[3] which is also confirmed by the recent numerical study.[10] The relation for non-self-dual cases can be checked as well. We estimated the location of the multicritical points for the $2D$ RBIM on triangular and hexagonal lattices by the non-equilibrium relaxation method.[4] For the hexagonal lattice $p_c^o = 0.930(5)$ which gives $0.347 < H(p_c^o) < 0.384$, and for the triangle $p_c^d = 0.835(5)$ which gives $0.634 < H(p_c^d) < 0.658$. The results yield $0.981 < H(p_c^o) + H(p_c^d) < 1.042$, which is consistent with our conjecture Eq. (2·10). The multicritical points for dual models in three dimensions were also estimated numerically. For the $3D$ RBIM $p_c^o = 0.7673(3)$[11] which gives $H(p_c^o) \approx 0.783$. For the $3D$ RPGM $p_c^d \approx 0.967$[12] which gives $H(p_c^d) \approx 0.209$. The sum $H(p_c^o) + H(p_c^d)$ is about 0.992, which is a reasonable value in view of our expectation.

§3. Conclusion

We investigated the locations of the multicritical points for various random spin models using the duality and the replica technique. From the dual expression of the replicated partition function, we obtained a conjecture which gives the relation between the location of two multicritical points of dual random model pairs. We checked the conjecture for self-dual models and dual pairs by the numerical results. The data were consistent with the conjecture, and we conclude that the conjecture for the multicritical points obtained by the duality are (not rigorously proved but) reliable. We can also estimate the accuracy thresholds of the toric codes in various dimensions because the accuracy thresholds are obtained from the values $1-p_c$ of the corresponding random spin models.[2] For the $2D$ toric code, the accuracy threshold is about 11% from the $2D$ RBIM,[5),9] and for the $4D$ code it is also about 11% from the $4D$ RPGM.[3),10] For the $3D$ code (or the $2D$ code with repeated observation) the accuracy threshold is $3 \sim 3.5\%$ from the $3D$ RPGM[2),12] (or also from the numerical study of the $3D$ RBIM[11] using the relation Eq. (2·10)).

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