Gauge/Gravity Correspondence and QCD

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We review the basic idea of gauge/gravity correspondence and its application to QCD.

§1. D-brane and QCD

String theory has experienced a revolutionary period since around 1995. One of the most important discovery is the D-brane. It has opened up profound relationship between string theory and quantum field theory. Now we have come to the stage to consider realistic physics by applying the techniques developed in string theory to QCD. The purpose of this note is to provide a brief overview on the application of the string theory to QCD, mainly based on Ref. 1).

Let us begin by explaining what D-branes are. D-branes are extended objects living in the ten dimensional space-time of superstring theory. A Dp-brane is defined as $(p+1)$-dimensional manifold on which the end points of open strings can be attached. A crucial fact is that there is a massless gauge field in the open string spectrum. If we consider N_c D-branes located at the same place, each of the two end points of each open string carries an index $a = 1, \cdots, N_c$ (Chan-Paton index) that labels which D-brane the end point is attached. In this case, the gauge field carries two indices and hence it is considered as an $N_c \times N_c$ matrix. In fact, it is interpreted as a $U(N_c)$ gauge field. Therefore, we can realize $(p+1)$-dimensional $U(N_c)$ gauge theory on the Dp-brane. This is the basic idea to construct non-abelian gauge theories in string theory. We can apply this construction to more complicated brane configurations to realize various gauge theories.

1.1. $\mathcal{N}=4$ *super Yang-Mills*

As described above, we can easily obtain a four dimensional gauge theory by considering N_c D3-branes. The massless fields created by the open strings attached on the D3-branes are a gauge field, six real scalar fields, four Weyl fermions. All of them are in the adjoint representation of the gauge group $U(N_c)$. This massless field content is nothing but that for $\mathcal{N}=4$ super Yang-Mills theory. In fact, if we take the decoupling limit, in which all the massive modes become infinitely massive, the system reduces to $\mathcal{N}=4$ super Yang-Mills theory. This theory is known as a superconformal field theory and gives an important example of AdS/CFT correspondence, which will be discussed in the next section.

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1.2. $\mathcal{N}=2$ *Super QCD with adjoint and fundamental matters*

In order to introduce matters in the fundamental representation, let us add N_f D7-branes to the D3-brane system considered above. Now we have N_c D3-branes extended along $x^0, \cdots, x^3$ directions and N_f D7-branes extended along $x^0, \cdots, x^7$ directions. The massless fields created by the open strings attached on the D3-branes are the same as above. In addition, there are open strings connecting between D3-brane and D7-brane. The massless fields created by such open string turn out to be two complex scalar fields and two Weyl fermions. These fields are in the bi-fundamental representation of $U(N_c) \times U(N_f)$ symmetry, since they carry Chan-Paton indices associated with both D3-branes and D7-branes. If we look at the fields localized on the D3-brane, we obtain a four dimensional $U(N_c)$ gauge theory with adjoint and fundamental matters. The $U(N_f)$ gauge symmetry on the D7-brane is considered as flavor symmetry. This symmetry becomes a global symmetry in the decoupling limit, since the coupling between the $U(N_f)$ gauge field and the four dimensional fields on the D3-brane vanishes in this limit.

1.3. *Pure Yang-Mills*

In order to obtain more realistic theory, we have to break supersymmetry. As an example, let us take N_c D4-branes compactified on an S^1, whose radius is denoted by $M_{\rm KK}^{-1}$, and impose the anti-periodic boundary condition along this S^1 to all the fermions of the system.[4] If we focus on the energy scale lower than the compactification scale, the system reduces to a four dimensional $U(N_c)$ gauge theory. Because of the anti-periodic boundary condition for the fermions, the fermions in the four dimensional effective theory become massive and the supersymmetry is completely broken. Furthermore, the adjoint scalar fields will also acquire mass via the quantum effect. As a result, we obtain four dimensional pure Yang-Mills theory at low energy.

1.4. *QCD with N_f massless quarks*

QCD is constructed by adding quarks to pure Yang-Mills theory. In order to add N_f massless quarks, we add N_f D8-brane - $\overline{\rm D8}$-brane pairs to the D4-brane system considered above.[1] Here the $\overline{\rm D8}$-brane is a D8-brane with opposite orientation. The brane configuration is as follows. We have N_c D4-branes extended along $x^0, \cdots, x^4$ directions and D8-$\overline{\rm D8}$ pairs extended along $x^0, \cdots, x^3$ and $x^5, \cdots, x^9$ directions. (See left side of Fig. 1.) The x^4 directions is compactified to S^1 with the anti-periodic boundary condition for the fermions. As explained above, the massless field created by the open strings attached on the D4-branes is just the gluon field. From the open strings connecting between the D4-brane and the D8-brane, we find Weyl fermions in the massless spectrum. Interestingly, if we replace the D8-brane with a $\overline{\rm D8}$-brane, the chirality of the massless fermion is flipped. Therefore, the massless field content in this system is the four dimensional $U(N_c)$ QCD with N_f massless quarks. It is important to note that the gauge symmetry for the D8-$\overline{\rm D8}$ pairs is $U(N_f)_{\rm D8} \times U(N_f)_{\overline{\rm D8}}$ and this is interpreted as the chiral symmetry $U(N_f)_L \times U(N_f)_R$ of massless QCD.

§2. Gauge/gravity correspondence

In general relativity, a heavy particle is represented by a curved background determined as a solution of the Einstein equation. The properties of the particle such as mass, charge, angular momentum, etc., are inherited in the background. Likewise, a D-brane system in string theory is represented by a curved background which is now a solution of the equations of motion for the ten dimensional supergravity theory. On the other hand, as explained in the previous section, a gauge theory is realized on the D-brane system. Then, it is not hard to imagine that the properties of the gauge theory is somehow hidden in the curved background. More ambitiously, one might conjecture that the gauge theory and the string theory in the corresponding curved background are actually equivalent. This is the basic idea of the gauge/gravity correspondence. To emphasize that the gravity side is string theory, it is also called gauge/string correspondence or gauge/string duality. One of the surprising feature of the correspondence is that the space-time dimension of the gravity side is higher than that of the dual gauge theory. From this reason, the gravity side is sometimes called a holographic dual description.

2.1. $\mathcal{N}=4$ *super Yang-Mills*

A typical example is given by considering N_c D3-branes. As explained in §1.1, the gauge theory realized on it is $\mathcal{N}=4$ super Yang-Mills theory. The supergravity solution corresponding to this D3-brane system is known as $AdS_5 \times S^5$. Motivated by this observation, $\mathcal{N}=4$ super Yang-Mills theory is conjectured to be equivalent to string theory in $AdS_5 \times S^5$ background.[2] One evidence that we can immediately find is the symmetry of the system. Since $\mathcal{N}=4$ super Yang-Mills theory is a conformal field theory, it has conformal symmetry $SO(2,4)$. In addition, there is an $SO(6) \simeq SU(4)$ symmetry that rotates six real scalar fields and four Weyl fermions. In the gravity side, $SO(2,4)$ and $SO(6)$ appear as the isometry groups of AdS_5 and S^5, respectively. Generalizing this consideration, one can show that a gravity dual of a conformal field theory (if it exists) always contains AdS space as a part of the space-time. This type of gauge/gravity correspondence is often called AdS/CFT correspondence.

Although there are numerous number of examples with highly non-trivial evidence supporting this idea, it is still a conjecture without any satisfying proof. It is therefore important to analyze both gauge theory side and gravity side to see if they are consistent with the conjectured correspondence. Such analysis have been done mostly in the cases with supersymmetry and conformal symmetry. It is more challenging to see if we can apply this idea to non-supersymmetric non-conformal field theory such as QCD.

2.2. *Pure Yang-Mills theory*

As we have learned in §1.3, Yang-Mills theory is constructed by D4-branes wrapped on an S^1. Fortunately, the corresponding supergravity solution is explicitly known.[4] Instead of writing down the explicit metric, here we briefly describe rough structure of the geometry. The topology of the background is $\boldsymbol{R}^{1,3} \times \boldsymbol{R}^2 \times S^4$. The

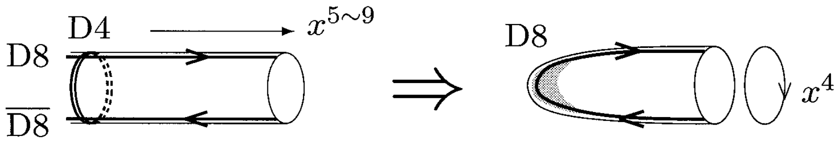

Fig. 1. Replacing D4-brane with the corresponding supergravity solution.

first $\boldsymbol{R}^{1,3}$ factor is the four dimensional Minkowski space where we are living. When we parameterize the second $\boldsymbol{R}^2$ factor with polar coordinates, the radial direction corresponds to the radial direction of the five dimensional space parameterized by $(x^5, \ldots, x^9)$ and the angular direction corresponds to the x^4 which is compactified to S^1. (See right side of Fig. 1.) The S^4 factor corresponds to the angular direction in the $(x^5, \ldots, x^9)$ plane.

The string theory in this background is considered as the gravity dual of pure Yang-Mills theory at low energy. Various physical quantities in Yang-Mills theory are computed using this gravity description and they are compared with lattice calculations. (See Ref. 5) for a review.) Although the approximation in the supergravity side is not good enough, they roughly agree well. We will see some of the results in the next section.

2.3. *QCD with N_f massless quarks*

In §1.4, we constructed QCD by adding D8-$\overline{\mathrm{D8}}$ pairs to the D4-brane system. Since the supergravity solution corresponding to the whole brane configuration is not known, we follow the strategy which was used in Ref. 3) for the D3-D7 system described in §1.2.

First, we assume the number of the D4-branes is much larger than the number of the D8-$\overline{\mathrm{D8}}$ pairs, i.e. $N_c \gg N_f$. Then, we replace only the D4-branes with the corresponding supergravity solution explained in the previous subsection and treat D8-$\overline{\mathrm{D8}}$ pairs as probe branes embedded in this curved background. This procedure is justified when all the $\mathcal{O}(N_f/N_c)$ corrections are neglected. The brane configuration is depicted in Fig. 1. Note that the N_f D8-$\overline{\mathrm{D8}}$ pairs are connected to form N_f D8-branes embedded in the curved background when we replace D4-brane with the corresponding supergravity solution. Therefore, four dimensional QCD with N_f massless quarks is conjectured to be dual to string theory in this background with N_f probe D8-branes (at low energy).

2.4. *Meson effective theory*

In the model considered in the previous subsection, we can easily find particles that correspond to hadrons in QCD. Obviously, closed strings and open strings are the candidates. The closed strings exist even when D8-branes are not added. They are actually interpreted as glueballs. The open strings are attached on the probe D8-branes. Since they carry flavor indices associated with the end points of the open strings, they are interpreted as mesons. In addition, if we wrap a D4-brane on the S^4 of the background, it behaves as a point-like particle in the four dimensional space-time since they are not extended along the non-compact spatial directions. Such D4-branes are interpreted as baryons.

The effective theory of the open strings on the D8-branes is a nine dimensional $U(N_f)$ gauge theory. Here, we only consider fields that are invariant under $SO(5)$ symmetry that acts as rotation of the S^4, and ignore all the higher Kaluza-Klein modes. Then, the effective theory is reduced to a five dimensional gauge theory. By inserting the supergravity background to the effective action of the D8-brane and integrating over the S^4, we obtain the five dimensional effective action*)

$$S_{\rm D8} \simeq \kappa \int d^4x\, dz\, {\rm Tr}\left(\frac{1}{2}K^{-1/3}F_{\mu\nu}^2 + M_{\rm KK}^2 K F_{\mu z}^2\right) + \frac{N_c}{24\pi^2}\int_5 \omega_5(A)\ , \qquad (2\cdot1)$$

where κ is a constant, $\omega_5(A)$ is the Chern-Simons 5-form, z is the fifth coordinate and $K(z) = 1 + z^2$. The claim is that this five dimensional $U(N_f)$ Yang-Mills - Chern-Simons theory is considered as the effective theory of mesons. It is quite interesting to note that this 5 dimensional description of mesons is closely related to 5 dimensional phenomenological models proposed in Ref. 6).

Let us next explain how to extract 4 dimensional physics from the 5 dimensional action (2·1). First we expand the gauge field (A_μ, A_z) using some complete sets $\{\psi_n(z)\}_{n\geq1}$ and $\{\phi_n(z)\}_{n\geq0}$ as

$$A_\mu(x^\mu, z) = \sum_{n\geq1} B_\mu^{(n)}(x^\mu)\psi_n(z)\ , \quad A_z(x^\mu, z) = \sum_{n\geq0}\varphi^{(n)}(x^\mu)\phi_n(z)\ . \qquad (2\cdot2)$$

These complete sets are chosen so that the kinetic and mass terms for the 4 dimensional fields $B_\mu^{(n)}$ and $\varphi^{(n)}$ become diagonal. We choose $\{\psi_n\}_{n\geq1}$ as eigenfunctions satisfying

$$-K^{1/3}\partial_z(K\partial_z\psi_n) = \lambda_n\psi_n\ , \quad \kappa\int dz\, K^{-1/3}\psi_n\psi_m = \delta_{nm}\ . \qquad (2\cdot3)$$

Here λ_n are the eigenvalues. Then we choose $\{\phi_n\}_{n\geq1}$ as $\phi_n = \partial_z\psi_n$ and, in addition, we have $\phi_0(z) = c/K(z)$, where the normalization constant c is given by $c = M_{\rm KK}^{-1}(\kappa\pi)^{-1/2}$.

Inserting the expansion (2·2) into the action (2·1) we obtain

$$S_{\rm D8} \sim \int d^4x {\rm Tr}\left[\partial_\mu\varphi^{(0)2} + \sum_{n\geq1}\left(\frac{1}{2}F_{\mu\nu}^{(n)2} + \lambda_n M_{\rm KK}^2\left(B_\mu^{(n)} - \partial_\mu\varphi^{(n)}\right)^2\right)\right]$$
$$+ \text{(interaction terms)}\ , \qquad (2\cdot4)$$

where $F_{\mu\nu}^{(n)} = \partial_\mu B_\nu^{(n)} - \partial_\nu B_\mu^{(n)}$. From this we see that $\varphi^{(n)}$ with $n \geq 1$ are eaten by $B_\mu^{(n)}$, which become massive vector fields. On the other hand, $\varphi^{(0)}$ does not have a partner vector field '$B_\mu^{(0)}$' in the expansion (2·2) and remain as a massless scalar field. We interpret $\varphi^{(0)}$ as the pion field, the lightest vector meson $B_\mu^{(1)}$ as the ρ meson, the second lightest vector meson $B_\mu^{(2)}$ as the a_1 meson and so on. The spin, parity and charge conjugation parity of π, ρ, a_1 mesons are consistent with this

*) There is also a scalar field on the D8-brane, but here we will omit it for simplicity.

interpretation. Actually, from the fact that ϕ_0 and ψ_{2k-1} are even functions while ψ_{2k} are odd function, it can be shown that $\varphi^{(0)}$ is a pseudo-scalar meson, $B_\mu^{(2k-1)}$ are vector mesons and $B_\mu^{(2k)}$ are axial-vector mesons for $k = 1, 2, \cdots$. In contrast to usual construction of effective theory of mesons, in which each meson field is introduced independently, various mesons π, ρ, a_1, $\cdots$ are now elegantly unified in the 5 dimensional gauge field (A_μ, A_z) in our description.

§3. Applications

Here we present some of the results obtained for pure Yang-Mills and QCD. Before going into the details, let us summarize how realistic our "QCD" is.

First, note that our analysis is based on supergravity approximation that is reliable only when the number of color N_c and the 't Hooft coupling λ are large. We also assumed $N_c \gg N_f$ in §2.3. The realistic value of N_c is 3 and it is not clear if the $1/N_c$ (or N_f/N_c) expansion really works. In the construction of Yang-Mills theory in §1.3, we compactified one of the direction and hence we know that the model deviates from QCD above the energy scale around the compactification scale M_{KK}. It is unfortunately difficult to take a limit $M_{\text{KK}} \to \infty$, since the asymptotic freedom of QCD requires $\lambda \to 0$ in this limit and supergravity approximation breaks down. Furthermore, since all the quarks in our model is massless, it is not possible to make good predictions for quantities that are sensitive to the quark mass. In the following, we mainly consider $N_f = 2$ case, in which we only take into account up and down quarks whose masses are reasonably small compared to Λ_{QCD}.

From these reasons, we should not be too serious about the agreement (or disagreement) with experiments. Our numerical results may only be useful to see if our model is too bad or not. But, believe it or not, the agreement is quite impressive as we will see below.

3.1. *Wilson (Polyakov) loop*

Wilson (Polyakov) loop can be calculated in the gravity side by considering a string world-sheet whose boundary condition is specified by the given loop.[7] For pure Yang-Mills theory, the area law behavior of the Wilson loop is found. It is also possible to introduce temperature to the system. It is found that there is a critical temperature above which the background geometry is changed. This transition corresponds to confinement/deconfinement phase transition.[4]

3.2. *Chiral symmetry breaking*

As explained in §2.3, the D8-brane and $\overline{\text{D8}}$-brane in the original brane configuration is forced to be connected when we replace the D4-brane with the corresponding supergravity solution. This phenomenon is interpreted as the chiral symmetry breaking in QCD. In fact, as explained in §1.4, $U(N_f)_{\text{D8}} \times U(N_f)_{\overline{\text{D8}}}$ symmetry associated with the N_f D8-$\overline{\text{D8}}$ pairs is interpreted as the chiral symmetry in QCD, but after replacing the D4-branes with the supergravity solution, we have only one connected component of D8-branes and hence we have only one factor of $U(N_f)$ left unbroken. This is nothing but the chiral symmetry breaking pattern for QCD.

3.3. *Quantitative tests for pure Yang-Mills*

There are a lot of works concerned with the supergravity analysis of pure Yang-Mills theory based on the construction given in §2.2. They were mostly done around 1998 and nicely reviewed in Ref. 5). Here we only list some of the results taken from Ref. 8) to show how it works. Glueball masses $M_{0^{\pm+}}$, QCD string tension, topological susceptibility χ_t, Gluon condensate are calculated and compared with the lattice calculation.

$$\frac{M_{0^{++}}^{\rm SUGRA}}{M_{0^{++}}^{\rm Lattice}}=1.2\ ,\quad \frac{M_{0^{-+}}^{\rm SUGRA}}{M_{0^{-+}}^{\rm Lattice}}=1.1\ ,\quad \left(\frac{(\text{QCD string tension})^{\rm SUGRA}}{(\text{QCD string tension})^{\rm Lattice}}\right)^{1/2}=2.0\ ,$$
$$\left(\frac{\chi_t^{\rm SUGRA}}{\chi_t^{\rm Lattice}}\right)^{1/4}=1.7\ ,\quad \left(\frac{(\text{Gluon condensate})^{\rm SUGRA}}{(\text{Gluon condensate})^{\rm Lattice}}\right)^{1/4}=0.9\ . \tag{3·1}$$

3.4. *Quantitative tests for QCD*

As we have seen in Eq. (2·4), the mass of the n-th vector meson is given by $m_n^2=\lambda_n M_{\rm KK}^2$ where λ_n is the eigenvalue of the eigenequation (2·3). It is not difficult to compute the eigenvalues numerically.

Here is the result:

	ρ	a_1	ρ'	(a_1')	ρ''
exp.(MeV)	776	1230	1459	(1647)	1720
our model	[776]	1189	1607	2023	2435

where we have fixed the value of $M_{\rm KK}$ to fit the rho meson mass.

We can also extract the interaction terms in (2·4). Here we skip all the details and show the results:

coupling	our model	fitting m_ρ and f_π	experiment
f_π	$1.13\cdot\kappa^{1/2}M_{\rm KK}$	[92.4 MeV]	92.4 MeV
L_1	$0.0785\cdot\kappa$	0.584×10^{-3}	$(0.1\sim0.7)\times10^{-3}$
L_2	$0.157\cdot\kappa$	1.17×10^{-3}	$(1.1\sim1.7)\times10^{-3}$
L_3	$-0.471\cdot\kappa$	-3.51×10^{-3}	$-(2.4\sim4.6)\times10^{-3}$
L_9	$1.17\cdot\kappa$	8.74×10^{-3}	$(6.2\sim7.6)\times10^{-3}$
L_{10}	$-1.17\cdot\kappa$	-8.74×10^{-3}	$-(4.8\sim6.3)\times10^{-3}$
$g_{\rho\pi\pi}$	$0.415\cdot\kappa^{-1/2}$	4.81	5.99
g_ρ	$2.11\cdot\kappa^{1/2}M_{\rm KK}^2$	0.164 GeV^2	0.121 GeV^2
$g_{a_1\rho\pi}$	$0.421\cdot\kappa^{-1/2}M_{\rm KK}$	4.63 GeV	$2.8\sim4.2$ GeV

The middle column of this table is the values obtained by fixing $M_{\rm KK}$ and κ to fit the rho meson mass and pion decay constant f_π. Though the agreement of our numerical results with the experimental data is not extremely good, we think it is much better than expected.

3.5. *More lessons*

We close this section by listing some other interesting topics in our paper[1] that are omitted here.

• *Structure of interaction*

It is shown that the structure of interaction is consistent with various phenomenological models proposed in the old days. For example, the model naturally reproduces hidden local symmetry, vector meson dominance and a phenomenological model for omega meson decay (GSW model).

• *Anomalies*

The chiral anomaly in QCD is elegantly reproduced from the Chern-Simons term in (2·1). It provides an easy derivation of the WZW-term. The $U(1)_A$ anomaly is also understood in the supergravity description. Taking this anomaly into account, the mass of the η' meson is shown to satisfy the Witten-Veneziano formula.

• *Baryon*

A baryon is constructed as a D4-brane wrapped on the S^4. as mentioned in §2.4. This wrapped D4-brane is realized as an instanton in the five dimensional Yang-Mills theory (2·1) and shown to be equivalent to the soliton (Skyrmion) in the Skyrme model. (See Ref. 9) for recent progress.)

§4. Discussion

The fact that open strings represent mesons reminds us of the old idea of string theory around late 60's when it was born as a theory of hadron. One of the problem at that time was that string theory requires unrealistic ten dimensional space-time. Our model seems to suggest that this idea was essentially correct. New ingredients added here are D-branes, curved background and holography. Using D-branes and a curved background, we have seen that four dimensional QCD can be described by ten dimensional string theory. Having ten dimensional space-time is no longer a problem once we accept the idea of holography.

Nowadays everyone believes that QCD is the fundamental theory of hadrons and the hadronic string is considered as an effective object. However, gauge/string correspondence states that QCD and its dual string theory (if it really exists) are actually equivalent, suggesting that both QCD and string theory could be fundamental theory of hadrons at the same time. In this way, it may provide a new perspective to the concept of "elementary particle" in the real world.

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