Closed Strings in Open String Field Theory

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We show that cubic, open bosonic string field theory contains a second nilpotent operator such that 1) its cohomology is isomorphic to the space of physical closed string states, 2) exact elements correspond to physically trivial deformations and infinitesimal shifts of the open string background and 3) this cohomology is independent of the open string background.

§1. Introduction and summary

Although open string field theory (OSFT) was formulated to provide a Lagrangian description of the interactions of open strings it is clear that it must contain information about the closed string excitations as well since the perturbative one-loop open string diagrams contain closed string poles. In this talk we shall argue that the closed string cohomology can in fact be identified at tree level in open string field theory. Concretely, we identify a second nilpotent operator, in addition to the open string BRST operator, Q. The existence of a second operator, d_H, can be understood as follows. For our purpose Witten's cubic open string field theory

$$S_W = \frac{1}{2}\langle \Psi, Q\Psi\rangle + \frac{1}{3}\langle \Psi, \Psi * \Psi\rangle \tag{1·1}$$

is defined in terms of a differential graded algebra $(A, Q, *)$ together with an invariant inner product $\langle \Psi_1 * \Psi_2, \Psi_3\rangle = \langle \Psi_1, \Psi_2 * \Psi_3\rangle$ of the string fields*) $\Psi_i \in A$. The operator $d_H = d_H(Q, *)$ is then constructed in terms of Q and the $*$-product of string fields. However, unlike the open string BRST operator Q, the operator d_H is not defined through its action on the open string fields but through its action on the maps $f_N : A^{\otimes N} \to A$. Thus while d_H is constructed entirely in terms of objects intrinsic to open string field theory, its cohomology is given by deformations of OSFT. In fact, it turns out that the cohomology of d_H is isomorphic to the space of, non-trivial and consistent infinitesimal deformations of OSFT and these turn out to be precisely the vertices of Zwiebach's linearized open closed string field theory[3),4)] with the closed string insertions given by physical closed string states.

§2. Applications

As an illustration we may consider an ordinary field theory, of a photon say, with Lagrangian

$$L[A_\mu] = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{2·1}$$

*) It is sometimes useful to include extra structure in OSFT such as the existence of an identity string field[1)] or Chan-Paton factors.[2)] However, this plays no role in the present discussion.

It is clear that the covariant form of the Lagrangian $L[A_\mu]$ in flat space already determines its consistent infinitesimal deformation that results in a linearized coupling to gravity through Noether's procedure. However, it does not imply the equation of motion for the graviton. In contrast, open string field theory, or more precisely d_H contains the wave equation for the graviton as well. On the other hand, d_H-exact deformations correspond to closed string gauge transformations, as well as other physically trivial deformations and infinitesimal shifts of the open string background.

We shall also see that the cohomology of d_H is independent of the open string background. In fact, for any solution, Ψ_0, of the open string equations of motion, $Q\Psi + \Psi * \Psi = 0$, one can construct the corresponding maps f_N in terms of those in the trivial open string background, $\Psi = 0$. This allows, in principle, to construct the linearized open-closed string field theory for any given open string background including Schnabl's solution where the open string cohomology is trivial.

§3. Derivation

We will sketch the derivation of the results described in the Introduction.*) For this we write Zwiebach's open-closed string field theory as

$$S = S_W + \sum_{M=1}^{\infty} C_M(\Psi, \ldots, \Psi), \tag{3·1}$$

describing the coupling of one closed string and M open strings on the disk. It is represented by the integrated correlator $C_M^\phi(\Psi_1, \Psi_2, \ldots, \Psi_M)$ defined as in Fig. 1. We have a disk with one puncture in the middle and M punctures on the boundary, and local coordinates around each puncture. The vertex operators ϕ and Ψ_i are inserted

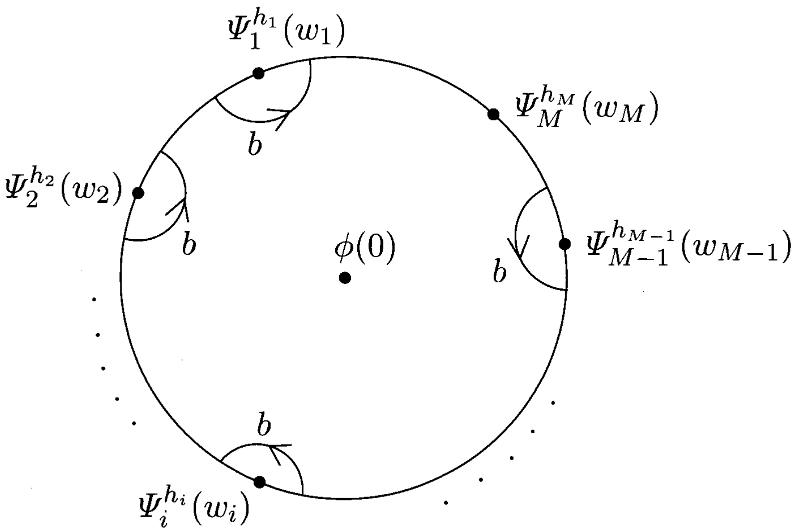

Fig. 1. The correlator $C_M^\phi(\Psi_1, \ldots, \Psi_M)$ on the disk D with global coordinate w. The open string punctures are at the points $w_i = e^{is_i}$.

*) See 5) for details and further references.

in the local coordinates. Since the moduli space of our punctured disk has dimension $M-1$, we need to insert $M-1$ antighosts that will determine the measure on the moduli space. We do that by inserting line integrals of b around each but one of the open string vertex operators. Using the doubling trick, we can write them as

$$b(v_i) = \frac{1}{2\pi i} \oint_{P_i} b(z)\, v_i(z)\, dz. \tag{3·2}$$

The vector $v(z)$ must correspond to a tangent vector in the moduli space, i.e. it moves the punctures. We can now write the definition of the integrated correlator:

$$C_M^{\phi}(\Psi_1, \Psi_2, \ldots, \Psi_M) \equiv \int_{\mathcal{T}_M} ds_1 \ldots ds_{M-1} \langle \phi(0)\, b(v_1) \ldots b(v_{M-1})\, \Psi_1(w_1) \ldots \Psi_M(w_M) \rangle_D. \tag{3·3}$$

Here $\mathcal{T}_M$ is the part of moduli space of the disk with one puncture in the bulk and M punctures on the boundary, not covered by Feynman diagrams obtained by lower order vertices. The s_i are $M-1$ real numbers parameterizing this moduli space. Concretely, we will take the coordinate on the disk to be

$$w = re^{is}, \tag{3·4}$$

so the s_i are the angles specifying the positions of the open string punctures, namely $w_i = e^{is_i}$. Finally, $\Psi_i(w_i)$ denotes the vertex operator inserted in the local coordinates by virtue of a suitable conformal map h_i. For simplicity we will assume that ϕ has ghost number 2 and

$$\begin{aligned} b_n|\phi\rangle &= \tilde{b}_n|\phi\rangle = 0, \\ L_n|\phi\rangle &= \tilde{L}_n|\phi\rangle = 0, \end{aligned} \tag{3·5}$$

for $n \geq 0$. This can always be achieved by a suitable gauge transformation. These conditions then ensure that the maps $C_M^{\phi} : A^{\otimes M} \to \boldsymbol{R}$ have the cyclic symmetry

$$C_M^{\phi}(\Psi_2, \ldots, \Psi_M, \Psi_1) = (-1)^{M-1+\Psi_1(\Psi_2+\ldots+\Psi_M)} C_M^{\phi}(\Psi_1, \ldots, \Psi_{M-1}, \Psi_M). \tag{3·6}$$

We denote the space of such maps by CC^M. We note in passing that $C_1^{\phi}(\Psi)$ reproduces the set of gauge invariant operators introduced in 6).

Let us now act with the BRST charge Q on the closed string field ϕ. This can be represented on the disk by inserting an integration of the BRST current j along a closed contour around the origin (the closed string puncture). We can further deform the contour by pushing it towards the boundary of the disk and then splitting it to M contours around the open string punctures and $b(v_i)$ insertions,

$$\begin{aligned} &C_M^{Q\phi}(\Psi_1, \ldots, \Psi_{M-1}, \Psi_M) \\ &= \int_{\mathcal{T}_M} ds_1 \ldots ds_{M-1} \langle Q\,\phi(0) b(v_1) \ldots b(v_{M-1})\, \Psi_1(w_1) \ldots \Psi_{M-1}(w_{M-1}) \Psi_M(w_M) \rangle_D \\ &= -\int_{\mathcal{T}_M} ds_1 \ldots ds_{M-1} \langle \phi(0)\, Q\, b(v_1) \ldots b(v_{M-1})\, \Psi_1(w_1) \ldots \Psi_{M-1}(w_{M-1}) \Psi_M(w_M) \rangle_D. \end{aligned} \tag{3·7}$$

Next we cross the $b(v_i)$ insertions, thereby producing a $T(v_i) \sim -\frac{\partial}{\partial s_i}$ so that we end up with

$$- \int_{\mathcal{T}_M} ds_1 \dots ds_{M-1} \sum_{l=1}^{M-1} (-1)^l \times \frac{\partial}{\partial s_l} \langle \phi(0)\, b(v_1) \dots \not{b}(v_l) \dots b(v_{M-1})\, \Psi_1(w_1) \dots \Psi_l(w_l) \dots \Psi_M(w_M) \rangle_D$$
$$- \sum_{l=1}^{M} (-1)^{M-1+\Psi_1+\dots+\Psi_{l-1}} C_M^\phi(\Psi_1, \dots, Q\Psi_l, \dots, \Psi_M). \qquad (3{\cdot}8)$$

The first term in (3·8) is a sum of total derivatives on the moduli space $\mathcal{T}_M$. On the other hand, when s_k is at a boundary of $\mathcal{T}_M$, then the punctured disk is identical to the Riemann surface given by the Feynman diagram obtained by gluing the Witten cubic vertex to the disk with $M-1$ punctures via a propagator of length zero. Therefore this contribution can be written equivalently as

$$(-1)^{\Psi_1(\Psi_2+\dots+\Psi_M)} C_{M-1}^\phi(\Psi_2, \dots, \Psi_M * \Psi_1) + \sum_{l=1}^{M-1} (-1)^l C_{M-1}^\phi(\Psi_1, \dots, \Psi_l * \Psi_{l+1}, \dots, \Psi_M). \qquad (3{\cdot}9)$$

This expression, in turn, is recognized as the Hochschild differential

$$\delta : \quad \mathrm{Hom}(A^{M-1}, \mathbb{R}) \to \mathrm{Hom}(A^M, \mathbb{R}) \qquad (3{\cdot}10)$$

for $\mathbb{Z}_2$ graded algebras. Adding the last term in (3·8) we can write

$$C_M^{Q\phi}(\Psi_1, \dots, \Psi_M) = -\Big((\delta C_{M-1}^\phi)(\Psi_1, \dots, \Psi_M) - (-1)^M (QC_M^\phi)(\Psi_1, \dots, \Psi_M) \Big), \qquad (3{\cdot}11)$$

where (QC_M^ϕ) is defined by (3·8). Equation (3·11) defines an operator, $d_H : CC^* \to CC^*$, where $CC^* = \oplus_{M=1}^\infty CC^M$. If we impose the condition that the closed string field be on-shell, $Q\phi = 0$, we then conclude that the maps $(C_1^\phi, C_2^\phi, \dots, C_M^\phi, \dots)$ are closed with respect to d_H. If the closed ghost number string field is off-shell, $Q\phi \neq 0$ then it is not hard to show that $C_M^{Q\phi}(\Psi_1, \dots, \Psi_M)$ is cyclic, so that the operator d_H takes cyclic elements into cyclic elements. Moreover, the above calculation also shows that if ϕ is pure gauge, i.e. $\phi = Q\Lambda$ for some Λ, then

$$C_M^\phi(\Psi_1, \dots, \Psi_M) = (\delta C_{M-1}^\Lambda)(\Psi_1, \dots, \Psi_M) - (-1)^M (QC_M^\Lambda)(\Psi_1, \dots, \Psi_M). \qquad (3{\cdot}12)$$

In other words, C_M^ϕ is exact with respect to d_H. Furthermore, $d_H^2 = 0$. We conclude that the on-shell closed string states are contained in the cyclic cohomology of d_H.

§4. Other d_H-exact elements

We should note that not all exact elements in the cyclic complex are closed string gauge transformations. In fact it can be shown that generic deformations of cubic

OSFT which do not correspond to an on-shell closed string state are d_H-exact. For illustration we give some examples below.

4.1. *Deformation of the open string background*

Let Ψ_0 be an infinitesimal marginal open string deformation. We express it as $\Psi_0 = O(\epsilon)$. And Ψ_0 is a solution of the equations of motion, therefore $Q\Psi_0 = 0 + O(\epsilon^2)$. Writing $\Psi = \Psi_0 + \tilde{\Psi}$, the action becomes

$$S' = S + C_2(\Psi, \Psi) + O(\epsilon^2), \tag{4·1}$$

where C_2 is defined by

$$C_2(\Psi_1, \Psi_2) = \frac{1}{2}\left(\langle \Psi_0, \Psi_1, \Psi_2\rangle + (-1)^{1+\Psi_1\Psi_2}\langle \Psi_0, \Psi_2, \Psi_1\rangle\right). \tag{4·2}$$

In fact the collection of maps formed by C_2 alone, namely $(0, C_2, 0, 0, \ldots)$, is exact. Indeed, let us define

$$D_1(\Psi) = -\frac{1}{2}\langle \Psi_0, \Psi\rangle. \tag{4·3}$$

We then have

$$(QD_1)(\Psi) = -\frac{1}{2}\langle \Psi_0, Q\Psi\rangle = -\frac{1}{2}\langle Q\Psi_0, \Psi\rangle = 0 + O(\epsilon^2). \tag{4·4}$$

And, straight from the definition of δ, we find that

$$C_2 = \delta D_1. \tag{4·5}$$

Thus we conclude that, to order ϵ, $(0, C_2, 0, 0, \ldots)$ is exact, namely

$$(0, C_2, 0, \ldots) = (\delta - (-1)^N Q)\,(D_1, 0, 0, \ldots). \tag{4·6}$$

4.2. *Strips*

As example of a geometric deformation of the 3-vertex which is d_H-exact we consider an infinitesimal version of Zwiebach's open-closed string field theory in which the vertices in the action include strips of length ϵ for the external open strings even in the absence of closed strings. This deformation should not be physical since it is merely a reorganization of the moduli space of the same theory. This has been shown in 7) using the BV formalism and also in 8) as a particular case of A_∞-quasi-isomorphisms. To see this in our context we define C_3^ϵ pictorially in Fig. 2 as the deviation from Witten's vertex. We can easily translate this picture to the algebraic expression

$$\begin{aligned} C_3^\epsilon(\Psi_1, \Psi_2, \Psi_3) &= \langle e^{-\epsilon L_0}\Psi_1, e^{-\epsilon L_0}\Psi_2, e^{-\epsilon L_0}\Psi_3\rangle - \langle \Psi_1, \Psi_2, \Psi_3\rangle \\ &= -\epsilon\left(\langle L_0\Psi_1, \Psi_2, \Psi_3\rangle + \langle \Psi_1, L_0\Psi_2, \Psi_3\rangle + \langle \Psi_1, \Psi_2, L_0\Psi_3\rangle\right) + O(\epsilon^2). \end{aligned} \tag{4·7}$$

A four-vertex C_4^ϵ is now needed to produce the part of moduli space of disks with four punctures on the boundary that is missed by the Feynman diagrams constructed with

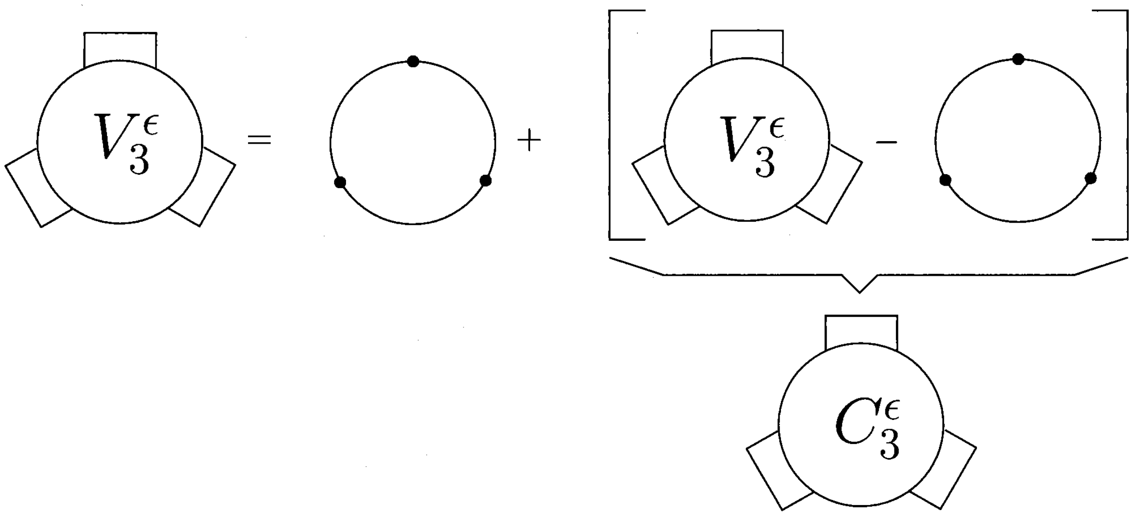

Fig. 2. Definition of C_3^ϵ. The symbol on the left-hand side represents V_3^ϵ, the three-vertex with strips. The first symbol on the right-hand side represents the three-vertex without strips.

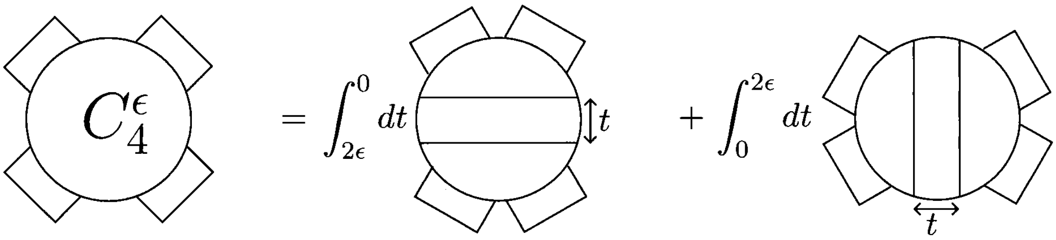

Fig. 3. Definition of C_4^ϵ.

C_3^ϵ. The missing surfaces are readily identified as the ones whose internal propagator has length smaller than 2ϵ. We can thus define C_4^ϵ pictorially as in Fig. 3. At $O(\epsilon)$ only C_3^ϵ and C_4^ϵ give non-vanishing contribution since the volume of the moduli space of C_M^ϵ is $O(\epsilon^{M-3})$. This can be seen by considering tree-level diagrams involving $M \geq 3$ vertices.

In order to show that $(0, 0, C_3^\epsilon, C_4^\epsilon, 0, \dots)$ is an exact element of the cyclic cohomology one neds to find a D_3^ϵ such that $QD_3^\epsilon = C_3^\epsilon$ and $\delta D_3^\epsilon = C_4^\epsilon$. We claim that the expression for D_3^ϵ is

$$D_3^\epsilon(\Psi_1, \Psi_2, \Psi_3) \equiv -\epsilon \left(\langle b_0\Psi_1, \Psi_2, \Psi_3 \rangle + (-1)^{\Psi_1} \langle \Psi_1, b_0\Psi_2, \Psi_3 \rangle + (-1)^{\Psi_1+\Psi_2} \langle \Psi_1, \Psi_2, b_0\Psi_3 \rangle \right). \tag{4·8}$$

Indeed one can show[5)] that

$$(0, 0, C_3^\epsilon, C_4^\epsilon, 0, \dots) = (\delta - (-1)^N Q)\,(0, 0, D_3^\epsilon, 0, 0, \dots). \tag{4·9}$$

Thus these geometric deformations do not contribute to the cyclic cohomology.

§5. Background independence

Let us expand the theory around a classical solution of the equations of motion, $\Psi \to \Psi_0 + \Psi$ with $Q\Psi_0 + \Psi_0 * \Psi_0 = 0$. The Witten action is then the same except for the appearance of a cosmological constant (irrelevant for us) and for the fact that Q

is replaced by a new BRST operator Q' given by

$$Q'\Psi = Q\Psi + \Psi_0 * \Psi + (-1)^{1+\Psi}\Psi * \Psi_0. \tag{5·1}$$

In order to show background independence it suffices to show that there exists a linear bijection $h : CC^* \to CC^*$ such that

$$(\delta(hC)_{M-1}) - (-1)^M (Q'(hC)_M) = 0 \iff (\delta C_{M-1}) - (-1)^M (QC_M) = 0. \tag{5·2}$$

It turns out that this map can be constructed explicitly as an infinite series in powers of Ψ_0.[5] Rather than displaying the explicit expression which is not very illuminating we illustrate the idea of the construction. Ignoring signs and combinatorial factor we can write for instance

$$(hC)_2(\Psi_1, \Psi_2) = \sum_{n=0}^{\infty} C_2^n(\Psi_1, \Psi_2), \tag{5·3}$$

where

$$\begin{aligned} C_2^0(\Psi_1, \Psi_2) &= C_2(\Psi_1, \Psi_2), \\ C_2^1(\Psi_1, \Psi_2) &= C_3(\Psi_0, \Psi_1, \Psi_2) + C_3(\Psi_1, \Psi_0, \Psi_2), \\ C_2^2(\Psi_1, \Psi_2) &= C_4(\Psi_0, \Psi_0, \Psi_1, \Psi_2) + C_4(\Psi_0, \Psi_1, \Psi_0, \Psi_2) + C_4(\Psi_1, \Psi_0, \Psi_0, \Psi_2). \\ &\cdots\cdots \end{aligned} \tag{5·4}$$

Taking the convergence of this series for granted this allows in principle to construct the maps C_M^ϕ and therefore the linearized open-closed theory in any open string background and, in particular, in the closed string vacuum where there are no open string physical states. This may provide a new probe to analyze the tachyon vacuum solution.

§6. Discussion

The natural question that arises then is whether there is more information to be uncovered by considering second order deformations in the closed string deformation. We think the answer is positive since a general property of the cohomology ring $HC(A)$ is that it possesses a natural Lie super-algebra structure, more precisely an L_∞-structure just like closed string field theory. So one can hope to learn more on closed strings by considering consistent deformations of OSFT. On the other hand there are certain obstructions in extending infinitesimal deformations to second order. This is, however, expected since not every marginal closed string deformation is exactly marginal.

Are there any applications of our results to open string field theory? For one thing the reformulation of closed string cohomology and in particular the open closed vertices in terms of cyclic cohomology of OSFT together with the background independence of the latter provides a formal definition of linearized open-closed string theory in any open string background. For instance, in Schnabl's vacuum solution where the open string cohomology is empty but the closed string cohomology is not we conclude that the closed string spectrum is still encoded in OSFT.

Finally we would like to mention another possible avenue for further investigation. Within boundary string field theory it has been argued[9] that certain deformations of the closed string background are equivalent to "collective excitations" of the open string (i.e. insertions of non-local boundary interactions on the boundary of the world sheet). At first sight this seems to be in contradiction with our present result that generic boundary deformations correspond to exact elements in the cyclic cohomology. However, we should note that the our argument for this is based on insertion of generic local operators on the boundary. It would be interesting to relax the latter condition.

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