Polyakov-Nambu-Jona Lasinio Model and Color-Flavor-Locked Phase of QCD

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The effect of Polyakov loop on the QCD phase diagram at high density is studied within the Nambu-Jona Lasinio model with Polyakov loop (PNJL model). We point out that the color neutrality is missing in the standard PNJL model at finite density. Moreover, we discuss how the color-flavor locked (CFL) phase is to be distorted by the inclusion of Polyakov loop.

§1. Introduction

The phase diagram of strongly interacting matter has been a subject of theoretical/experimental work since the foundation of Quantum Chromodynamics (QCD). The perturbative QCD can be of some help in the extremely high density regime, but it is no longer reliable at density of physical interest. The lattice QCD is a powerful tool to study such a strongly interacting regimes. However there is a well-known difficulty in simulating QCD on lattice at finite density. So exploring phase structure at intermediate density, where neither lattice simulations nor perturbative calculations can be trusted, remains the subject of various model studies which mimic some of basic features in QCD. The Nambu-Jona Lasinio (NJL) model is one of them[1)] and it nicely predicts the chiral restoration of QCD at extreme conditions.[2)]

The main defect of the NJL model had been the lack of the notion of the confinement. To improve this point, Fukushima included the Polyakov loop dynamics into the NJL model,[3)] and the model is now called "Polyakov-Nambu-Jona Lasinio" (PNJL) model. This model has two order parameters, $q\bar{q}$ for the chiral restoration, and the Polyakov loop Φ for the deconfinement. Even though these two serve as the exact order parameters only in the different limits, ($m_q \to 0$ and $m_q \to \infty$), the model enables to interpret nicely some bulk properties of matter observed on the lattice on the field theoretical ground.[4)]

In this work, we will extend the application of PNJL model to color superconducting phases at high density 5). In particular, we are interested in: (a) how the phase structure in $(T, m_s^2/\mu)$-plane will be modified by the inclusion of the Polyakov loop, and (b) what is the consequence of imposing the color/electrical neutralities on the PNJL model with and without diquark condensations. The purpose (a) is regarded as the extension of the earlier work,[6),7)] while (b) is considered as the extension of Refs. 8) and 9). We note there is a parallel development on the role of electrical neutrality in the PNJL model at low density.[10)]

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§2. The model

The Wilson line operator is a key quantity whose expectation value plays a role of order parameter for deconfinement transition in a pure gauge theory. It can be expressed by the background Euclidean temporal gauge field $A_4(\tau, \boldsymbol{x}) \equiv igA_0^\alpha(\tau, \boldsymbol{x})T_\alpha$ ($T_\alpha = \frac{\lambda_\alpha}{2}$; $\{\lambda_\alpha\}$ are the standard Gell-Mann matrices for $SU(3)_c$) as

$$L_Q = P_\tau \exp\left(i\int_0^{1/T} d\tau A_4(\tau, \boldsymbol{x})\right). \qquad (2\cdot1)$$

The Wilson line in the anti-triplet representation can be defined as $L_{\bar{Q}} \equiv L_Q^\dagger$. In a pure gauge theory with zero chemical potential for quarks ($\mu = 0$), these are regarded as the operators associated with heavy (anti)quark excitation in the gluonic heat bath at temperature T; loop $\Phi = \frac{1}{N_c}\langle \mathrm{tr}L_Q\rangle_T$ (anti-triplet loop $\bar{\Phi} = \frac{1}{N_c}\langle \mathrm{tr}L_{\bar{Q}}\rangle_T$) is related to the Free energy of single (anti)quark excitation in the gluon medium by $\Phi = e^{-F_Q/T}$ and $\bar{\Phi} = e^{-F_{\bar{Q}}/T}$. In this case, one finds eventually $\bar{\Phi} = \Phi = \Phi^*$. In the full gauge theory with dynamical quarks and with a finite chemical potential for quarks, however, this is no longer the case because quark and antiquark propagate differently in each direction of imaginary time. The detailed analysis of a matrix model shows that Φ and $\bar{\Phi}$ certainly differ from each other but still both stay real; this is due to the imaginary piece in the action proportional to the imaginary part of $\frac{1}{N_c}\mathrm{tr}L_Q$ which comes from integration of dynamical quarks and is C-odd quantity.[11)]

In the PNJL model, the loop Φ can be parametrized by eight *real* parameters $\{\varphi_1, \varphi_2, \cdots, \varphi_8\}$ each of which has a dimension of energy, as

$$\Phi[A_4] = \frac{1}{N_c}\mathrm{tr}e^{iA_4/T}, \quad A_4 = \sum_{\alpha=1}^{N_c^2-1}\varphi_\alpha T_\alpha. \qquad (2\cdot2)$$

Using this A_4 field, the PNJL model is given by the following Lagrangian density

$$\mathcal{L}[q,\bar{q};A_4] = \bar{q}(i(\not{\mathcal{D}}[A_4] + \gamma_0(\mu + \delta\mu_{\mathrm{eff}}))q + \frac{G}{4}\bar{q}P_{\eta\eta'}\bar{q}^T q^T \bar{P}_{\eta\eta'}q - \mathcal{U}(T, \Phi[A_4]). \quad (2\cdot3)$$

$q = (q_{ur}, q_{dg}, q_{sb},\, q_{ug}, q_{dr},\, q_{sr}, q_{ub},\, q_{db}, q_{sg})^T$ is the quark field. $\mathcal{D}_\mu = \partial_\mu - \delta_{\mu 0}A_4$ is the covariant derivative through which the Polyakov loop can change the nature of propagation of dynamical quarks. G parametrizes the strength of attractive coupling in the color-flavor channel $P_{\eta\eta'} = C\gamma_5\epsilon_{\eta ij}\epsilon_{\eta' ab}$ ($\bar{P}_{\eta\eta'} = \gamma_0 P_{\eta\eta'}^\dagger\gamma_0$). In this work, we take the CFL type *diagonal* ansatz for diquark condensation,[12)] i.e.,

$$\frac{G}{2}\langle q^T\bar{P}_{\eta\eta'}q\rangle = \begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix}_{\eta\eta'} \equiv \hat{\boldsymbol{\Delta}}_{\eta\eta'}. \qquad (2\cdot4)$$

η (η') stands for the flavor (color) index. We work within the chiral $SU(2)$ limit setting $m_u = m_d = 0$, and take into account the strange quark mass m_s within the high density approximation. So we set $\delta\mu_{\mathrm{eff}} = -\mu_e Q + \mu_3 T_3 + \mu_8 T_8 - \frac{m_s^2}{2\mu}\mathrm{diag.}(0,0,1)_{\mathrm{f}} \times \mathbf{1}_{\mathrm{c}}$, where $Q = \mathrm{diag.}(2/3, -1/3, -1/3)_{\mathrm{f}} \times \mathbf{1}_{\mathrm{c}}$, $T_3 = \mathbf{1}_{\mathrm{f}} \times \frac{1}{2}\lambda_3$, and $T_8 = \mathbf{1}_{\mathrm{f}} \times \frac{1}{\sqrt{3}}\lambda_8$. $\mathcal{U}(T, \Phi)$ is the Polyakov loop potential which controls the confinement/deconfinement transition in the pure gauge sector, whose detailed form will be given later.

The real part of effective potential*) within the high density effective theory (HDET) comes out to be

$$\begin{aligned}\Re\Omega(\Delta_\eta,\varphi_\alpha) &= \mathcal{U}(T,\Phi)-\tfrac{\mu_e^4}{12\pi^2}-\tfrac{\mu_e^2T^2}{6}-\tfrac{7\pi^2T^4}{180}\\ &-\textstyle\sum_{A=1}^9\left[\tfrac{(\mu+\delta\mu_{\rm eff}^A)^4}{12\pi^2}+\int\tfrac{(\mu+l_\parallel)^2dl_\parallel}{2\pi^2}2T\ln(\|1+e^{-E_A(l_\parallel)/T}\|)\right]\\ &+\textstyle\sum_\eta\tfrac{\Delta_\eta^2}{G}-\sum_{A=1}^9\int_{-\omega_c}^{\omega_c}\tfrac{(\mu+l_\parallel)^2dl_\parallel}{2\pi^2}\left[\Re E_A(l_\parallel)-|l_\parallel-\delta\mu_{\rm eff}^A|\right].\end{aligned} \tag{2·5}$$

$l_\parallel$ is the quark momentum measured from the Fermi surface $p=\mu$. ω_c is the ultraviolet cutoff needed to regularize the divergent third line. The complex energies $\{E_1,E_2,\cdots,E_9\}$ are defined by choosing the eigenvalues of non-hermitian matrix

$$\mathcal{H}=\begin{pmatrix} l_\parallel-\delta\mu_{\rm eff}+iA_4 & \Delta_\eta\epsilon_{\eta ab}\epsilon_{\eta ij}\\ \Delta_\eta\epsilon_{\eta ab}\epsilon_{\eta ij} & -l_\parallel+\delta\mu_{\rm eff}^t-iA_4^t\end{pmatrix}, \tag{2·6}$$

such that $E_A\to|l_\parallel-\mu_{\rm eff}^A|$ when $\Delta_\eta\to 0$ is satisfied. We use the cutoff (ω_c) dependent coupling $\frac{1}{G}=\frac{2\mu^2}{\pi^2}\ln\left(\frac{2\omega_c}{2^{1/3}\Delta_0}\right)$ where Δ_0 is the magnitude of CFL gap parameter at $T=0$ in the chiral limit ($m_s=0$). With this convention, the effective potential is only weakly (logarithmically) divergent, so the gap equations and neutrality conditions have well-defined finite limit as $\omega_c\to\infty$.

Parameter reduction via gauge invariance: The effective potential is a function of three gap parameters $\{\Delta_\eta\}$ and eight parameters $\{\varphi_\alpha\}$ for the Wilson line matrix. In certain cases, the gauge invariance is helpful to reduce the dynamical variables. $\Delta_\eta=0$ is such a case; the number of parameters for A_4 can be reduced from N_c^2-1 down to N_c-1 as we see below. Since the Wilson line transforms as $L_Q\to gL_Qg^{-1}$ by the gauge transformation with g being the arbitrary $SU(3)$ matrix, we see

$$\Omega(A_4)=\Omega(gA_4g^{-1}), \tag{2·7}$$

from the gauge invariance. We can always make A_4 diagonal by choosing the suitable g as $gA_4g^{-1}=\phi_3T_3+\phi_8T_8$, so we can work with this simplified ansatz for A_4 without loss of generality. The effective potential as well as the Polyakov loop Φ is a function of $\{\phi_3,\phi_8\}$ in this case. This procedure is just like the change of integration variable from $SU(3)$ to its eigenvalues by integrating out six phase variables. However, once diquarks come into the problem, this simple reduction does no longer work. In fact, if we try to diagonalize A_4, the diquark condensate also suffers from the gauge rotation

$$A_4\to A_4'\equiv gA_4g^{-1},\quad \hat{\boldsymbol{\Delta}}_{\eta\eta'}\to\hat{\boldsymbol{\Delta}}'_{\eta\eta'}\equiv\left(\hat{\boldsymbol{\Delta}}g^{-1}\right)_{\eta\eta'}. \tag{2·8}$$

We note that the condensate matrix $\Delta'_{\eta\eta'}$ is no longer restricted to be of diagonal form in the color-flavor space. A_4 can be diagonalized by g which can be parametrized by six "phase" parameters; at the same time the diquark condensate acquires this phase

*) We have also an imaginary part of the effective potential, which may viewed as a sign problem. It can cause a splitting of Φ and $\bar{\Phi}$, but here we simply discard it.[8)] Accordingly, we have $\Phi=\bar{\Phi}$.

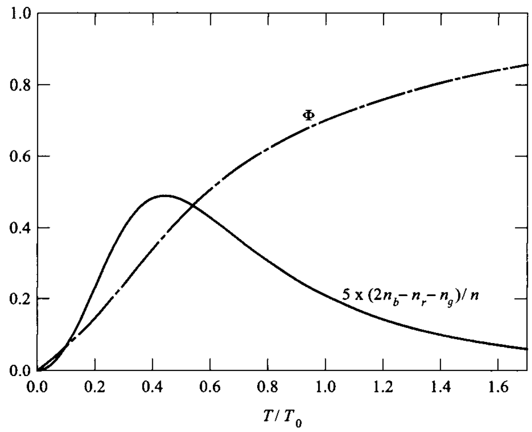

Fig. 1. T_8 color density with $m_s = \varDelta_\eta = \mu_{e,3,8} = 0$ as a function of T.

rotation, so the new diquark condensate is to be parametrized by nine parameters. Thus in principle, we have two choices; one is (1) to work in the diagonal form of A_4 with the generalized off-diagonal ansatz for diquark condensate. The other is (2) to work in the standard diquark ansatz with $\varDelta_\eta$ but with a general A_4 parametrized by eight parameters. These proper treatments require us to deal with all eleven variational parameters. Leaving these proper arguments to our future plan, we here work within the simplified *ansatz* for the ground state that the diquark condensate is diagonal $\{\varDelta_\eta\}$ even after fixing the gauge which diagonalizes A_4; this means we take only the diagonal entries of A_4, $\{\varphi_3, \varphi_8\}$, as the variational variables.

Although the continuous gauge freedom should be considered to have gone away to diagonalize A_4, there remain six discrete gauge transformations each of which leaves $\varPhi$ unchanged; these are the elements of permutation of fundamental color indices. Also in accordance with discarding $\Im\varOmega$, we further put $\varPhi$ to be real. For this, we take $\varphi_8 = 0$, so in this way, the gauge is completely fixed in our calculations.

Finally, we specify the Polyakov loop potential. We adopt the following form 4)

$$\frac{\mathcal{U}(T,\varPhi)}{T^4} = -\frac{b_2(T)}{2}\varPhi^*\varPhi + b(T)\log\left(1 - 6\varPhi^*\varPhi + 4(\varPhi^{*3} + \varPhi^3) - 3(\varPhi^*\varPhi)^2\right), \tag{2·9}$$

with $b_2(T) = a_0 + a_1\left(\frac{T_0}{T}\right) + a_2\left(\frac{T_0}{T}\right)^2$, $b(T) = b_3\left(\frac{T_0}{T}\right)^3$. T_0 is the value of the transition temperature for deconfinement in pure gauge, i.e., $T_0 = 270\,\mathrm{MeV}$. The logarithmic term was first proposed in 3) and it is nothing but the Vandermonde determinant, i.e., the Jacobian associated with the change of dynamical variables from the $SU(3)$ matrix to its eigenvalues.[13] For more details for parameter setting used in our numerical analysis, the readers are referred to Ref. 5).

§3. Results and discussion

Color density associated with color symmetry breaking: Before going into the full calculation, we study the simplified case with unpaired matter in the chiral limit, $\varDelta_\eta = m_s = 0$, just to illustrate the importance of imposing color neutrality in the PNJL model at finite μ. In Fig. 1, we show the color T_8 density, $\langle q^\dagger T_8 q\rangle$, and the Polyakov loop $\varPhi$ as a function of T. Surprisingly, T_8 color density takes

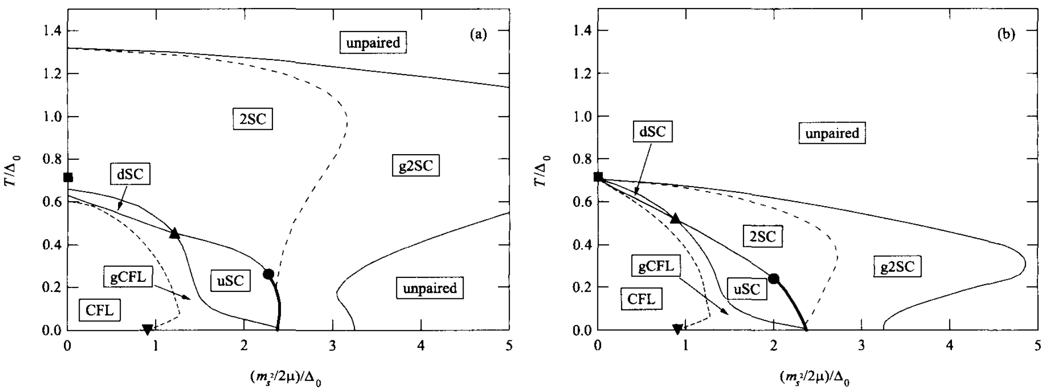

Fig. 2. (Reproduced figure with permission from [H. Abuki et al., Phys. Rev. D **77** (2008), 074018] Copyright (2008) by the American Physical Society). (a): Phase diagram in $(\frac{m_s^2}{2\mu}, T)$-plane at $\Delta_0 = 60\,\mathrm{MeV}$, $\mu = 500\,\mathrm{MeV}$. (b): The same as (a) but without the Polyakov loop.

nonzero value.*) It can be shown that this is the case except for two different limits, $T \to 0$ and $T \to \infty$ ($\Phi \to 1$).[5] The Polyakov loop Φ is the colorless object, so one might think it is strange to have nonvanishing color density. The reason is simple; we are breaking color symmetry in addition to Z_3 center symmetry by introducing the constant A_4 background field. In fact, $\Phi = \frac{1}{N_c}\mathrm{tr}L_Q$ is invariant under the color rotation $L_Q \to gL_Qg^{-1}$; it changes its value only under color transformation which is not exactly periodic in imaginary time but only up to Z_3, i.e., $\Phi \to z\Phi$ under

$$L_Q \to gL_Q(zg)^{-1}, \quad \text{where} \quad z \in Z_3. \tag{3·1}$$

Since we *assumed* the constant A_4 background to parametrize Φ, and A_4 in contrast to Φ is not invariant under color rotation, we have broken the color symmetry in addition to Z_3 symmetry. This is unexpected, undesirable feature of the PNJL model and may be considered as the model artifact. It could be dangerous for theoretical foundation of the model itself, but we do not discuss further this problem here. Instead, we simply assume that the model is still useful once we impose the vanishing color density as the constraint by tuning color chemical potentials $\{\mu_3, \mu_8\}$.**)

The phase diagrams: The phase diagram coming out from our high density PNJL model is displayed in Fig. 2(a). For comparison, we have also shown in Fig. 2(b), the phase diagram calculated with the model without the Polyakov loop. From these figures, the impact of the Polyakov loop dynamics on the quark Cooper pairing is clear; it has two major effects.

(a) First, we notice that the Polyakov loop dynamics stabilizes the 2SC phase significantly. In fact, the critical temperature for the 2SC-to-unpaired phase tran-

*) It should be noted that the color density itself is a gauge dependent quantity and thus should depend on the choice of the gauge. With our diagonal representation of A_4 with $\varphi_8 = 0$, the T_8 color density becomes finite as we observed above. If we selected a different gauge, the other entries of octet color density $\{\langle q^\dagger T_\alpha q\rangle\}$ should have appeared. The important thing is, however, whichever gauge we choose, some color density should become finite; in fact the squared sum of the octet color densities is shown to be the gauge independent quantity.[14]

**) We checked that off-diagonal color charge densities $\langle q^\dagger T_\alpha q\rangle$ (for $a \neq 3, 8$) are automatically vanishing for all the situations we are interested in here.

sition at $m_s = 0$ is almost doubled by inclusion of the Polyakov-loop dynamics. This point can be understood by the observation that the Polyakov loop suppresses the thermal excitation of colored quasiquarks which tend to break the Cooper pair condensate.[5)] Numerically, the factor of enhancement of T_c is 1.8 which is in good agreement with our analytical estimate 1.79.[5)]

(b) Second, as a consequence of the effect (a), we have the color-flavor unlocking transition even at $m_s = 0$. One may wonder why the $SU(3)$ flavor symmetry should be broken down to the *isospin* $SU(2)$ in the (u, d)-sector, and why not either in (s, u) or (d, s) sector. This is strange because at $m_s = 0$ the flavor $SU(3)$ symmetry is perfect so how can the flavor-blind Wilson line distinguish them? Actually, the fact that we have the isospin symmetry intact is directly attributed to our model assumption mentioned in §2; we are limiting ourselves to treat only two out of eight parameters for the $SU(3)_c$ matrix, L_Q. As noted, in principle this cannot be justified in our case because the gauge is already fixed in the diquark sector once we put the ansatz for diquarks to the diagonal form $\hat{\Delta}_{\eta\eta'} = \mathrm{diag.}(\Delta_1, \Delta_2, \Delta_3)$; therefore there remains no continuous gauge freedom to rotate L_Q to a diagonal form. So this limitation should be rather viewed as *an ansatz* for many possible ground states, such as the color-flavor locking ansatz. For the proper treatment, we should take into account all the eight parameters to represent the Wilson line L_Q. It is possible after making such a proper treatment, either that the ground state prefers the diagonal form of L_Q or that the ground state we obtained here turns out to be one of several degenerated ground states. We defer this task in the future.

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