Effective Theory for Black Branes

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Black branes, like other branes, can bend elastically as well as have worldvolume oscillations. I describe the long-wavelength effective theory that captures this dynamics. This is the theory of an effective fluid coupled to a dynamical worldvolume. It allows to study a vast new regime of higher-dimensional black holes, and it also captures in a very simple manner the Gregory-Laflamme instability of black branes, with impressive accuracy if the effective shear and bulk viscosities of the black brane are included.

§1. Introduction

1.1. *Motivation: General Relativity as a tool*

Why should anyone be interested in studying General Relativity and its black holes in dimension $D \geq 4$?

My own main motivation is that advances since the late 1990's make it clear that General Relativity must be regarded as theoretical tool — much like, say, Quantum Field Theory — that is useful in areas of physics that have little or nothing to do with its traditional fields of application in astrophysics and cosmology. Indeed, General Relativity seems to be *the* tool for studying a large class of strongly coupled systems, which do not involve gravity in the ordinary sense, but which in certain regimes admit a semiclassical description with emergent diffeomorphism invariance. This is the subject of correspondences derived from AdS/CFT, such as AdS/QCD, AdS/QGP, AdS/cond-mat, or the fluid/gravity correspondence, which is many cases involve black holes in spacetimes with dimensionality different, often larger, than four. Besides this, there is of course the fact that higher-dimensional General Relativity is indispensable in String Theory and also in TeV-gravity scenarios, which if realized would give rise to the production of small higher-dimensional black holes at colliders.

The aim is then to develop and understand better this multi-purpose tool. To this effect I will focus on the most basic set up, namely General Relativity in vacuum, $R_{\mu\nu} = 0$. In this case, the theory contains only one parameter that can be adjusted, namely, the number of spacetime dimensions D, and therefore we are motivated to investigate how the theory behaves as this parameter is changed. To gain a deep understanding of the theory it makes sense to study the properties of its most basic objects. These are D-dimensional black holes, and in particular higher-dimensional ones ($D \geq 4$).

The emphasis will be more on developing the fundamentals of the subject, without worrying, for the time being, about possible quick application to any of the fields mentioned above. It is good to remember here that, when first found, black holes have always been "answers waiting for a question".

1.2. *Black brane dynamics*

In this contribution I will focus on a very basic aspect of higher-dimensional General Relativity: the dynamics of its black branes. These are well-known solutions, which can be easily constructed, and of which it might be natural to ask:

- Can a black brane bend, flap, vibrate, like other familiar extended objects (strings, membranes, p-branes...) are known to do?

The answer is *yes*, and the theory that describes this shares in fact many aspects of the more familiar theories used to describe the long-wavelength dynamics of solitonic objects, such as Nambu-Goto strings, D-branes, or small black holes when they can be treated as point-particles in a background, or in trajectory with small acceleration.

The theory has several areas of application. The one that is most developed, and which in fact motivated it in the first place, is the study of novel dynamical regimes of higher-dimensional black holes, which do not have a counterpart in four dimensions. It has been known that in $D > 4$ there are black holes which in some regimes can be appropriately characterized as black branes whose worldvolume wraps a compact submanifold of a spacetime. These have been dubbed *blackfolds*, and the applications of these techniques has greatly enlarged our understanding of higher-dimensional black holes. Other applications, when the brane is charged, include the study of thermally excited D-branes.

This contribution is based on the articles,[1)–4)] and we refer the interested reader to them for further details.

§2. Effective worldvolume theory

We present the effective theory of blackfolds trying to highlight the similarities with the field-theoretical effective description of other extended objects, such as cosmic strings or D-branes. The main differences with these are, first, that the short-distance degrees of freedom that are integrated out are not those of an Abelian Higgs model nor massive string modes, but rather purely gravitational degrees of freedom. Second, the extended objects —curved black branes— possess black hole horizons. We obtain the equations using general symmetry and conservation considerations, rather than doing a detailed derivation from first principles.

2.1. *Collective coordinates for a black brane*

Schematically, the degrees of freedom of General Relativity are split into long and short wavelength components,

$$g_{\mu\nu} = \left\{ g_{\mu\nu}^{(\text{long})}, g_{\mu\nu}^{(\text{short})} \right\}. \tag{2·1}$$

The Einstein-Hilbert action is then approximated as

$$I_{EH} = \frac{1}{16\pi G}\int d^D x\sqrt{-g}R \approx \frac{1}{16\pi G}\int d^D x\sqrt{-g^{(\text{long})}}R^{(\text{long})} + I_{\text{eff}}[g_{\mu\nu}^{(\text{long})}, \phi], \tag{2·2}$$

where $I_{\rm eff}[g_{\mu\nu}^{\rm (long)},\phi]$ is an effective action obtained after integrating-out the short-wavelength gravitational degrees of freedom (precisely what we mean by this will be made clear in §2.2). The coupling of these to the long-wavelength component of the gravitational field is captured through a set of 'collective coordinates' that we denote schematically by ϕ. Our first task is to identify these effective field variables and the length scales that allow this splitting of degrees of freedom.

The main clue to the nature of the effective theory comes from the observation that the limit $\ell_M/\ell_J \to 0$ of known black holes, when it exists, results in flat black branes. Thus we shall take the effective theory to describe the collective dynamics of a black p-brane. Its geometry in $D=3+p+n$ spacetime dimensions is

$$ds^2_{p-\rm brane} = -\left(1-\frac{r_0^n}{r^n}\right)dt^2+\sum_{i=1}^{p}(dz^i)^2+\frac{dr^2}{1-\frac{r_0^n}{r^n}}+r^2d\Omega^2_{n+1}\,. \qquad (2\cdot3)$$

The coordinates $\sigma^a=(t,z^i)$ span the brane worldvolume. A more general form of the metric is obtained by boosting it along the worldvolume. If the velocity field is u^a, with $u^au^b\eta_{ab}=-1$ then

$$ds^2_{p-\rm brane} = \left(\eta_{ab}+\frac{r_0^n}{r^n}u_au_b\right)d\sigma^ad\sigma^b+\frac{dr^2}{1-\frac{r_0^n}{r^n}}+r^2d\Omega^2_{n+1}\,. \qquad (2\cdot4)$$

The parameters of this black brane solution consist of the 'horizon thickness' r_0, the p independent components of the velocity u (say, its spatial components u^i), and the $D-p-1$ coordinates that parametrize the position of the brane in directions transverse to the worldvolume, which we denote collectively by $X^\perp$. The D collective coordinates of the black brane are

$$\phi(\sigma^a)=\{X^\perp(\sigma^a),r_0(\sigma^a),u^i(\sigma^a)\} \qquad (2\cdot5)$$

and in the long-wavelength effective theory one allows $\partial X^\perp$, $\ln r_0$ and u_i to vary slowly along the worldvolume, $\mathcal{W}_{p+1}$, over a length scale R much longer than the size-scale of the black brane,

$$R\gg r_0\,. \qquad (2\cdot6)$$

Typically the scale R is set by the smallest intrinsic or extrinsic curvature radius of the worldvolume. Observe that we require slow variations of $\partial X^\perp$, not of $X^\perp$. Like the longitudinal velocities u^a, the transverse 'velocities' $\partial X^\perp$ can be arbitrary.

In order to preserve manifest diffeomorphism invariance it is convenient to introduce some gauge redundancy and enlarge the set of embedding coordinates of the worldvolume of the black brane to include all the spacetime coordinates $X^\mu(\sigma^a)$. From this embedding we can compute an induced metric

$$\gamma_{ab}=g_{\mu\nu}^{\rm (long)}\partial_aX^\mu\partial_bX^\nu\,. \qquad (2\cdot7)$$

This is naturally interpreted as the geometry induced on the worldvolume of the brane. To understand what this means, regard the split between degrees of freedom as follows: the long-wavelength degrees of freedom live in a 'far-zone' $r\gg r_0$, and they describe the background geometry in which the (thin) brane lives. Then (2·7) is the metric induced on the brane worldvolume. The short-wavelength degrees of freedom live in the 'near-zone' $r \ll R$. In the strict limit where $R \to \infty$, the near-zone solution is (2·4), but when R is large but finite, the collective coordinates depend on σ. Also, the long and short degrees of freedom interact together in the 'overlap' or 'matching-zone' $r_0 \ll r \ll R$, where the metrics $g_{\mu\nu}^{(\text{long})}$ and $g_{\mu\nu}^{(\text{short})}$ must match. Then the near-zone metric for the black brane must be of the form

$$ds^2_{(\text{short})} = \left(\gamma_{ab}(\sigma) + \frac{r_0^n(\sigma)}{r^n} u_a(\sigma) u_b(\sigma)\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n(\sigma)}{r^n}} + r^2 d\Omega^2_{n+1} + \cdots . \quad (2\cdot8)$$

The dots here indicate that, without additional terms, in general this is not a solution to the Einstein equations. These equations contain terms with gradients of $\ln r_0$, u^a and γ_{ab}. However these terms can be seen to come multiplied by powers of r_0 so they are small when $r_0/R \ll 1$. Then we can consider an expansion of the equations in derivatives and add a correction to (2·8) to find a solution to the Einstein equations to first order in the derivative expansion. A subset of the resulting Einstein equations can be rewritten as equations on the collective field variables $\phi(\sigma)$. An important requirement is that the perturbations preserve the regularity of the horizon, and to this effect working in a set of coordinates (Eddington-Finkelstein type) different than the ones above may be more appropriate.

The development of this line of argument, which can be regarded as a blend of the ideas for the effective descriptions of black hole dynamics in 5), 6) (and references therein), and in 7), produces a systematic derivation of the blackfold equations. This is however a technically involved approach. Here we shall instead follow a less rigorous but quicker and physically well-motivated path, relying on general effective-theory-type of arguments that allow us to readily obtain the blackfold formalism valid to lowest order in the derivative expansion. As we will see, this is the 'perfect fluid' and 'generalized geodesic' approximation.

2.2. *Effective stress tensor*

By the phrase 'integrating out the short-distance dynamics' we mean that the Einstein equations are solved at distances $r \ll R$ and then the effects of the solution at distances $r \gg r_0$ are encoded in a stress-energy tensor that depends only on the collective coordinates. The stress tensor is such that its effect on the long-wavelength field $g_{\mu\nu}^{(\text{long})}$ is the same as that of the black brane at distances $r \gg r_0$. For reasons that will become apparent as we proceed, it is both simpler and more convenient to work with an effective stress-energy tensor rather than with an effective action. In any case, nothing is lost since we work at the classical level.

The effective equations from (2·2) are

$$R_{\mu\nu}^{(\text{long})} - \frac{1}{2} R^{(\text{long})} g_{\mu\nu}^{(\text{long})} = 8\pi G T_{\mu\nu}^{\text{eff}} , \quad (2\cdot9)$$

where the effective worldvolume stress tensor is

$$T_{\mu\nu}^{\text{eff}} = - \frac{2}{\sqrt{-g_{(\text{long})}}} \frac{\delta I_{\text{eff}}}{\delta g^{\mu\nu}_{(\text{long})}} \Bigg|_{\mathcal{W}_{p+1}} . \quad (2\cdot10)$$

We now argue that the appropriate notion for this effective stress-tensor that captures the coupling of the short-wavelength degrees of freedom to the long-wavelength ones, is the quasilocal stress-energy tensor introduced by Brown and York.[8)] This is defined by considering a timelike hypersurface that lies away from the black brane and encloses it by extending along the worldvolume directions and the angular directions $\Omega_{(n+1)}$, i.e., the hypersurface acts as a boundary. The angular directions are integrated over in our description (and to leading order they do not play any role), so we can simplify the discussion by focusing exclusively on the worldvolume directions of the boundary. If the boundary metric (along worldvolume directions) is γ_{ab} then the quasilocal stress tensor is

$$T_{ab}^{(\text{quasilocal})} = -\frac{2}{\sqrt{-\gamma}}\frac{\delta I_{cl}}{\delta\gamma^{ab}}\,, \tag{2·11}$$

where I_{cl} is the classical on-shell action of the solution. For our purposes, this is the action where the short-distance gravitational degrees of freedom, $r \ll R$, are integrated and so it must be the same function of the collective variables as I_{eff}. Together with the relation (2·7) this implies that we can identify (2·10) with (2·11).

It is shown in 8) that the Einstein equations with an index orthogonal to the boundary are first-order equations equivalent to the equation of conservation of the quasilocal stress tensor,

$$D_a T^{ab}_{(\text{quasilocal})} = 0\,, \tag{2·12}$$

where D_a is the covariant derivative associated to the boundary metric γ_{ab}. Hence, solving Eq. (2·12) is equivalent to solving (a subset of) the Einstein equations.

Since we identify the stress tensors (2·10) and (2·11), henceforth we drop the superscripts from them. We also drop the superscript $^{(\text{long})}$ from the background metric $g_{\mu\nu}$.

The effective stress tensor is computed in the zone $r_0 \ll r \ll R$, where the gravitational field is weak and the quasilocal stress tensor T^{ab} is, to leading order in r_0/R, the same as the ADM stress tensor. For the boosted black p-brane (2·4) one can readily compute it and find

$$T^{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left(n u^a u^b - \eta^{ab}\right)\,. \tag{2·13}$$

After introducing a slow variation of the collective coordinates the stress tensor becomes

$$T^{ab}(\sigma) = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n(\sigma) \left(n u^a(\sigma) u^b(\sigma) - \gamma^{ab}(\sigma)\right) + \cdots\,, \tag{2·14}$$

where the dots stand for terms with gradients of $\ln r_0$, u^a, and γ_{ab}, which we are neglecting.

§3. Blackfold dynamics

The general effective theory of classical brane dynamics can be formulated as a theory of a fluid on a dynamical worldvolume. The fluid variables must satisfy the intrinsic equation (2·12), and they will be coupled to the 'extrinsic' equations for the dynamics of the worldvolume geometry, which we still have to determine. To this effect, in the next subsection we introduce a few notions about the geometry of worldvolume embeddings.

3.1. *Embedding and worldvolume geometry*

Given the induced metric on $\mathcal{W}_{p+1}$, (2·7), the first fundamental form of the submanifold is

$$h^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \gamma^{ab} . \tag{3·1}$$

Indices $\mu\,,\nu$ are raised and lowered with $g_{\mu\nu}$, and a, b with γ_{ab}. Defining

$$\perp_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu} \tag{3·2}$$

it is easy to see that the tensor $h^\mu{}_\nu$ acts as a projector onto $\mathcal{W}_{p+1}$, and $\perp^\mu{}_\nu$ along directions orthogonal to $\mathcal{W}_{p+1}$.

Background tensors $t^{\mu\cdots}{}_{\nu\dots}$ with support on $\mathcal{W}_{p+1}$ can be converted into worldvolume tensors $t^{a\cdots}{}_{b\dots}$ and viceversa using $\partial_a X^\mu$. For instance, the velocity field

$$u^\mu = \partial_a X^\mu \, u^a \,, \tag{3·3}$$

preserves its negative-unit norm under this mapping.

The covariant differentiation of tensors that live in the worldvolume is well defined only along tangential directions, which we denote by an overbar,

$$\overline{\nabla}_\mu = h_\mu{}^\nu \nabla_\nu \,. \tag{3·4}$$

Note that in general $\overline{\nabla}_\rho t^{\mu\cdots}{}_{\nu\dots}$ has both orthogonal and tangential components. The tangentially projected part is essentially the same as the worldvolume covariant derivative $D_c t^{a\cdots}{}_{b\dots}$ for the metric γ_{ab}, both tensors being related via the pull-back map $\partial_a X^\mu$. In particular, the divergence of the stress-energy tensor

$$T^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \, T^{ab} \tag{3·5}$$

satisfies

$$h^\rho{}_\nu \overline{\nabla}_\mu T^{\mu\nu} = \partial_b X^\rho D_a T^{ab} \,. \tag{3·6}$$

The *extrinsic curvature tensor*

$$K_{\mu\nu}{}^\rho = h_\mu{}^\sigma \overline{\nabla}_\nu h_\sigma{}^\rho \tag{3·7}$$

is tangent to $\mathcal{W}_{p+1}$ along its (symmetric) lower indices μ, ν, and orthogonal to $\mathcal{W}_{p+1}$ along ρ. Its trace is the *mean curvature vector*

$$K^\rho = h^{\mu\nu} K_{\mu\nu}{}^\rho = \overline{\nabla}_\mu h^{\mu\rho} \,. \tag{3·8}$$

Explicit expressions for the extrinsic curvature tensor in terms of the embedding functions $X^\mu(\sigma^a)$ can be found in the appendix.

3.2. *Blackfold equations*

The general extrinsic dynamics of a brane has been analyzed by Carter in 9). The equations are formulated in terms of a stress-energy tensor with support on the $p+1$-dimensional worldvolume $\mathcal{W}_{p+1}$ satisfying the tangentiality condition

$$\perp^{\rho}{}_{\mu} T^{\mu\nu} = 0 \,. \tag{3·9}$$

The basic assumptions are that (i) this effective stress-energy tensor derives from an underlying conservative dynamics (in our case, General Relativity), even if the macroscopic (= long-wavelength) dynamics may be dissipative; and that (ii) spacetime diffeomorphism invariance holds, or equivalently, the worldvolume theory can be consistently coupled to the long-wavelength gravitational field $g_{\mu\nu}$. Under these assumptions, the stress tensor must obey the conservation equations

$$\overline{\nabla}_{\mu} T^{\mu\rho} = 0 \,. \tag{3·10}$$

These are in fact the generic equations of motion for the entire set of worldvolume field variables $\phi(\sigma^a)$, both intrinsic and extrinsic: we can decompose (3·10) along directions parallel and orthogonal to $\mathcal{W}_{p+1}$ as

$$\begin{aligned}\overline{\nabla}_{\mu} T^{\mu\rho} &= \overline{\nabla}_{\mu}(T^{\mu\nu} h_{\nu}{}^{\rho}) = T^{\mu\nu}\overline{\nabla}_{\mu} h_{\nu}{}^{\rho} + h_{\nu}{}^{\rho}\overline{\nabla}_{\mu} T^{\mu\nu} \\ &= T^{\mu\nu} h_{\nu}{}^{\sigma}\overline{\nabla}_{\mu} h_{\sigma}{}^{\rho} + h_{\nu}{}^{\rho}\overline{\nabla}_{\mu} T^{\mu\nu} \\ &= T^{\mu\nu} K_{\mu\nu}{}^{\rho} + \partial_b X^{\rho} D_a T^{ab} \,,\end{aligned} \tag{3·11}$$

where in the last line we used (3·6) and (3·7). Thus the D equation (3·10) separate into $D-p-1$ equations in directions orthogonal to $\mathcal{W}_{p+1}$ and $p+1$ equations parallel to $\mathcal{W}_{p+1}$,

$$T^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0 \,, \qquad (\textit{extrinsic equations}) \tag{3·12}$$

$$D_a T^{ab} = 0 \,. \qquad (\textit{intrinsic equations}) \tag{3·13}$$

Let us now apply Eq. (3·10) onto the specific stress tensor of a neutral black brane, (2·13). After a little manipulation one finds

$$\dot{u}^{\mu} + \frac{1}{n+1} u^{\mu}\overline{\nabla}_{\nu} u^{\nu} = \frac{1}{n} K^{\mu} + \overline{\nabla}^{\mu} \ln r_0 \,. \tag{3·14}$$

These *blackfold equations* describe the general collective dynamics of a neutral black brane.

Blackfolds differ from other branes in that they represent objects with black hole horizons. In the long-distance effective theory we lose sight of the horizon, since its thickness is of the order of the scale r_0 that we integrate out. But the presence of the horizon is reflected in the effective theory in the existence of an entropy and in the local thermodynamic equilibrium of the effective fluid.

3.3. *The metric at all length scales: Matched asymptotic expansion*

Under the splitting in (2·2), the set of field variables in the system are the collective worldvolume fields, intrinsic and extrinsic, and the background gravitational field $g_{\mu\nu}$. The complete set of equations are the extrinsic equation (3·12), intrinsic equation (3·13), and backreaction equation (2·9). Since they are a consequence of general symmetry and conservation principles, these equations retain their form at any perturbative order.*) The specific form of the stress tensor, as well as the background metric, will in general be corrected at higher orders.

The only equations that one has to solve at the lowest order are those that suffice to ensure that $T_{\mu\nu}$ can be consistently coupled to the long-wavelength gravitational field. These are just the intrinsic and extrinsic equations, and backreaction is neglected. The explicit blackfold equation (3·14) that result are valid only for test branes.

Using this approach, Ref. 3) has managed to construct large new classes of higher-dimensional black holes. In the following we will concentrate on a different application of the method.

§4. Gregory-Laflamme instability in blackfolds

The blackfold approach must capture the perturbative dynamics of a black hole when the perturbation wavelength λ is long,

$$\lambda \gg r_0 \,. \tag{4·1}$$

These perturbations can be either intrinsic variations in the thickness r_0 and local velocity u, or extrinsic variations in the worldvolume embedding geometry X. In general, these two kinds of perturbations are coupled. Here we extract some simple but important consequences for perturbations with wavelength

$$r_0 \ll \lambda \ll R \,, \tag{4·2}$$

i.e., perturbations for which the worldvolume looks essentially flat, $K_{\mu\nu}{}^{\rho} \approx 0$. In this case it is easy to see that the intrinsic and extrinsic perturbations decouple.

4.1. *Perfect fluid approximation: GL as a sound-mode instability*

It is instructive to perform the analysis for a general perfect fluid, and then particularize to the neutral blackfold fluid (2·13). For simplicity we consider a fluid initially at rest $u^a = (1, 0 \ldots)$, with uniform equilibrium energy density ε and pressure P. The flat worldvolume metric is parametrized, in 'static gauge', using orthonormal coordinates $X^0 = t$, $X^i = z^i$, $i = 1, \ldots p$ and the transverse coordinates X^m are held at constant values. Introduce small perturbations

$$\delta\varepsilon \,, \qquad \delta P = \frac{dP}{d\varepsilon}\delta\varepsilon \,, \qquad \delta u^a = (0, v^i) \,, \qquad \delta X^m = \xi^m \,, \tag{4·3}$$

and work to linearized order in them.

*) This, however, is a somewhat formal statement due to the appearance of gravitational self-force divergences on the worldvolume that must be dealt with carefully.[5] Reference 10) shows how the equation of stress tensor conservation can be used as the basis to obtain these corrections to particle motion.

The extrinsic equation (3·12) then become

$$\left(\varepsilon\partial_t^2 + P\partial_i^2\right)\xi^m = 0\,. \tag{4·4}$$

Thus transverse, elastic oscillations of the brane propagate with speed

$$c_T^2 = -\frac{P}{\varepsilon}\,. \tag{4·5}$$

The intrinsic equation (3·13) give

$$\left(\partial_t^2 - \frac{dP}{d\varepsilon}\partial_i^2\right)\delta\varepsilon = 0\,, \tag{4·6}$$

so longitudinal, sound-mode oscillations of the fluid propagate with speed

$$c_L^2 = \frac{dP}{d\varepsilon}\,. \tag{4·7}$$

These Eqs. (4·5) and (4·7) are hardly new: they are the conventional speeds of elastic and sound waves. They have a remarkable consequence: a brane with a worldvolume fluid equation of state such that

$$\frac{P}{\varepsilon}\frac{dP}{d\varepsilon} > 0 \tag{4·8}$$

has

$$c_L^2\, c_T^2 < 0 \tag{4·9}$$

and so is unstable to either longitudinal or transverse oscillations with wavelengths in the range (4·2). For instance this happens in the simple case $P = w\varepsilon$ with constant w, where the interpretation is easy (we assume $\varepsilon > 0$): positive *tension* is required for elastic stability, but positive *pressure* is needed to prevent that the fluid clumps under any density perturbation.

Neutral blackfolds have

$$c_L^2 = -c_T^2 = -\frac{1}{n+1} \tag{4·10}$$

and therefore are generically unstable to longitudinal sound-mode oscillations and stable to elastic oscillations in the range of wavelengths (4·2).

This instability is not unexpected. Black branes suffer from the Gregory-Laflamme instability,[11] which makes the horizon radius vary as

$$\delta r_0 \sim e^{\Omega t + i k_i z^i}\,. \tag{4·11}$$

Here Ω is positive real and thus the frequency is imaginary. The threshold mode for the instability, with $\Omega = 0$ and $k = \sqrt{k^i k^i} \neq 0$, has 'small' wavelength $\lambda = 2\pi/k \sim r_0$ and therefore cannot be seen in the blackfold approximation. But the GL instability extends to arbitrarily small k, i.e., arbitrarily long wavelengths, and when k is very small it should be captured by the blackfold dynamics.

The sound-mode instability corresponds precisely to this long-wavelength part, $\Omega, k \to 0$, of the GL instability. Observe that sound waves in a blackfold produce $\delta\varepsilon \sim \delta P \sim \delta r_0$ i.e., variations in the horizon thickness. Equation (4·10) tells us that these are unstable, of the form (4·11) with dispersion relation

$$\Omega = \frac{1}{\sqrt{n+1}}\, k + O(k^2)\,. \tag{4·12}$$

4.2. *Inclusion of black brane viscosity*

In 4) we have analyzed long wavelength perturbations of the black brane and their effect on the stress tensor measured near spatial infinity. From this study we obtain

$$\eta = \frac{s}{4\pi}\,, \qquad \zeta = 2\eta\left(\frac{1}{p} - c_L^2\right)\,, \tag{4·13}$$

where s is the entropy density of the fluid, i.e., $1/4G$ times the area density of the black brane.

Using these results for η and ζ we can include the viscous damping of sound waves in the effective black brane fluid. The linear perturbation equations are now

$$-i\Omega\delta\rho + (\rho + P)k_i\delta u^i + O(k^3) = 0\,, \tag{4·14}$$

$$\Omega(\rho + P)\delta u^j + ic_s^2 k^j\delta\rho + \eta k^2\delta u^j + k^j\left(\left(1 - \frac{2}{p}\right)\eta + \zeta\right)k_l\delta u^l + O(k^3) = 0\,. \tag{4·15}$$

Applying our results above, any solution to these equations can be used to obtain an explicit black brane solution with a small, long-wavelength fluctuation of r_0 and u^a. If we eliminate $\delta\rho$ we find that non-trivial sound waves require

$$\Omega - c_s^2\frac{k^2}{\Omega} + \frac{k^2}{Ts}\left(2\left(1 - \frac{1}{p}\right)\eta + \zeta\right) + O(k^3) = 0\,, \tag{4·16}$$

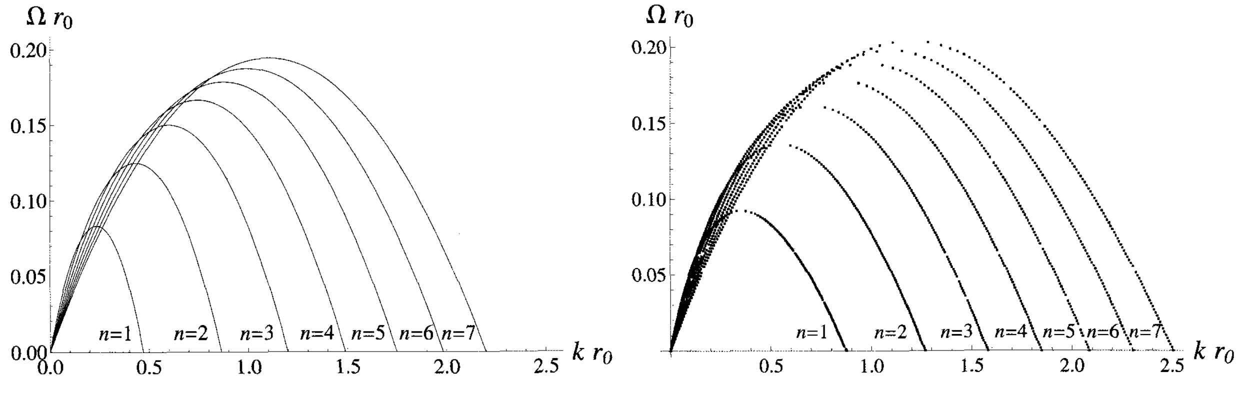

Fig. 1. Left: dispersion relation $\Omega(k)$, Eq. (4·18), for unstable sound waves in the effective black brane fluid (normalized relative to the thickness r_0). Right: $\Omega(k)$ for the unstable Gregory-Laflamme mode for black branes (numerical data courtesy of P. Figueras). For black p-branes in D spacetime dimensions, the curves depend only on $n = D - p - 3$.

where $k = \sqrt{k_i k_i}$ and we have used the Gibbs-Duhem relation $\rho + P = Ts$. This equation determines the dispersion relation $\Omega(k)$ as

$$\Omega = \sqrt{-c_s^2}k - \left(\left(1 - \frac{1}{p}\right)\frac{\eta}{s} + \frac{\zeta}{2s}\right)\frac{k^2}{T} + O(k^3)\,. \qquad (4\cdot17)$$

For the specific black p-brane fluid this yields the dispersion relation

$$\Omega = \frac{k}{\sqrt{n+1}}\left(1 - \frac{n+2}{n\sqrt{n+1}}\,kr_0\right)\,, \qquad (4\cdot18)$$

which is valid up to corrections $\propto k^3$. Figure 1 compares this dispersion relation to the numerical results obtained from linearized perturbations of a black p-brane. Zooming in on small values of kr_0, the match is excellent. When kr_0 is of order one we have no right to expect agreement, but the overall qualitative resemblance of the curves is nevertheless striking. The quantitative agreement improves with increasing n and indeed, as Fig. 2 shows, at large n it becomes impressively good over all wavelengths: for $n = 100$ the numerical values are reproduced to better than 1% accuracy up to the maximum value of k.

The ordinary derivation of the GL instability involves a complicated analysis of linearized gravitational perturbations of a black brane and the numerical resolution of a boundary value problem for a differential equation (which is moreover compounded at small k since larger grids are required to avoid finite-size problems). Here we have shown that the long-wavelength component of the instability, (4·12), can be obtained by an almost trivial calculation of the sound speed in a fluid. Including the effective viscosity of the fluid refines this calculation to the point of capturing in a simple manner all of the main features of the dispersion relation. In our opinion these results are striking evidence of the power of the blackfold approach.

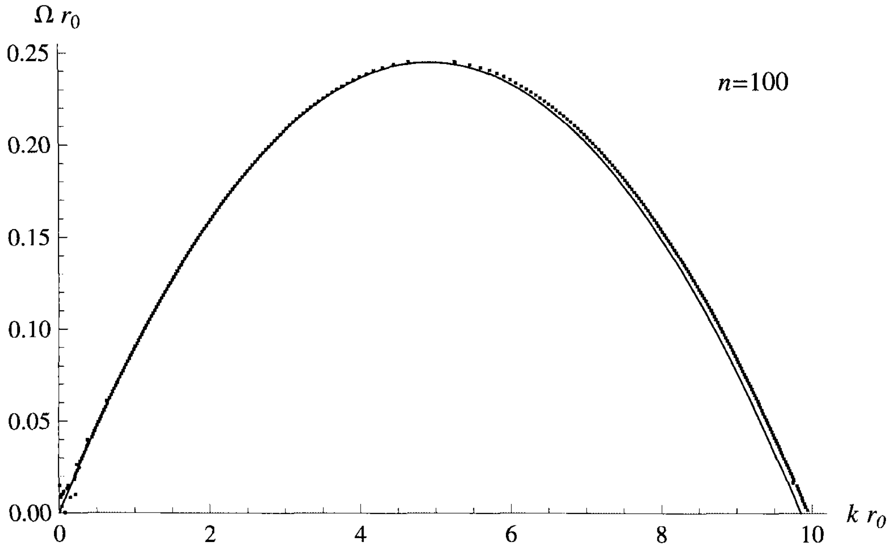

Fig. 2. Dispersion relation $\Omega(k)$ of unstable modes for $n = 100$: the solid line is our analytic approximation Eq. (4·18); the dots are the numerical solution of the Gregory-Laflamme perturbations of black branes (numerical data courtesy of P. Figueras).

§5. Discussion

The effective worldvolume theory that describes the long-wavelength dynamics of the black brane is quite similar to the effective Dirac-Born-Infeld theory for D-branes in open string theory, or the Nambu-Goto effective description for Nielsen-Olesen vortices. In our opinion, the blackfold approach should be regarded and judged in much the same way as one does in the case of these other effective theories, both in terms of its validity and of its utility. Blackfolds provide the leading order description of objects for which an exact account is very probably out of practical reach. Corrections to this leading order are more often than not very complicated too, but unless there is good reason to do so, one does not doubt the validity of this approximation to a full, physical solution describing the object in the complete theory.

There is however one significant respect in which blackfolds differ from DBI branes or NG strings (or indeed any other dynamical branes that we are aware of): the worldvolume theory of blackfolds features proper hydrodynamical behavior, in the sense of requiring local thermodynamical equilibrium. In contrast, NG strings have constant energy density and pressure, so their intrinsic dynamics is trivial, while the worldvolume dynamics of DBI branes is a nonlinear electrodynamics that does not involve thermal features. This is the main reason that in the blackfold theory configurations in stationary equilibrium are particularly singled out.

This approach is a powerful tool for the identification of new solutions and their properties. But its utility should not be reduced to only describing novel classes of stationary black holes, but also to analyzing their dynamics in specific physical situations.

The metric at all length scales

The blackfold approach might be regarded with scepticism since, although it is claimed that new black holes are found, no explicit black hole metric appears to be produced. Expressed in this crude form, this criticism is unwarranted. First, let us emphasize again the similarity to the fact that in general a solution of the DBI action does not provide an explicit solution to the full open string theory (indeed quite often it is not even known how to solve string theory in the backgrounds where this effective theory is applied), and a similar situation occurs for vortex strings and their Nambu-Goto description. Second, it is not quite true that no metric for the new black hole is given. It actually is, to leading order: far from the black hole, it is the background metric, with a submanifold singled out as the location of the blackfold; and near the black hole, it is the metric of a boosted black brane.

Nevertheless we admit that there is a point in this criticism, since traditionally black holes have been regarded as embodying a non-trivial geometry, and it might be desirable to see how a new metric is obtained for the new black hole, at least in principle. Indeed the first application of the blackfold methodology included a long and detailed analysis of the next-to-leading order metric for higher-dimensional black rings.[12] A more general analysis can be performed and is underway.

As explained in detail in 2),3),12), the method of matched asymptotic expansions

(MAE's) systematically produces an explicit solution for the geometry of the black hole spacetime at all scales, including the region near the horizon, with the effects of the bending of the black brane in an expansion in r_0/R. It should be appreciated that the leading order MAE is a rather involved technical task even in the very symmetric situations that we have studied, a fact that emphasizes once again the virtue of having a universal, long-distance effective theory that captures in a simple manner most of the physically interesting features of the solution.

Dynamical aspects: Stability and time-dependence

Many of the emerging new solutions of higher-dimensional gravity exhibit regions of instability in the blackfold regime. For example, ultraspinning MP black holes and thin black rings have been argued[13)] to be unstable under GL-type instabilities. Corresponding statements can be made for more general blackfolds. As we have seen, the blackfold approach does easily capture this instability for a generic blackfold in a regime in which the wavelength λ of the instability lies in the range $r_0 \ll \lambda \ll R$.

This is one example where one can decouple the extrinsic equations from the stability analysis of the intrinsic sector. Conversely, there are situations where the extrinsic stability can be analyzed while guaranteeing that the intrinsic equations remain solved. A simple instance is the study of stability against variations of the radius of round odd-sphere blackfolds. These solutions extremize a potential $V = -I/\beta$ where I is the action for the stationary configurations. This V is *minimized* by these solutions, implying that they are stable to variations of R (this was in fact already known for black rings[14)]). An explicit analysis of time-dependent perturbations confirms this result.

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