Heavy Quark Diffusion in AdS/CFT

Derek TEANEY

Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794-3800, USA

I elucidate the origin of thermal noise in AdS/CFT by studying quantum fluctuations. The Schwinger-Keldysh contour plays an essential role in this calculation. The results show how a quark diffuses in the context of AdS/CFT.

§1. Motivation

The purpose of this short note is to clarify how thermal noise arises within the context of AdS/CFT.*) Recently the AdS/CFT correspondence has been used to calculate several items of great interest to the heavy ion physics community. This list includes: the drag of heavy quarks,[1)-3)] higher order hydrodynamics,[4),5)] and radiative processes.[6)] All of these processes are intrinsically related to noise in thermal field theory. However, noise does not appear naturally within the AdS/CFT context. This is essentially manifest since it is difficult to acquire stochastic terms by solving linearized classical equations of motion. However it turns out that the quantum mechanics of AdS_5 produces the required thermal noise.

To address the question of noise we will consider the simplest stochastic process — Brownian motion. General considerations of heavy quarks would indicate that the heavy quark should obey a Langevin equation of motion

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{p_i}{M}, \\ \frac{dp_i}{dt} &= \xi_i(t) - \eta_D p_i\,, \qquad \langle \xi_i(t)\xi_j(t')\rangle = \kappa\delta_{ij}\delta(t-t')\,. \end{aligned} \tag{1·1}$$

(For a brief review of the Langevin equation see Ref. 7).) The drag and fluctuation coefficients are related in turn by the Einstein relation

$$\eta_D = \frac{\kappa}{2MT}\,. \tag{1·2}$$

The drag coefficient η_D can be related to the diffusion coefficient

$$D = \frac{T}{M\eta_D} = \frac{2T^2}{\kappa}\,. \tag{1·3}$$

For a brief review of the Langevin equation and a derivation of these results, see Ref. 7).

*) This is a preliminary account of work done in collaboration with D. T. Son and J. Casalderrey-Solana.

Previously the drag coefficient and the noise coefficient have been computed using the AdS/CFT correspondence and the results obey the expected Einstein relation of Eq. (1·2).[2),3),8)] However if the gravity theory is dual to the gauge theory then the whole Langevin process should emerge from the within the gravity theory and not just this or that coefficient. This will require including the quantum mechanical motion of a string in AdS_5.

§2. String equations of motion

Strings in AdS_5 are dual to heavy quarks and the dynamics of these objects is dictated by the Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\ell_s^2}\int d\tau d\sigma \sqrt{-h_{ab}}\,. \tag{2·1}$$

AdS_5 with a black hole is given by the canonical form

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + d\boldsymbol{x}^2\right) + \frac{R^2 dz^2}{f(z)z^2} + R^2 d\Omega_5^2\,, \tag{2·2}$$

where $z = \frac{R^2}{r}$, $z_H = \frac{1}{\pi T}$, and $f(z) = 1-(z/z_H)^4$. Changing coordinates to $\bar{z} = z/z_H$ and subsequently dropping the "bar" gives

$$ds^2 = (\pi T)^2\frac{R^2}{z^2}\left(-f(z)dt^2 + d\boldsymbol{x}^2\right) + \frac{R^2 dz^2}{f(z)z^2} + R^2 d\Omega_5^2\,, \tag{2·3}$$

which is the form adopted for these proceedings.

Initially a static quark string at rest at the origin is given by the map

$$(\tau,\sigma) \mapsto (t=\tau, z=\sigma, \boldsymbol{x}=\boldsymbol{0}, \Omega_5 = \text{const})\,. \tag{2·4}$$

Considering now the dynamics of small fluctuations we have $\boldsymbol{x} \to \boldsymbol{x}(t,z)$ and the action for these small fluctuations is

$$S = -\frac{(\pi T)R^2}{2\pi\ell_s^2}\int\frac{dtdz}{z^2}\left[1 - \frac{1}{2}\frac{(\dot{\boldsymbol{x}})^2}{f} + \frac{1}{2}(\pi T)^2 f(\boldsymbol{x}')^2\right]\,. \tag{2·5}$$

From this action we derive an equation of motion in Fourier space

$$\frac{\mathfrak{w}^2}{z^2 f}\boldsymbol{x} + \partial_z\left(\frac{f}{z^2}\partial_z\boldsymbol{x}\right) = 0\,, \tag{2·6}$$

with $\mathfrak{w} = \omega/(\pi T)$.

The usual AdS_5 prescription[9)] is to find the retarded solution to the equation of motion for the source in bulk and to associate the corresponding retarded correlator of the field theory operators with the boundary limit. In this case the source is the fluctuation in position of the string and the operator is the subsequent force on the quark. Extracting out the overall constant from the solution of the above equation (i.e. $\boldsymbol{x}(\omega, z) = \boldsymbol{x}_0(\omega) X(\omega, z)$ with $X(\omega, 0) = 1$) the usual AdS$_5$ prescription is the following

$$G_R(\omega) = \lim_{z \to 0} A(z) X(-\mathfrak{w}, z) \partial_z X(\mathfrak{w}, z) , \tag{2·7}$$

where $A(z)$ is the coefficient of the kinetic term. Usually the limit diverges and these divergences need to be renormalized by the process of holographic renormalization.

In the present context the solution Eq. (2·6) for small frequency close to the boundary is (after Appendix A)

$$G_R(\omega) = \lim_{z \to 0} A(z) X(-\mathfrak{w}, z) \partial_z X(\mathfrak{w}, z) = -M_Q^0 \omega^2 + \bar{G}_R(\omega) , \tag{2·8}$$

where M_Q^0 is the zero temperature mass of the quark. The mass term is the "divergent" term of the retarded correlator and the regular piece is the retarded force-force correlator. In the Appendix it is shown that to order ω^2 inclusive this correlator is

$$-M_Q^0 \omega^2 + \bar{G}_R(\omega) \simeq -M_{\rm kin}(T) \omega^2 - i \frac{\omega}{2T} \kappa + O(\omega^3) , \tag{2·9}$$

where we have defined the kinetic mass and the momentum broadening coefficient

$$M_{\rm kin}(T) = M_Q^0 - \frac{\sqrt{\lambda} T}{2} \qquad \kappa = \sqrt{\lambda} \pi T^3 . \tag{2·10}$$

The notation and notion of kinetic mass is taken from Ref. 1). In the next sections we will show how this correlator appears in a dynamical context.

§3. Brownian motion in thermal quantum mechanics

To start we review how Brownian motion occurs in quantum mechanics (see for example 10) and 11), and references therein). Schematically the real time quantum mechanical partition function for a heavy particle coupled to a bath through an ensemble of forces is

$$Z_Q = \left\langle \int Dx_1 Dx_2 \, e^{i \int \frac{1}{2} M v_1^2 - i \int \frac{1}{2} M v_2^2} \, e^{i \int dt_1 F_1 x_1} \, e^{-i \int dt_2 F_2 x_2} \right\rangle_{\rm Bath} . \tag{3·1}$$

Here the integration over the "1" coordinates represents the amplitude while the integration over the "2" coordinates represents the conjugate amplitude for the quark. When we integrate over the bath coordinates we are left with a reduced action for the dynamics of the heavy particle. Using the fact that the force term is small compared to the inertial term we make the following approximation:

$$\left\langle e^{i \int dt_1 F_1 x_1} \, e^{-i \int dt_2 F_2 x_2} \right\rangle_{\rm bath} \simeq e^{-\frac{1}{2} \int dt \, dt' s_a s_b \, x_a(t) \, \langle F_a(t) F_b(t') \rangle \, x_b(t')} , \tag{3·2}$$

where the a and b denote run over 1 and 2 and $s_a = \pm$ respectively. The contour ordered averages of forces select different time orderings of force operators. For instance

$$iG_{11}(t-t') \equiv \langle F_1(t)F_1(t')\rangle = \left\langle T[\hat{F}(t)\hat{F}(t']\right\rangle , \tag{3·3}$$

$$iG_{12}(t-t') \equiv \langle F_1(t)F_2(t')\rangle = \left\langle \hat{F}(t')\hat{F}(t)\right\rangle = iG_{21}(t'-t) \tag{3·4}$$

$$iG_{22}(t-t') \equiv \langle F_2(t)F_2(t')\rangle = \left\langle \tilde{T}[\hat{F}(t)\hat{F}(t')]\right\rangle . \tag{3·5}$$

In equilibrium these correlators are related to the retarded correlator by

$$iG_{11}(\omega) = i\mathrm{Re}\, G_R(\omega) - \coth\left(\frac{\omega}{2T}\right) \mathrm{Im}G_R(\omega)\,, \tag{3·6}$$

$$iG_{12}(\omega) = -2n(\omega)\mathrm{Im}G_R(\omega) = iG_{21}(-\omega)\,, \tag{3·7}$$

$$iG_{22}(\omega) = -i\mathrm{Re}\, G_R(\omega) - \coth\left(\frac{\omega}{2T}\right) \mathrm{Im}G_R(\omega)\,, \tag{3·8}$$

with $n(\omega) = 1/(\exp(\omega/T)-1)$.

Rather than integrating over the $x_1(t_1)$ and $x_2(t_2)$ we change variables to the "ra" basis

$$x_r = (x_1(t)+x_2(t))/2\,, \qquad x_a(t) = (x_1(t)-x_2(t))/2\,, \tag{3·9}$$

which leads to the following reduced path integral:

$$Z_{HQ} = \int Dx_r Dx_a e^{iS_{\rm eff}}\,, \tag{3·10}$$

with the action

$$iS_{\rm eff} = -i\int dt x_a(t) M_Q^0 \ddot{x}_r(t) - \int dt \int dt' x_a(t) iG_R(t-t')x_r(t') \\ -\frac{1}{2}\int dt \int dt' x_a(t) G_{\rm sym}(t-t')x_a(t')\,. \tag{3·11}$$

In writing this result we have made liberal use of the identities

$$G_{\rm sym}(t-t') = \frac{i}{4}\left[G_{11}+G_{22}+G_{12}+G_{21}\right] = \left\langle \left\{\hat{F}(t),\hat{F}(0\right\}\right\rangle , \tag{3·12}$$

$$iG_R(t) = \frac{i}{2}\left[G_{11}-G_{22}-G_{12}+G_{21}\right] = \theta(t)\left\langle [\hat{F}(t),\hat{F}(0)]\right\rangle . \tag{3·13}$$

To proceed further we introduce the Fourier transform of the gaussian term

$$\exp\Big(-\frac{1}{2}\int dt \int dt' x_a(t) G_{\rm sym}(t-t')x_a(t')\Big) \\ = \int D\xi \exp\left(i\int dt \xi(t)x_a(t) - \frac{1}{2}\xi(t)G^{-1}_{\rm sym}(t-t')\xi(t')\right) , \tag{3·14}$$

and integrate over $x_a(t)$ which gives a functional δ-function. The result is

$$Z = \int Dx_r D\xi \prod_t \delta_t\Big(M_Q^0\ddot{X}_r(t) + \int^t dt' G_R(t-t')X_r(t') - \xi(t)\Big) \\ \times \exp\Big(-\frac{1}{2}\int dt\int dt' \xi(t)G^{-1}_{\rm sym}(t-t')\xi(t')\Big)\,. \tag{3·15}$$

The meaning of this path integral[10] is that one is supposed to solve the classical equation

$$M_Q^0 \ddot{x}_r(t) + \int^t dt' G_R(t-t') x_r(t') - \xi(t) = 0\,, \tag{3·16}$$

with the colored noise

$$\left\langle \xi(t)\xi(t') \right\rangle = G_{\rm sym}(t-t')\,. \tag{3·17}$$

In equilibrium the noise is related to the imaginary part of the retarded greens function by the fluctuation dissipation relation

$$G_{\rm sym}(\omega) = -2\,{\rm Im}G_R(\omega)\left(\frac{1}{2} + n(\omega)\right)\,. \tag{3·18}$$

Equation (3·16) is a finite memory version of the Brownian dynamics discussed in the introduction.

§4. AdS/CFT

Clearly the contour structure was essential to determining the noise in thermal quantum mechanics. The contour structure real time partition function of the field theory has been associated with the full Kruskal plane in the gravity dual.[8),12)] The string which was originally mapped as in Eq. (2·4) actually passes right through the event horizon of the black hole and fills the full Kruskal plane.[2)] This is shown in Fig. 1.

We then wish to consider the real time partition function of string path integral

$$Z_{HQ} = \int \prod_{t_1} d\boldsymbol{x}_1^o(t_1) \prod_{t_2} d\boldsymbol{x}_2^o(t_2) \prod_{t,z} d\boldsymbol{x}_1(t,z) d\boldsymbol{x}_2(t,z) e^{iS_{NG}}\,. \tag{4·1}$$

Here we have written the path integral explicitly; $\boldsymbol{x}_1^o(t)$ is the endpoint of the string in the right quadrant and $\boldsymbol{x}_2^o(t)$ is the endpoint in the left quadrant. $\boldsymbol{x}_1(t,z)$ labels bulk fluctuations in the right quadrant and $\boldsymbol{x}_2(t,z)$ labels the fluctuation in the left. Here S_{NG} is the small fluctuations path integral written above in Eq. (2·5). Imagine integrating out the bulk fluctuations to yield an effective action for the boundary variables $\boldsymbol{x}_1^o$ and $\boldsymbol{x}_2^o$. Since the integral is gaussian (for small fluctuations) the integration over the bulk coordinates is simply

$$\int [D\boldsymbol{x}_1^o][D\boldsymbol{x}_2^o] {\rm det}[D]\, e^{iS_{\rm eff}[X_{\rm cl}(\boldsymbol{x}_1^o(t_1), \boldsymbol{x}_2^o(t_2))]}\,, \tag{4·2}$$

where $S_{\rm eff}$ is the Nambu-Goto action evaluated with the classical path $X_{\rm cl}$ which passes through the end points $\boldsymbol{x}_1^o(t_1)$ and $\boldsymbol{x}_2^o(t_2)$. For a classical path this reduces to a boundary term as usual in AdS/CFT.

Following the logic of the Herzog-Son construction for the real time path integral in AdS/CFT this boundary term reduces to the following:[2),8),13)]

$$iS_{\rm eff} = -\frac{1}{2}\int \frac{d\omega}{2\pi}$$

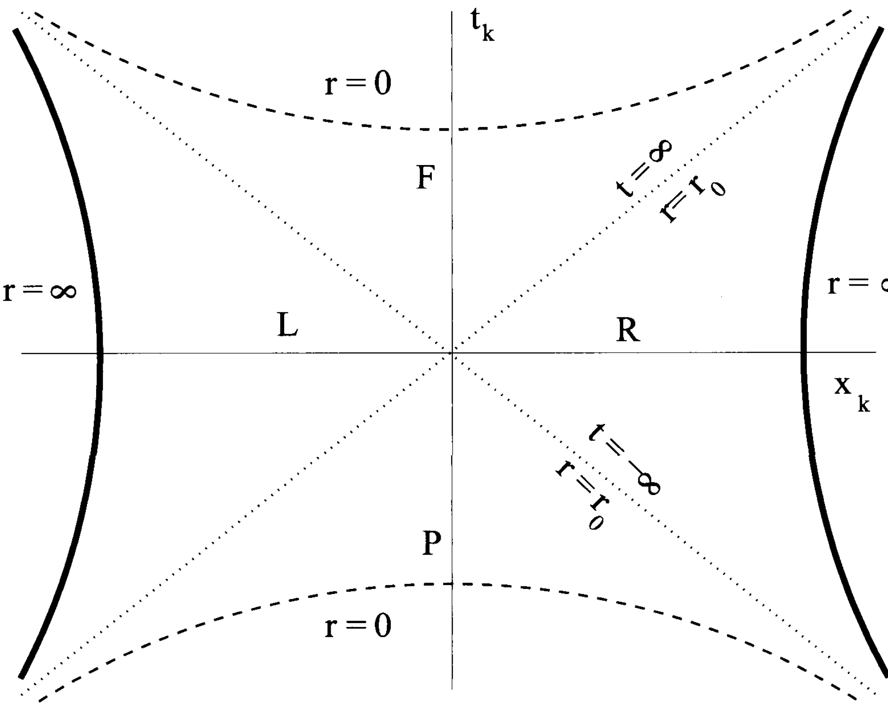

Fig. 1. Kruskal diagram for the AdS black hole. The coordinates (t, z) span the left (L) quadrant. The dotted lines and the dashed hyperbolas represent the future and past horizons and singularities. The thick hyperbolas on the sides are the two boundaries ($z = 0$). The quark propagates along the 1 and 2 axes of the Schwinger-Keldysh contour which corresponds to a string whose endpoints follow these boundaries. The minimal surface with these boundary conditions is the full Kruskal plane.

$$\times\boldsymbol{x}_1^o(-\omega)\left[-iM_Q^0\omega^2+i\bar{G}_{11}(\omega)\right]\boldsymbol{x}_1^o(\omega)-\boldsymbol{x}_1^o(-\omega)\left[e^{+\omega\sigma}i\bar{G}_{12}(\omega)\right]\boldsymbol{x}_2^o(\omega)$$
$$+\boldsymbol{x}_2^o(-\omega)\left[+iM_Q^0\omega^2+i\bar{G}_{22}(\omega)\right]\boldsymbol{x}_2^o(\omega)-\boldsymbol{x}_2^o(-\omega)\left[e^{-\omega\sigma}i\bar{G}_{21}(\omega)\right]\boldsymbol{x}_1^o(\omega)\,, \quad (4\cdot3)$$

Here the different functions ($\bar{G}_{11}(\omega)$ etc.) are related to the AdS/CFT retarded correlator ($\bar{G}_R(\omega)$ in Eq. (2·9)) via the KMS relations given in Eqs. (3·6) – (3·8). The determinant

$$\det[D]=\det\left[+\frac{\mathfrak{w}^2}{z^2 f}+\partial_z\left(\frac{f}{z^2}\partial_z\cdot\right)\right]\,, \quad (4\cdot4)$$

is an opaque object but the important point is that it is independent of $\boldsymbol{x}_1^o$ and $\boldsymbol{x}_2^o$ and can safely be ignored. After a change of variables and reverting to time

$$\boldsymbol{x}_r(t)=\frac{\boldsymbol{x}_1^o(t)+\boldsymbol{x}_2^o(t+i\sigma)}{2}\,,\qquad \boldsymbol{x}_a(t)=\frac{\boldsymbol{x}_1^o(t)-\boldsymbol{x}_2^o(t+i\sigma)}{2}\,, \quad (4\cdot5)$$

the partition function for a string in AdS$_5$ Eq. (4·3) reduces to the partition function of a heavy particle coupled to a thermal bath as in Eq. (3·10).

At this point the procedure is identical to the usual Caldeira&Legget and Feynman&Vernon procedure in quantum mechanics.[10),11)] In the low frequency limit we have

$$\bar{G}_R(\omega)=+\frac{\sqrt{\lambda}T}{2}\omega^2-i\frac{\omega}{2T}\kappa\,, \quad (4\cdot6)$$

or

$$\bar{G}_R(t-t') = -\frac{\sqrt{\lambda}T}{2}\frac{d^2}{dt^2}\delta(t-t') + \frac{\kappa}{2T}\frac{d}{dt}\delta(t-t')\,. \tag{4·7}$$

The symmetrized correlator is also given by the fluctuation dissipation relation of Eq. (3·18)

$$\bar{G}_{\rm sym}(t-t') = \kappa\delta(t-t')\,. \tag{4·8}$$

Taking Eqs. (4·7) and (4·8) and substituting into the generalized Langevin equation (3·16) yields a Langevin equation of motion for the average endpoint of the string $\boldsymbol{x}_r$

$$M_{\rm kin}(T)\frac{d^2\boldsymbol{x}_r}{dt^2} + \frac{\kappa}{2T}\frac{d\boldsymbol{x}_r}{dt} = \xi(t) \qquad \langle\xi(t)\xi(t')\rangle = \kappa\delta(t-t')\,, \tag{4·9}$$

with

$$M_{\rm kin}(T) = M_Q^0 - \frac{\sqrt{\lambda}T}{2} \qquad \text{and} \qquad \kappa = \sqrt{\lambda}\pi T^3\,. \tag{4·10}$$

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Appendix A
—— *Small Fluctuations* ——

Our starting point is Eq. (2·6). Near the boundary $z=0$ the solution to this equation is

$$\boldsymbol{x}(\omega,z) = \boldsymbol{x}_o(\omega) + \boldsymbol{x}_o(\omega)\,\mathfrak{w}^2\frac{z^2}{2} + \boldsymbol{x}_o(\omega)\frac{C(\mathfrak{w})}{3}z^3 + O(z^4)\,. \tag{A·1}$$

Here we have used the fact that $f(z)=1-z^4$ to solve these equations The constant C (a function of ω) remains to be determined from the boundary condition. The coefficient in front of the kinetic term of Eq. (2·5) is to order z^4

$$A(z) = -\frac{(\pi T)^3R^2}{2\pi\ell_s^2}\frac{1}{z^2}\,. \tag{A·2}$$

We recall that the solution is written $\boldsymbol{x}(\omega,z) = \boldsymbol{x}_o(\omega)X(\mathfrak{w},z)$ with $\boldsymbol{X}(\mathfrak{w},0)=1$ and substitute

$$G_R(\omega) = \lim_{z\to 0} A(z)X(-\mathfrak{w},z)\partial_z X(\mathfrak{w},z)\,, \tag{A·3}$$

$$= -\frac{(\pi T)^3R^2}{2\pi\ell_s^2}\left[\frac{\mathfrak{w}^2}{z} + C(\mathfrak{w})\right]\,, \tag{A·4}$$

$$= -M_Q^0\omega^2 - \sqrt{\lambda}\frac{(\pi T)^3}{2\pi}C(\mathfrak{w})\,. \tag{A·5}$$

To get the the last line we have reinstated the "bar" in $\bar{z} = z/z_H$, recalled definitions ($z_H = 1/(\pi T)$, $\mathfrak{w} = \omega/(\pi T)$, $z = R^2/r$), expressed the string tension as $T_o = \frac{1}{2\pi \ell_s^2}$, exploited fundamental relation $R^2/\ell_s^2 = \sqrt{\lambda}$, and finally used the zero temperature result for the energy of a string as tension times length, $M_Q^o = T_o r_m$. Thus we see that the divergent boundary terms conspire to give the zero temperature kinetic term

$$G_R(\omega) = -M_Q^0 \omega^2 + \bar{G}_R(\omega)\,, \tag{A·6}$$

where $\bar{G}_R(\omega)$ is the regular (in z) part of the retarded propagator.

To determine the regular part of the retarded propagator we must analyze the solutions of Eq. (2·6). The basic procedure is straightforward and standard: (1) Near the horizon $z = 1$, a short exercise shows that the solution behaves as $(1-z)^{\pm i\mathfrak{w}/4}$; the infalling $(-i\mathfrak{w}/4)$ solution should be selected for retarded boundary conditions. (2) Then writing $X(z,\mathfrak{w})$ as $(1-z)^{-i\mathfrak{w}/4} g(z,\mathfrak{w})$ one solves for $g(z,\mathfrak{w})$ demanding regularity at the horizon. (3) In practice the resulting equation for $g(z,\mathfrak{w})$ can only be solved order by order in the frequency. Following this procedure one finds that

$$\boldsymbol{x}(\omega, z) = \boldsymbol{x}_0(\omega)(1-z^4)^{-i\mathfrak{w}/4}\left[1 + +i\mathfrak{w}\frac{z^3}{3} + \mathfrak{w}^2\frac{z^2}{2} - \frac{\mathfrak{w}^2}{3}z^3\right] + O(z^4, \mathfrak{w}^3)\,. \tag{A·7}$$

Substituting this solution into Eq. (A·3) we determine the retarded correlator quoted in the text Eqs. (2·9) and (2·10).

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