Symmetry Properties of Black Holes in Higher Dimensional General Relativity*)

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We discuss symmetry properties of black holes in general relativity—known as *black hole rigidity*—of which basic assertion is that the event horizon of an asymptotically flat, stationary black hole with certain matter fields must be a Killing horizon and is rephrased (combined together with staticity results) that such a black hole must be either static, or axisymmetric. A precise formulation of the rigidity theorem for black holes with non-degenerate event horizon in arbitrary spacetime dimensions has been recently made by Hollands, Wald and the present author. [Hollands, S., Ishibashi, A. and Wald, R. M., Commun. Math. Phys. **271** (2007), 699.] In our formulation, no assumptions concerning the topology of cross-sections of event horizon (other than the compactness) are made. Therefore, different from Hawking's original proof given in 4-dimensions, our proof applies also to non-spherical black holes, which are known to occur in higher dimensions but not in 4-dimensions.

§1. Introduction

Many attempts to unify the forces in nature, such as string-theories, require more than 4-dimensions to formulate. Also, recent phenomenological ideas, such as braneworld models, have renewed interest in extra-dimensions. In order to develop such higher dimensional theories and derive their physical consequences, black hole solutions in higher dimensions play an important role. Below is a very brief overview of recent results concerning basic, mathematical properties of asymptotically flat, stationary black holes in higher dimensions (within the context of general relativity) and comparison with corresponding, established results concerning stationary black holes in 4-dimensions.

(1) Exact solutions: In 4-dimensions, due to the black hole uniqueness theorem, possible exact black hole solutions are rather restricted: Astrophysically relevant solutions are given essentially by the Kerr metric. In contrast, there seems to be a much larger variety of exact black hole solutions in higher dimensions. For example, in addition to a natural higher dimensional generalization of the Kerr metric[37)] (see also, e.g., Ref. 16)), one has rotating black-ring solutions in 5-dimensions.[10)] (See also Refs. 32),38),46).) Furthermore, combining black rings and/or a rotating hole, one can construct more complicated configurations of multi black solutions, such as di-rings, multi-ring saturns, orthogonal di-rings/bi-rings etc (see e.g., Refs. 7),8),27) and references therein).

(2) Stability: Once one obtains some exact solutions, the next main concern would be whether the solutions are stable or not since the stability of a solution implies that the solution may describe a possible final state of dynamical process, e.g., grav-

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itational collapse. It has now been established that the Kerr metric is stable against mass-less scalar fields and linear gravitational perturbations at least, at the level of mode-by-mode analysis.[48)]

In higher dimensions, the stability has not been fully studied yet, but there have appeared some partial results. It was shown, for example, that static, vacuum black holes (i.e., higher dimensional Schwarzschild type black holes) are stable against gravitational perturbations.[24),28),29)] For rotating black holes, gravitational perturbations have been considered so far only on the backgrounds of odd dimensional Myers-Perry holes with all angular momenta being equal. (Such a background possesses enhanced isometries and is said to be *co-homogeneity one*.) For co-homogeneity one Myers-Perry holes in odd $d \geqslant 7$, some restricted class of tensor-type perturbations with respect to $(d-3)$ base space has been analysed.[31)] For the same type Myers-Perry holes in $d=5$, decoupled master equations for gravitational perturbations have recently been obtained by Murata and Soda.[36)]

(3) Topology: Apart from analysis of exact solutions, one can show in more general context that spatial cross-sections of the event horizon of stationary black holes must be topologically 2-sphere,[6),17),18)] on the assumptions of stationarity, asymptotic flatness, and certain energy conditions.

In higher dimensions, we have more variety in the topology of cross-sections of the event horizon, having already, as an explicit example, black-ring solutions with non-spherical horizon topology $S^1 \times S^2$ in 5-dimensions. But other than ring solutions, no black objects of which connected component of the event horizon has non-spherical topology have not been discovered yet. There are, however, certain restrictions on the possible topology: Galloway and Schoen[12),13)] have shown that the horizon topology must be such that its cross-section admits a metric with positive scalar curvature.

(4) Symmetry: In 4-dimensions, one can prove a theorem concerning Killing symmetry which states that the event horizon of a stationary black hole must be a Killing horizon. It then follows from this result that if such a black hole is rotating, then the spacetime must be axisymmetric. This symmetry property is called the *rigidity property*.[3),4),17),18)] Combined together with some other results,[45)] this theorem yields that a stationary black hole must be either static, or axisymmetric. This symmetry/rigidity theorem has recently been generalized to arbitrary spacetime dimensions by Hollands, Wald, and the present author.[19)] We will present precise statements of our symmetry/rigidity theorems and provide a brief sketch of our proof in the next section.

(5) Uniqueness: In 4-dimensions, using the theorems concerning topology (3) and symmetry properties (4) mentioned above (and with the help of non-trivial identities), one arrives at the celebrated black hole uniqueness theorem[1),2),30),40)] which states that the Kerr metric is the only vacuum solution that describes an asymptotically flat, rotating black hole. For static case, see Refs. 25), 26). (This has been generalized to the case including electro-magnetic field.) Together with the (weak version of) cosmic censorship conjecture and the stability result (2), the uniqueness implies that astrophysically relevant black holes formed via gravitational collapse in our universe are described very well by the Kerr metric.

In higher dimensions, uniqueness theorem of this type (i.e., type that an isolated gravitating system is completely specified by the set of *global charges*) no longer holds as it stands. Since in 5-dimensions we have a rotating hole and (two) rotating ring solutions with exactly the same mass and angular momentum, clearly, the uniqueness theorem fail to hold. However, for some restricted case, such as static, vacuum holes[14),15)] or 5-dimensional stationary, vacuum black holes with multiple rotational symmetries, certain types of uniqueness theorems have been proven.[20),33),34)] (See also Refs. 41),42) and references therein.)

(6) Thermodynamics: The correspondence between black hole mechanics and thermodynamics has been generalized to higher dimensions almost as it stands within the context of general relativity. The relevant parameters appeared in the 1st-law of thermodynamics depend on configurations of black objects. (See, e.g., Ref. 44) and references therein.)

In the subsequent sections we will take a closer look at the symmetry properties of black holes in higher dimensional general relativity.

§2. Symmetry/rigidity property

We begin with recalling that in general relativity, a black hole horizon is defined as the *event horizon*, $\mathcal{H}$, namely, a boundary of the causal past of idealized distant observers or a (future) null infinity. There is another distinguished notion of a horizon, a *Killing* horizon, which is defined as a null hypersurface $\mathcal{N}$ to which a Killing symmetry vector K^a is normal. (Note here that K^a is assumed to generate a one-parameter group of isometries, and hence, in particular, that the orbits of K^a are complete.) It is, in general, not at all obvious when an event horizon can be a Killing horizon. In 4-dimensions it can be shown that

(A) The event horizon of a stationary, electro-vacuum black hole must be a Killing horizon, and **(B)** if, furthermore, the stationary black hole is rotating, then that black hole spacetime must be axi-symmetric.

This assertion implies that the event horizon is *rigidly* rotating with respect to infinity. For this reason, this symmetry property is called the *black hole rigidity*.[3)] The rigidity theorem was proven for the first time by Hawking[17),18)] in 4-dimensions.

There are good reasons why the symmetry/rigidity is interesting:

(i) The rigidity property connects a global notion of event horizon to a local notion of Killing horizon. This is especially relevant when one wants to explicitly compute physical, locally defined quantities associated with the event horizon.

(ii) It helps to establish (a part of) the foundation of black hole thermodynamics. This is because one can define a surface gravity for a Killing horizon, and the surface gravity corresponds to a black hole temperature. Since we can also show that the surface gravity is constant over the Killing horizon, it implies that the temperature of the event horizon is constant. Thus, it proves the 0th-law of black hole thermodynamics.

(iii) It implies that when black hole is rotating, starting from the existence of merely a single (stationary) symmetry, one can get an additional symmetry, i.e., axial-symmetry.

(iv) In 4-dimensions, having two Killing symmetries of stationary symmetry and axial-symmetry, one can reduce the Einstein equations into a certain simple form, and then applying non-trivial identities, one can show the black hole uniqueness. Thus, the rigidity theorem is a critical step toward a proof of the uniqueness theorem in 4-dimensions.

(v) As briefly mentioned above, the uniqueness theorem no longer holds as it stands in higher dimensions, and there seems to be a much larger variety of possible exact black hole solutions, whose classification has not been fully achieved yet. If the rigidity holds also in higher dimensions, then one may be able to place some important restrictions on possible exact solutions, on the symmetry ground.

It should be noted that the proof given by Hawking relies heavily on the fact that event horizon cross-section Σ is topologically 2-sphere, and therefore does not work in higher dimensions. A generalization of the rigidity theorem to higher dimensions needs some new idea so that proof would not require any assumptions concerning the topology of spatial cross-sections of the event horizon, other than that they are compact, connected. This has been recently done in Ref. 19). The precise statement of our rigidity theorems and brief sketch of the proof are as follows:

Theorem 1: Consider an asymptotically flat, stationary analytic, vacuum black hole solution to the Einstein's equations in arbitrary dimensions. Assume that the event horizon $\mathcal{H}$ be analytic, non-degenerate, and topologically $\boldsymbol{R} \times \Sigma$ with cross-sections Σ being compact, connected. There exits a Killing field K^a in the entire exterior of the black hole such that K^a is normal to $\mathcal{H}$ and commutes with the stationary Killing vector filed t^a.

This establishes that the event horizon is a *"Killing horizon"*.

Theorem 2: If t^a is *not* normal to $\mathcal{H}$ (i.e., $t^a \neq K^a$), then there exist mutually commuting Killing vector fields $\varphi^a_{(1)}$ $, \cdots,$ $\varphi_{(j)}$ $(j \geq 1)$ with period 2π and $t^a = K^a + \Omega_{(1)}\varphi^a_{(1)} + \cdots + \Omega_{(j)}\varphi^a_{(j)}$, where $\Omega_{(j)}$'s constants.

This establishes, *"axisymmetry"* of a stationary, rotating black holes in arbitrary dimensions. Theorem 2 implies that if the orbits of $s^a \equiv t^a - K^a$ on a cross-section Σ fail to be closed, then the spacetime has to possess at least two linearly independent rotational Killing vector fields $\varphi_{(i)}$.

Our proof of Theorem 1 consists of the following two steps:

1.—Find a "candidate" Killing vector field on the event horizon $\mathcal{H}$

2.—Extend the candidate Killing vector field defined on $\mathcal{H}$ to the entire spacetime.

In Step 1., we wish to find a candidate Killing vector field K^a which possesses the following properties:

(i) K^a should be null on the horizon $\mathcal{H}$ (i.e., normal to $\mathcal{H}$) and commute with t^a

(ii) K^a should satisfy the Killing equation, $\pounds_K g_{ab} = 0$, on $\mathcal{H}$
(iii) K^a should have a constant surface gravity α with $K^c\nabla_c K^a = \alpha K^a$ over $\mathcal{H}$.

How can we find such a candidate Killing vector field? Since we assume that our spacetime is stationary, we already have, at least, one Killing symmetry, namely, t^a. Since it generates an isometry group, it must be tangent to the event horizon. But in general, it is not necessarily null, as the hole may be rotating. So, we consider the case that t^a is spacelike on the horizon. Then, if we choose a foliation, or a cross-section of the horizon, Σ, in an appropriate way,[5] so that each orbit of t^a intersects Σ only at a single point, we can decompose t^a into a tangential direction, s^a, to the cross-section Σ and the null direction, K^a, normal to the cross-section, i.e., $t^a = s^a + K^a$. Since t^a is a Killing field, the decomposed components s^a and K^a also satisfy a type of equations similar to the Killing equation. In fact, it turns out that s^a becomes a Killing vector field with respect to the chosen cross-section Σ. One may consider the decomposition, K^a, as a possible candidate Killing field. A key issue at this point is that decomposition of t^a and hence K^a depends on the choice of a cross-section Σ. Thus, although it immediately follows that K^a satisifies the properties (i) and (ii), K^a in general fails to satisfy (iii); there is no reason why α need be constant. So, we would like to find a desired $\tilde{K}^a$ on $\mathcal{H}$ by choosing a "correct cross-section $\tilde{\Sigma}$" so that the corresponding surface gravity, $\tilde{\alpha}$, becomes constant over $\mathcal{H}$. Since both K^a and $\tilde{K}^a$ are null, the desired $\tilde{K}^a$ should be proportional to the original K^a, i.e., $\tilde{K}^a = fK^a$ with some function $f(x)$. Our task is to find a solution to equation for the change of cross-sections from the original Σ to the desired, correct one, $\tilde{\Sigma}$. The transformation equaion is given on Σ by

$$-\pounds_s f(x) + \alpha(x) f(x) = \tilde{\alpha} =: \kappa\,. \tag{2·1}$$

When one solves this equation, the spacetime dimensionality comes to play a key role. Let us consider the 4-dimensional case first. Since in 4-dimensions, the topology theorem states that cross-section Σ must be topologically 2-sphere. Then, the action, ϕ_s, of the Killing field s^a on Σ must have a fixed point, and thus ϕ_s has closed orbits with some period, P. Thus, any point on this Σ is mapped by the action ϕ_s to the same point after the period, P. Since each point of Σ corresponds to a null geodesic generator of the event horizon $\mathcal{H}$, there is a *discrete isometry* Γ which maps each null generator into itself. This discrete isometry, Γ, helps to set the surface gravity κ to be

$$\kappa = \frac{1}{P}\int_0^P \alpha[\phi_s(x)]\mathrm{d}s$$

and, furthermore, helps to find the desired correct $\tilde{\Sigma}$ and show Step 2.

In higher dimensions $d > 4$, however, there is no reason that the isometry s^a need have closed orbits on Σ, and in general, there is no discrete isometry analogous to Γ. This can be seen in 5-dimensional Myers-Perry black hole with 2-rotations $\Omega_{(1)}$, $\Omega_{(2)}$. In this case, Σ is topologically 3-sphere, and s^a is given by a linear combination of two rotational symmetries $\varphi^a_{(1)}$ and $\varphi^a_{(2)}$,

$$s^a = \Omega_{(1)}\varphi^a_{(1)} + \Omega_{(2)}\varphi^a_{(2)}\,. \tag{2·2}$$

Of course, each rotation Killing vector $\varphi^a_{(i)}$ has closed orbits, but s^a itself does not when $\Omega_{(1)}$ and $\Omega_{(2)}$ are incommensurable. So, there is, in general, no dersiered descrete isometry in higher dimensions. (Note that this could be the case even in 4-dimensions if the horizon topology was not 2-sphere but was, for example, 2-torus.) So, we need some new method to show the rigidity in higher dimensions.

The idea is when s^a does not have closed orbits, we apply the *von Neumann ergodic theorem* (see e.g., Ref. 47)), which yields, in the present case, that

$$\textit{Long-time average:}\quad \alpha^*(x) := \lim_{T\to\infty}\frac{1}{T}\int_0^T \alpha[\phi_s(x)]\mathrm{d}s \quad \textit{exists}. \tag{2·3}$$

Then, with the help of the vacuum Einstein's equations one can show that α^* is, in fact, constant and is to be identified with

$$\textit{Spatial average:}\quad \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma \alpha(x)\mathrm{d}\Sigma =: \kappa\,. \tag{2·4}$$

Furthermore, with the help of the ergodic theorem, one can find a solution to Eq. (2·1) as

$$f(x) = \kappa\int_0^\infty P(x,T)\mathrm{d}T\,,\quad P(x,T) = \exp\left(-\int^T \alpha[\phi_s(x)]\mathrm{d}s\right)\,,$$

which is, in fact, well-defined since for any small $\epsilon > 0$, one has $P(x,T) < e^{(\epsilon-\kappa)T}$, for sufficiently large T, and thus one can obtain a candidate Killing field $\tilde{K}^a$ with desired properties on $\mathcal{H}$. Furthermore one can find a correct foliation $\tilde{\Sigma}$ and complete Step 1.

In Step 2., by using vacuum Einstein's equations and making inductive arguments, one can show

$$\underbrace{\pounds_\ell\pounds_\ell\cdots\pounds_\ell}_{m\ \text{times}}(\pounds_{\tilde{K}}g_{ab}) = 0, \quad m = 0,1,2,\ldots \quad \text{on } \mathcal{H}\,. \tag{2·5}$$

Then, by invoking the analyticity, one can extend $\tilde{K}^a$ to the entire spacetime and complete the proof of Theorem 1.

Our proof of Theorem 2 is briefly sketched as follows. By Theorem 1, we have the horizon Killing vector field $\tilde{K}^a$ and preferred cross-section $\tilde{\Sigma}$. Since $\tilde{s}^a = t^a - \tilde{K}^a$ generates an abelian group, $\mathcal{G}$, of isometries on horizon cross-sections $\tilde{\Sigma}$, if $\tilde{s}^a$ has a closed orbit, then there exists $U(1)$ isometry and we are done. If not, we can invoke the following proposition (see, e.g., Ref. 23)): The closure of $\mathcal{G}$ on a compact space—in our case, $\tilde{\Sigma}$—must be a torus of dimension N, so $\mathcal{G}$ is written as the direct product of N factors of $U(1) \approx U(1)^N$, where $N = \dim(\bar{\mathcal{G}}) \geqslant 2$. Thus, we have N commuting Killing fields on $\tilde{\Sigma}$. Then, we Lie-drag them into $\mathcal{H}$ by $\tilde{K}^a$ and then as in Step 2., we extend $U(1)^N$ into the entire spacetime by invoking the analyticity.

§3. Remarks

A few remarks are in order.

It is immediate to generalize the present results obtained for vacuum black holes to the case of Einstein-Λ-Maxwell system. Combined with *staticity theorems* (see, e.g., Refs. 43),45)) the rigidity theorems above are rephrased that *stationary, non-extremal black holes in $d \geqslant 4$ Einstein-Maxwell system are either static or axisymmetric.*

The theorems above apply not only to black hole horizon but also any horizon defined as the "boundary" of causal past of a complete timelike orbit of some Killing vector field. So, for example, the theorems can apply to cosmological horizon.

We assume that spacetime is analytic, so that we can use the analytic continuation of the candidate horizon Killing vector to the black hole exterior. Actually one can remove this analyticity assumption for the black hole interior, following similar strategy of Refs. 11),39). Since the horizon turns out to be a part of bifurcate horizon for non-degenerate case, we can use a certain part of the horizon as an initial data surface for the candidate horizon Killing field. Then, applying charachteristic initial value formulation to extend the candidate horizon Killing field into the interior of the black hole, which is the domain of dependence for the initial date surface. Although this type of characteristic initial value problem is ill-defined toward the exterior region and therefore would not appear to be useful to remove the analyticity assumption for the black hole exterior. Nevertheless, a remarkable progress has recently been made along this direction.[21),22)]

The present proof relies on Einstein's equations. Therefore a generalization to black holes in other theories, such as theories with higher derivative terms, seems highly non-trivial. Also, as inspired by higher dimensional theories one may be interested in black holes with compactified extra-dimensions. However, if infinity or black hole's exterior has some non-trivial topology, then it is likely that horizon Killing field K^a (if exits) may not have a single-valued analytic extension. For such a case, one would need to do case-by-case analysis.

Finally, we are also interested in extremal black holes with a degenerate horizon, i.e., $\mathcal{H}$ with $\kappa = 0$. We believe that similar rigidity results should hold also for extremal black holes, but we have not fully analysed this case yet.

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